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# COMMON FIXED POINTS OF TWO FINITE FAMILIES OF NONEXPANSIVE MAPPINGS IN KOHLENBACH HYPERBOLIC SPACES

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**Abstract.** In this paper, we use a two-step iterative process for two finite families of nonexpansive mappings in a hyperbolic space defined by Kohlenbach. Strong and Δ-convergence theorems are established for this two-step iterative process being used for the first time in hyperbolic spaces.

**Keywords.** Hyperbolic space; Nonexpansive map; Common fixed point; iterative process; Semi-compactness.

## 1. Introduction

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers and $I = \{1, 2, \ldots, N\}$, the set of first $N$ natural numbers. Let $(X,d)$ be a metric space and $K$ be a nonempty subset of $X$. A selfmap $T$ on $K$ is said to be nonexpansive if $d(Tx,Ty) \leq d(x,y)$. Denote by $F(T)$ the set of fixed points of $T$ and by $F = \cap_{i=1}^{N} (F(T_i) \cap F(S_i))$ the set of common fixed points of two finite families of mappings $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$.

Takahashi [1] introduced the notion of convex metric spaces and studied the fixed point theory for nonexpansive mappings in this setting. Later on, several attempts have been made to

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introduce a convex structure on a metric space. One such convex structure is available in a hyperbolic space introduced by Kohlenbach [2]. Kohlenbach hyperbolic space [2] is restrictive than the hyperbolic type introduced in [3] and more general than the concept of hyperbolic s-pace in [5]. Spaces like CAT(0) and Banach are special cases of hyperbolic space. The class of hyperbolic spaces also contains Hadamard manifolds, Hilbert ball equipped with the hyperbolic metric [4], $\mathbb{R}$-trees and Cartesian products of Hilbert balls, as special cases.

A hyperbolic space [2] is a triple $(X,d,W)$ where $(X,d)$ is a metric space and $W: X^2 \times [0,1] \to X$ is such that

W1 $d(u,W(x,y,\alpha)) \leq (1-\alpha)d(u,x)+\alpha d(u,y)$,

W2 $d(W(x,y,\alpha),W(x,y,\beta)) = |\alpha-\beta| d(x,y)$,

W3 $W(x,y,\alpha) = W(y,x,(1-\alpha))$,

W4 $d(W(x,z,\alpha),W(y,w,\alpha)) \leq (1-\alpha)d(x,y)+\alpha d(z,w)$,

for all $x,y,z,w \in X$ and $\alpha,\beta \in [0,1]$. If $(X,d,W)$ satisfies only (W1), then it coincides with the convex metric space introduced by Takahashi [1]. A subset $K$ of a hyperbolic space $X$ is convex if $W(x,y,\alpha) \in K$ for all $x,y \in K$ and $\alpha \in [0,1]$.

A hyperbolic space $(X,d,W)$ is said to be:

i. strictly convex [1] if for any $x,y \in X$ and $\lambda \in [0,1]$, there exists a unique element $z \in X$ such that

$$d(z,x) = \lambda d(x,y) \text{ and } d(z,y) = (1-\lambda)d(x,y);$$

ii. uniformly convex [9] if for all $u,x,y \in X$, $r>0$ and $\varepsilon \in (0,2]$, there exists a $\delta \in (0,1]$ such that

$$\left.\begin{array}{r} d(x,u) \leq r \\ d(y,u) \leq r \\ d(x,y) \geq \varepsilon r \end{array}\right\} \Rightarrow d\left(W\left(x,y,\frac{1}{2}\right),u\right) \leq (1-\delta)r.$$

A map $\eta : (0,\infty)\times(0,2] \to (0,1]$ which provides such a $\delta = \eta(r,\varepsilon)$ for given $r>0$ and $\varepsilon \in (0,2]$, is called modulus of uniform convexity. We call $\eta$ monotone if it decreases with $r$ (for a fixed $\varepsilon$). A uniformly convex hyperbolic space is strictly convex (see [10]).

The concept of $\Delta$-convergence in a metric space was introduced by Lim [6] and its analogue in CAT(0) spaces has been investigated by Dhompongsa and Panyanak [7]. In [8], Khan et al. continued the investigation of $\Delta$-convergence in the general setup of hyperbolic spaces.

Now, we collect some basic concepts.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space $X$. For $x \in X$, define a continuous functional $r(., \{x_n\}) : X \to [0, \infty)$ by:

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius $\rho = r(\{x_n\})$ of $\{x_n\}$ is given by:

$$\rho = \inf \{r(x, \{x_n\}) : x \in X\}.$$

The asymptotic center of a bounded sequence $\{x_n\}$ with respect to a subset $K$ of $X$ is defined as follows:

$$A_K(\{x_n\}) = \{x \in X : r(x, \{x_n\}) \leq r(y, \{x_n\}) \text{ for any } y \in K\}.$$

If the asymptotic center is taken with respect to $X$, then it is simply denoted by $A(\{x_n\})$. It is known that uniformly convex Banach spaces and even CAT(0) spaces enjoy the property that "bounded sequences have unique asymptotic centers with respect to closed convex subsets".

The following lemma is due to Leustean [11].

**Lemma 1.1.** [11] *Let $(X, d, W)$ be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity. Then every bounded sequence $\{x_n\}$ in $X$ has a unique asymptotic center with respect to any nonempty closed convex subset $K$ of $X$.*

Recall that a sequence $\{x_n\}$ in $X$ is said to $\Delta$-converge to $x \in X$ if $x$ is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_n x_n = x$ and call $x$ as $\Delta$ - limit of $\{x_n\}$.

Recently, Khan *et al.* [8] proved the generalized version of Lemma 1.3 of Schu [13] and Lemma 2.9 of Laowang and Panyanak [12].

**Lemma 1.2.** [8] *Let $(X, d, W)$ be a uniformly convex hyperbolic space with monotone modulus of uniform convexity $\eta$. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[b, c]$ for some $b, c \in (0, 1)$. If*

$\{x_n\}$ *and* $\{y_n\}$ *are sequences in* $X$ *such that* $\limsup_{n\to\infty} d(x_n,x) \leq r, \limsup_{n\to\infty} d(y_n,x) \leq r$ *and* $\lim_{n\to\infty} d(W(x_n,y_n,\alpha_n),x) = r$ *for some* $r \geq 0$*, then* $\lim_{n\to\infty} d(x_n,y_n) = 0$.

Also, Khan *et al.* [8] gave and proved a metric version of a result due to Bose and Laskar [14] as follows. It plays a crucial role in proving $\Delta$-convergence of the our algorithm.

**Lemma 1.3.** [8] *Let $K$ be a nonempty closed convex subset of a uniformly convex hyperbolic space and $\{x_n\}$ a bounded sequence in $K$ such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in $K$ such that $\lim_{n\to\infty} r(y_m,\{x_n\}) = \rho$, then $\lim_{n\to\infty} y_m = y$.*

We know that Picard and Mann iteration processes are defined for given $x_1$ in $K$ (a subset of Banach space) respectively as:

$$x_{n+1} = Tx_n \tag{1.1}$$

and

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \tag{1.2}$$

where $\{\alpha_n\}$ is in $(0,1)$.

Very recently, Thianwan [15] considered a new class of iterative schemes in Banach spaces as follows:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)y_n + \alpha_n S y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \ n \in \mathbb{N}. \end{aligned} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0,1)$.

The process (1.3) reduces to Mann process (1.2) when $S = I$.

The two-step algorithm (1.3) can be defined for two finite families of nonexpansive selfmaps in a hyperbolic space as:

$$\begin{aligned} x_{n+1} &= W(y_n, S_n y_n, \alpha_n), \\ y_n &= W(x_n, T_n x_n, \beta_n), \ n \geq 1, \end{aligned} \tag{1.4}$$

where $T_n = T_{n(modN)}$ and $S_n = S_{n(modN)}$.

The purpose of this article is to investigate $\Delta-$ convergence as well as strong convergence of algorithm (1.4) for two finite families of nonexpansive maps in the more general setup of hyperbolic spaces. Our results can be viewed as refinement and generalization of several well-known results in uniformly convex Banach spaces.

## 2. Preliminaries

From now on for two finite families $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ of maps, we set $F = \cap_{i=1}^{N} (F(T_i) \cap F(S_i)) \neq \varnothing$.

**Lemma 2.1.** *Let $K$ be a nonempty closed convex subset of a hyperbolic space $X$ and let $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of nonexpansive selfmaps on $K$ such that $F \neq \emptyset$. Then for the sequence $\{x_n\}$ defined in* (1.4)*, we have* $\lim_{n\to\infty} d(x_n, p)$ *exists for $p \in F$.*

**Proof.** Let $p \in F$. Using (1.4), we have

$$\begin{aligned} d(y_n, p) &= d(W(x_n, T_n x_n, \beta_n), p) \\ &\leq (1-\beta_n) d(x_n, p) + \beta_n d(T_n x_n, p) \\ &\leq (1-\beta_n) d(x_n, p) + \beta_n d(x_n, p) \\ &= d(x_n, p) \end{aligned}$$

and so

$$\begin{aligned} d(x_{n+1}, p) &= d(W(y_n, S_n y_n, \alpha_n), p) \\ &\leq (1-\alpha_n) d(y_n, p) + \alpha_n d(S_n y_n, p) \\ &\leq (1-\alpha_n) d(y_n, p) + \alpha_n d(y_n, p) \\ &= d(y_n, p) \\ &\leq d(x_n, p). \end{aligned} \tag{2.1}$$

Thus $\lim_{n\to\infty} d(x_n, p)$ exists exists for each $p \in F$. The proof is completed.

**Lemma 2.2.** *Let $K$ be a nonempty closed convex subset of a uniformly convex hyperbolic space $X$ with monotone modulus of uniform convexity $\eta$ and let $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of nonexpansive selfmaps of $K$ such that $F \neq \emptyset$. Then for the sequence $\{x_n\}$ defined in* (1.4)*, we have*

$$\lim_{n\to\infty} d(x_n, T_l x_n) = \lim_{n\to\infty} d(x_n, S_l x_n) = 0 \text{ for each } l = 1, 2, \cdots, N.$$

**Proof.** Let $p \in F$. By Lemma 2.1, we see that $\lim_{n\to\infty} d(x_n, p)$ exists. Assume that $\lim_{n\to\infty} d(x_n, p) = c$. Using (1.4), we get

$$d(y_n, p) \leq d(x_n, p). \tag{2.2}$$

Taking the lim sup on both sides in the inequality (2.2), we have

$$\limsup_{n\to\infty} d(y_n, p) \le c. \tag{2.3}$$

In addition, $d(S_n y_n, p) \le d(y_n, p)$, taking the lim sup on both sides in this inequality, we have

$$\limsup_{n\to\infty} d(S_n y_n, p) \le c. \tag{2.4}$$

Moreover,

$$\lim_{n\to\infty} d(x_{n+1}, p) = \lim_{n\to\infty} d(W(y_n, S_n y_n, \alpha_n), p) = c. \tag{2.5}$$

By using (2.3), (2.4) and (2.5) and Lemma 1.2, we have

$$\lim_{n\to\infty} d(y_n, S_n y_n) = 0. \tag{2.6}$$

In addition, $d(T_n x_n, p) \le d(x_n, p)$ , and taking the lim sup on both sides in this inequality, we have

$$\limsup_{n\to\infty} d(T_n x_n, p) \le c. \tag{2.7}$$

Using (1.4), we have

$$\begin{aligned} d(x_{n+1}, p) &= d(W(y_n, S_n y_n, \alpha_n), p) \\ &\le (1-\alpha_n) d(y_n, p) + \alpha_n d(S_n y_n, p) \\ &\le (1-\alpha_n) d(y_n, p) + \alpha_n d(S_n y_n, y_n) + \alpha_n d(y_n, p) \\ &\le d(y_n, p) + d(S_n y_n, y_n). \end{aligned} \tag{2.8}$$

Taking the lim inf on both sides in the inequality (2.8), by (2.6) and $\lim_{n\to\infty} d(x_n, p) = c$, we have

$$\liminf_{n\to\infty} d(y_n, p) \ge c. \tag{2.9}$$

It follows from (2.3) and (2.9) that $\lim_{n\to\infty} d(y_n, p) = c$. This implies that

$$\lim_{n\to\infty} d(y_n, p) = \lim_{n\to\infty} d(W(x_n, T_n x_n, \beta_n), p) = c. \tag{2.10}$$

Using (2.7), (2.10) and Lemma 1.2,

$$\lim_{n\to\infty} d(x_n, T_n x_n) = 0. \tag{2.11}$$

From $y_n = W(x_n, T_n x_n, \beta_n)$ and (2.11), we have

$$\begin{aligned} d(y_n, x_n) &= d(W(x_n, T_n x_n, \beta_n), x_n) \\ &\leq \beta_n d(T_n x_n, x_n) \\ &\to 0 \text{ (as } n \to \infty). \end{aligned} \tag{2.12}$$

In addition,

$$\begin{aligned} d(x_n, S_n x_n) &\leq d(x_n, y_n) + d(y_n, S_n x_n) \\ &\leq d(x_n, y_n) + d(y_n, S_n y_n) + d(S_n y_n, S_n x_n) \\ &\leq d(x_n, y_n) + d(y_n, S_n y_n) + d(y_n, x_n). \end{aligned}$$

Thus, it follows from (2.6) and (2.12) that

$$\lim_{n\to\infty} d(x_n, S_n x_n) = 0. \tag{2.13}$$

By using (1.4), we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(W(y_n, S_n y_n, \alpha_n), x_n) \\ &\leq (1-\alpha_n) d(y_n, x_n) + \alpha_n d(S_n y_n, x_n) \\ &\leq (1-\alpha_n) d(y_n, x_n) + \alpha_n d(S_n y_n, y_n) + \alpha_n d(y_n, x_n) \\ &\leq d(y_n, x_n) + d(S_n y_n, y_n). \end{aligned}$$

It follows from (2.6) and (2.12) that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{2.14}$$

This implies that

$$\lim_{n\to\infty} d(x_{n+l}, x_n) = 0 \text{ for each } l \in I. \tag{2.15}$$

Clearly,

$$\begin{aligned} d(x_n, T_{n+l} x_n) &\leq d(x_n, x_{n+l}) + d(x_{n+l}, T_{n+l} x_{n+l}) + d(T_{n+l} x_{n+l}, T_{n+l} x_n) \\ &\leq d(x_n, x_{n+l}) + d(x_{n+l}, T_{n+l} x_{n+l}) + d(x_{n+l}, x_n) \\ &\leq 2d(x_n, x_{n+l}) + d(x_{n+l}, T_{n+l} x_{n+l}). \end{aligned}$$

Taking lim on both sides of the above inequality, we get from (2.11) and (2.15)

$$\lim_{n\to\infty} d\left(x_n, T_{n+l}x_n\right) = 0 \text{ for each } l \in I.$$

Since for each $l \in I$, the sequence $\{d\left(x_n, T_l x_n\right)\}$ is a subsequence of $\bigcup_{i=1}^{N}\{d\left(x_n, T_{n+l}x_n\right)\}$ and $\lim_{n\to\infty} d\left(x_n, T_{n+l}x_n\right) = 0$ for each $l \in I$, therefore

$$\lim_{n\to\infty} d\left(x_n, T_l x_n\right) = 0 \text{ for each } l \in I. \tag{2.16}$$

Similarly, we have

$$\lim_{n\to\infty} d\left(x_n, S_{n+l}x_n\right) = 0 \text{ for each } l \in I,$$

and hence

$$\lim_{n\to\infty} d\left(x_n, S_l x_n\right) = 0 \text{ for each } l \in I. \tag{2.17}$$

This completes the proof.

## 3. Main results

In this section, we prove theorems of strong and $\Delta-$convergence of the iterative scheme given in (1.4) to a common fixed point for two finite families of nonexpansive maps in a hyperbolic spaces.

### 3.1 Strong convergence theorems

For further development, we need the following concepts and technical results.

A sequence $\{x_n\}$ in a metric space $X$ is said to be Fejér monotone with respect to $K$ (a subset of $X$) if $d\left(x_{n+1}, p\right) \leq d\left(x_n, p\right)$ for all $p \in K$ and for all $n \geq 1$. A map $T : K \to K$ is semi-compact if any bounded sequence $\{x_n\}$ satisfying $d\left(x_n, Tx_n\right) \to 0$ as $n \to \infty$, has a convergent subsequence.

Two mappings $T, S : K \to K$ are said to satisfy condition $(A')$ [16, 17] if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$ such that

$$\frac{1}{2}\left(d\left(x, Tx\right) + d\left(x, Sx\right)\right) \geq f\left(d\left(x, F\right)\right)$$

for all $x \in K$, where $d\left(x, F\right) = \inf\{d\left(x, p\right) : p \in F := F(T) \cap F(S)\}$.

We can modify this definition for two finite families of mappings as follows. Let $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of nonexpansive selfmaps on $K$ with $F \neq \emptyset$. Then the two families are said to satisfy condition (B) on $K$ if

$$\max_{1 \leq l \leq N} \{(d(x, T_l x), d(x, S_l x))\} \geq f(d(x, F)) \text{ for all } x \in K.$$

**Lemma 3.1.** [18] *Let $K$ be a nonempty closed subset of a complete metric space $(X, d)$ and $\{x_n\}$ be Fejér monotone with respect to $K$. Then $\{x_n\}$ converges to some $p \in K$ if and only if* $\lim_{n\to\infty} d(x_n, K) = 0$.

**Lemma 3.2.** *Let $K$ be a nonempty closed convex subset of a complete uniformly convex hyperbolic space $X$ with monotone modulus of uniform convexity $\eta$ and let $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of nonexpansive selfmaps on $K$ such that $F \neq \emptyset$. Then the sequence $\{x_n\}$ defined in $(1.4)$ converges strongly to $p \in F$ if and only if* $\lim_{n\to\infty} d(x_n, F) = 0$.

**Proof.** It follows from (2.1) that $\{x_n\}$ is Fejér monotone with respect to $F$ and $\lim_{n\to\infty} d(x_n, F)$ exists. Hence, the result follows from Lemma 3.1.

**Theorem 3.3.** *Let $K$ be a nonempty closed convex subset of a complete uniformly convex hyperbolic space $X$ with monotone modulus of uniform convexity $\eta$ and let $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of nonexpansive selfmaps on $K$ such that $F \neq \emptyset$. Suppose that $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ satisfy condition $(B)$. Then the sequence $\{x_n\}$ defined in $(1.4)$ converges strongly to $p \in F$.*

**Proof.** By Lemma 2.1, we see that $\lim_{n\to\infty} d(x_n, F)$ exists for all $p \in F$. Also, by Lemma 2.2, we find that $\lim_{n\to\infty} d(x_n, T_l x_n) = \lim_{n\to\infty} d(x_n, S_l x_n) = 0$ for each $l \in I$. By using condition $(B)$, we get $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since $f$ is nondecreasing with $f(0) = 0$, it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. Therefore, Lemma 3.2 implies that $\{x_n\}$ converges strongly to a point $p$ in $F$.

Note that the Condition (B) is weaker than both the compactness of $K$ and the semicompactness of the nonexpansive mappings $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$, therefore we already have the following theorem proved. However, for the sake of completeness, we include its proof in the following.

**Theorem 3.4.** *Let $K$ be a nonempty closed convex subset of a complete uniformly convex hyperbolic space $X$ with monotone modulus of uniform convexity $\eta$ and let $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of nonexpansive selfmaps on $K$ such that $F \neq \emptyset$. Suppose that either $K$ is compact or one of the map in $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ is semi-compact. Then the sequence $\{x_n\}$ defined in* (1.4) *converges strongly to $p \in F$.*

**Proof.** Suppose that $T_{i_0}$ and $S_{j_0}$ are semicompact for some $i_0, j_0 \in I$. It follows from Lemma 2.2 that $\lim_{n\to\infty} d(x_n, T_{i_0}x_n) = \lim_{n\to\infty} d\left(x_n, S_{j_0}x_n\right) = 0$. By semicompactness of the mappings $T_{i_0}$ and $S_{j_0}$, there exist a subsequence $\left\{x_{n_j}\right\}$ of $\{x_n\}$ such $\lim_{j\to\infty} x_{n_j} = p \in K$ and

$$\lim_{n\to\infty} d\left(x_{n_j}, T_{i_0}x_{n_j}\right) = \lim_{n\to\infty} d\left(x_{n_j}, S_{j_0}x_{n_j}\right) = 0.$$

Now by Lemma 2.2, we obtain that $\lim_{n\to\infty} d\left(x_{n_j}, T_l x_{n_j}\right) = 0$ for all $l \in I$, this implies that $d(p, T_l p) = 0$ for all $l \in I$. Thus $p$ is a common fixed point of finite family of mappings $\{T_i : i \in I\}$. Similarly we can prove that $p$ is a common fixed point of finite family of mappings $\{S_i : i \in I\}$. Then $p \in F$, which leads to $\lim_{n\to\infty} d(x_n, F) = 0$. By Lemma 3.2, we have $\{x_n\}$ converges strongly to a common fixed point in $F$. This completes the proof.

### 3.2 Δ-Convergence theorems

Finally, we give our Δ-convergence theorems.

**Theorem 3.5.** *Let $K$ be a nonempty closed convex subset of a complete uniformly convex hyperbolic space $X$ with monotone modulus of uniform convexity $\eta$ and let $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$ be two finite families of nonexpansive selfmaps on $K$ such that $F \neq \emptyset$. Then the sequence $\{x_n\}$ defined by* (1.4) *$\Delta-$converges to a common fixed point of $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$.*

**Proof.** It follows from Lemma 2.1 that $\{x_n\}$ is bounded. Therefore by Lemma 1.1, $\{x_n\}$ has a unique asymptotic center, that is, $A(\{x_n\}) = \{x_n\}$. Assume that $\{u_n\}$ is any subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u_n\}$. Then by Lemma 2.2, we have $\lim_{n\to\infty} d(u_n, T_l u_n) = \lim_{n\to\infty} d(u_n, S_l u_n) = 0$ for each $l = 1, 2, \cdots, N$. We claim that $u$ is the common fixed point of $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$.

Now, we define a sequence $\{v_n\}$ in $K$ by $v_m = T_m u$, where $T_m = T_{m(modN)}$.

On the other hand, we have

$$\begin{aligned} d(v_n, u_n) &\leq d(T_m u, T_m u_n) + d(T_m u_n, T_{m-1} u_n) + \cdots + d(T u_n, u_n) \\ &\leq d(u, u_n) + \sum_{i=1}^{m-1} d(u_n, T_i u_n). \end{aligned}$$

Therefore, we have

$$r(v_m, \{u_n\}) = \limsup_{n\to\infty} d(v_m, \{u_n\}) \leq \limsup_{n\to\infty} d(u, u_n) = r(u, u_n).$$

This implies that $|r(v_m, \{u_n\}) - r(u, u_n)| \to 0$ as $m \to \infty$. By Lemma 1.3, we get $T_{m(modN)} u = u$. Thus $u$ is the common fixed point of $\{T_i : i \in I\}$. By the same argument, we can show that $u$ is the common fixed point of $\{S_i : i \in I\}$. Therefore $u$ is the common fixed point of $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$. Moreover, $\lim_{n\to\infty} d(x_n, u)$ exists by Lemma 2.1.

Assume $x \neq u$. By the uniqueness of asymptotic centers,

$$\begin{aligned} \limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x) \\ &\leq \limsup_{n\to\infty} d(x_n, x) \\ &< \limsup_{n\to\infty} d(x_n, u) \\ &= \limsup_{n\to\infty} d(u_n, u). \end{aligned}$$

a contradiction. Thus $x = u$. Since $\{u_n\}$ is an arbitrary subsequence of $\{x_n\}$, therefore $A(\{u_n\}) = \{u\}$ for all subsequences $\{u_n\}$ of $\{x_n\}$. This proves that $\{x_n\}$ $\Delta-$converges to a common fixed point of $\{T_i : i \in I\}$ and $\{S_i : i \in I\}$.

Although the following can be obtained as a corollary from our above theorem by putting $S_i = T_i$ for all $i \in I$, yet it is new in itself.

**Corollary 3.6.** *Let $K$ be a nonempty closed convex subset of a complete uniformly convex hyperbolic space $X$ with monotone modulus of uniform convexity $\eta$ and let $\{T_i : i \in I\}$ be a finite family of nonexpansive selfmaps on $K$ such that $F \neq \emptyset$. Then the sequence $\{x_n\}$ defined by*

$$\begin{aligned} x_{n+1} &= W(y_n, T_n y_n, \alpha_n), \\ y_n &= W(x_n, T_n x_n, \beta_n), \ n \geq 1, \end{aligned}$$

$\Delta-$*converges to a common fixed point of* $\{T_i : i \in I\}$ *.*

**Remark 3.7.** Corollaries like Corollary 3.6 can now be obtained from Theorem 3.3 and Theorem 3.4.

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