COEFFICIENT ESTIMATES FOR BI-UNIVALENT FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

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Abstract. In this paper, we introduce certain subclasses of bi-univalent functions with respect to symmetric points in the unit disc \( E = \{ z : |z| < 1 \} \) and obtain coefficient estimates for the functions in these classes.

Keywords: Starlike functions with respect to symmetric points, convex functions with respect to symmetric points, univalent functions, bi-univalent functions, coefficient estimate, subordination.

1. Introduction and preliminaries

Let \( U \) be the class of bounded functions of the form

\[ u(z) = \sum_{k=1}^{\infty} c_k z^k \]

which are analytic in the unit disc \( E = \{ z : |z| < 1 \} \) and satisfying the conditions \( u(0) = 0 \) and \( |u(z)| < 1 \).

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Let $A$ be the class of analytic functions $f(z)$ in $E$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

Further, let $S$ be the class of functions $f(z) \in A$ and univalent in $E$.

It is well known that every function $f \in S$ has an inverse $f^{-1}$, defined by

$$f^{-1}(f(z)) = z(z \in E)$$

and

$$f(f^{-1}(w)) = w \left( |w| < r_0(f) : r_0(f) \geq \frac{1}{4} \right)$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + ...$$

Let $f$ and $g$ be two analytic functions in $E$. Then $f$ is said to be subordinate to $g$ (symbolically $f \prec g$) if there exists a bounded function $u(z) \in U$ such that $f(z) = g(u(z))$. This result is known as principle of subordination.

A function $f \in A$ is said to be bi-univalent in $E$ if both $f$ and $f^{-1}$ are univalent in $E$.

Let $\Sigma$ denote the class of bi-univalent functions in $E$ given by (1).

Lewin [6] investigated the class $\Sigma$ of bi-univalent functions and obtained the bound for the second coefficient. Brannan and Taha [2] considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced bi-starlike functions and bi-convex functions and obtained estimates on the initial coefficients. Bounds for the initial coefficients of several sub-classes of bi-univalent functions were also investigated in [3,5-10].
In this paper, estimates on the initial coefficients for bi-univalent functions with respect to symmetric points are obtained. This class was motivated by Rosihan M. Ali et al. [1].

In the sequel, to avoid repetition, we assume that $0 < \alpha \leq 1$, $\lambda \geq 0$, $-1 \leq B < A \leq 1$, $z \in E$, $w \in E$ and $g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_3^2 - 5a_2a_3 + a_4)w^4 + ...$.

**Definition 1.1.** A function $f(z) \in A$ is said to be in the class $M_{\Sigma}^\alpha(\alpha, \lambda; A, B)$ if the following conditions are satisfied:

$$f \in \Sigma \text{ and } (1 - \lambda) \frac{2zf'(z)}{f(z) - f(-z)} + \lambda \frac{2zf'(z)}{(f(z) - f(-z))'} \prec \left( \frac{1 + Az}{1 + Bz} \right)^\alpha$$

and

$$(1 - \lambda) \frac{2wg'w}{g(w) - g(-w)} + \lambda \frac{2wg'w}{(g(w) - g(-w))'} \prec \left( \frac{1 + Aw}{1 + Bw} \right)^\alpha.$$

In particular, $M_{\Sigma}^\alpha(\alpha, \lambda; 1, -1) \equiv M_{\Sigma}^\alpha(\alpha, \lambda)$ and $M_{\Sigma}^\alpha(1, \lambda; 1 - 2\beta, -1) \equiv M_{\Sigma}^\alpha(\beta, \lambda)$.

**Definition 1.2.** A function $f(z) \in A$ is said to be in the class $N_{\Sigma}^\alpha(\alpha, \lambda; A, B)$ if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \frac{2zf'(z) + \lambda 2z^2 f''(z)}{(1 - \lambda)(f(z) - f(-z)) + \lambda z(f(z) - f(-z))'} \prec \left( \frac{1 + Az}{1 + Bz} \right)^\alpha$$

and

$$\frac{2wg'w + \lambda 2w^2 g''(w)}{(1 - \lambda)(g(w) - g(-w)) + \lambda w(g(w) - g(-w))'} \prec \left( \frac{1 + Aw}{1 + Bw} \right)^\alpha.$$

Particularly, $N_{\Sigma}^\alpha(\alpha, \lambda; 1, -1) \equiv N_{\Sigma}^\alpha(\alpha, \lambda)$ and $N_{\Sigma}^\alpha(1, \lambda; 1 - 2\beta, -1) \equiv N_{\Sigma}^\alpha(\beta, \lambda)$. 
Definition 1.3. A function \( f(z) \in A \) is said to be in the class \( P^s_\Sigma(\alpha, \lambda; A, B) \) if the following conditions are satisfied:

\[
f \in \Sigma \text{ and } \left( \frac{2zf'(z)}{f(z) - f(-z)} \right)^{(1-\lambda)} + \left( \frac{2zf'(z)'}{(f(z) - f(-z))'} \right)^\lambda \prec \left( \frac{1 + Az}{1 + Bz} \right)^\alpha
\]

and

\[
\left( \frac{2wg'(w)}{g(w) - g(-w)} \right)^{(1-\lambda)} + \left( \frac{2wg'(w)'}{(g(w) - g(-w))'} \right)^\lambda \prec \left( \frac{1 + Aw}{1 + Bw} \right)^\alpha.
\]

Obviously, \( P^s_\Sigma(\alpha, \lambda; 1, -1) \equiv P^s_\Sigma(\alpha, \lambda) \) and \( P^s_\Sigma(1, \lambda; 1 - 2\beta, -1) \equiv P^s_\Sigma(\beta, \lambda) \).

For deriving our main results, we need to the following lemma:

Lemma 1.1. If \( p(z) = \frac{1 + Au(z)}{1 + Bu(z)} = 1 + \sum_{k=1}^{\infty} p_k z^k \), \( u(z) \in U \),

then \( |p_n| \leq (A - B), n \geq 1 \).

This result is due to Goel and Mehrok [4].

2. Coefficient bounds for the function class \( M^s_\Sigma(\alpha, \lambda; A, B) \)

Theorem 2.1. If \( f \in M^s_\Sigma(\alpha, \lambda; A, B) \), then

\[
|a_2| \leq \frac{\alpha \sqrt{(A - B)}}{\sqrt{2[(1 + \lambda)^2 - \alpha \lambda^2]}}
\]

and

\[
|a_3| \leq \frac{\alpha^2(A - B)^2}{4(1 + \lambda)^2} + \frac{\alpha(A - B)}{2(1 + 2\lambda)}.
\]

Proof. From definition 1.1, by principle of subordination, we have

\[
(1 - \lambda) \frac{2zf'(z)}{f(z) - f(-z)} + \lambda \frac{2zf'(z)'}{(f(z) - f(-z))'} = \left( \frac{1 + Au(z)}{1 + Bu(z)} \right)^\alpha = [p(z)]^\alpha, u \in U
\]
and

$$(5) \quad (1 - \lambda) \frac{2wg'(w)}{g(w) - g(-w)} + \lambda \frac{2(wg'(w))'}{(g(w) - g(-w))'} = \left( \frac{1 + Av(w)}{1 + Bv(w)} \right)^\alpha = [q(w)]^\alpha, v \in U,$$

where $p(z) = 1 + p_1 z + p_2 z^2 + ...$ and $q(w) = 1 + q_1 w + q_2 w^2 + ...$.

On expanding and equating the coefficients of $z$ and $z^2$ in (4) and of $w$ and $w^2$ in (5), we obtain

$$(6) \quad 2(1 + \lambda)a_2 = \alpha p_1,$$

$$(7) \quad 2(1 + 2\lambda)a_3 = \alpha p_2 + \frac{\alpha(\alpha - 1)p_1^2}{2}$$

and

$$(8) \quad -2(1 + \lambda)a_2 = \alpha q_1,$$

$$(9) \quad 2(1 + 2\lambda)(2a_2^2 - a_3) = \alpha q_2 + \frac{\alpha(\alpha - 1)q_1^2}{2}.$$

(6) and (8) together gives

$$\quad p_1 = -q_1$$

and

$$(11) \quad 8(1 + \lambda)^2 a_2^2 = \alpha^2(p_1^2 + q_1^2).$$

Adding (7) and (9) and using (11), it yields

$$\quad 4(1 + 2\lambda)a_2^2 = \alpha(p_2 + q_2) + \frac{4(\alpha - 1)(1 + \lambda)^2 a_2^2}{\alpha}.$$

(12) gives

$$(13) \quad a_2^2 = \frac{\alpha^2(p_2 + q_2)}{4\alpha(1 + 2\lambda) + 4(1 - \alpha)(1 + \lambda)^2}.$$

On applying Lemma 1.1 to the coefficients $p_2$ and $q_2$, we can easily obtain (2).

Now subtracting (9) from (7), we get

$$\quad 4(1 + 2\lambda)a_3 - 4(1 + 2\lambda)a_2^2 = \alpha(p_2 - q_2) + \frac{\alpha(\alpha - 1)(p_1^2 - q_1^2)}{2}.$$
Using (10) and (11), (14) yields

\begin{equation}
    a_3 = \frac{\alpha^2 p_1^2}{4(1+\lambda)^2} + \frac{\alpha (p_2 - q_2)}{4(1+2\lambda)}.
\end{equation}

Applying Lemma 1.1 to the coefficients \(p_2, q_2\) and \(p_1\) in (15), (3) is obvious.

For \(A = 1, B = -1\), Theorem 2.1 gives the following result:

**Corollary 2.1.** If \(f(z) \in M_S^\alpha(\alpha, \lambda)\), then

\[
|a_2| \leq \frac{\alpha}{\sqrt{(1+\lambda)^2 - \alpha^2}}
\]

and

\[
|a_3| \leq \frac{\alpha^2}{(1+\lambda)^2} + \frac{\alpha}{1+2\lambda}.
\]

Putting \(A = 1 - 2\beta, B = -1\) and \(\alpha = 1\) in Theorem 2.1, we obtain the following result:

**Corollary 2.2.** If \(f(z) \in M_S^\beta(\beta, \lambda)\), then

\[
|a_2| \leq \sqrt{\frac{1-\beta}{1+2\lambda}}
\]

and

\[
|a_3| \leq \frac{(1-\beta)^2}{(1+\lambda)^2} + \frac{(1-\beta)}{1+2\lambda}.
\]

3. **Coefficient bounds for the function class \(N_S^{\alpha}(\alpha, \lambda; A, B)\)**

**Theorem 3.1.** If \(f \in N_S^{\alpha}(\alpha, \lambda; A, B)\), then

\[
|a_2| \leq \frac{\alpha \sqrt{(A-B)}}{\sqrt{2[(1+\lambda)^2 - \alpha^2]}}
\]

and

\[
|a_3| \leq \frac{\alpha^2 (A-B)^2}{4(1+\lambda)^2} + \frac{\alpha (A-B)}{2(1+2\lambda)}.
\]

The proof of this theorem is similar to as in Theorem 2.1.
4. Coefficient bounds for the function class $P_s^\Sigma(\alpha, \lambda; A, B)$

**Theorem 4.1.** If $f \in P_s^\Sigma(\alpha, \lambda; A, B)$, then

\begin{equation}
|a_2| \leq \frac{\alpha \sqrt{(A - B)}}{\sqrt{2(1 + \lambda)^2 - \alpha \lambda}}
\end{equation}

and

\begin{equation}
|a_3| \leq \frac{\alpha^2 (A - B)^2}{4(1 + \lambda)^2} + \frac{\alpha (A - B)}{2(1 + 2\lambda)}.
\end{equation}

**Proof.** From definition 1.3, by principle of subordination, we have

\begin{equation}
\left( \frac{2zf'(z)}{f(z) - f(-z)} \right)^{(1-\lambda)} + \left( \frac{2zf'(z)}{f(z) - f(-z)} \right)^\lambda = \left( \frac{1 + Au(z)}{1 + Bu(z)} \right)^\alpha = [p(z)]^\alpha, u \in U
\end{equation}

and

\begin{equation}
\left( \frac{2wg'(w)}{g(w) - g(-w)} \right)^{(1-\lambda)} + \left( \frac{2wg'(w)}{g(w) - g(-w)} \right)^\lambda = \left( \frac{1 + Av(w)}{1 + Bv(w)} \right)^\alpha = [q(w)]^\alpha, v \in U,
\end{equation}

where $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ and $q(w) = 1 + q_1 w + q_2 w^2 + \ldots$.

On expanding and equating the coefficients of $z$ and $z^2$ in (18) and of $w$ and $w^2$ in (19), we obtain

\begin{equation}
2(1 + \lambda) a_2 = \alpha p_1,
\end{equation}

\begin{equation}
2(1 + 2\lambda) a_3 - 2\lambda(1 - \lambda) a_2^2 = \alpha p_2 + \frac{\alpha(\alpha - 1)p_1^2}{2}
\end{equation}

and

\begin{equation}
-2(1 + \lambda) a_2 = \alpha q_1,
\end{equation}

\begin{equation}
2(1 + 2\lambda)(2a_2^2 - a_3) - 2\lambda(1 - \lambda) a_2^2 = \alpha q_2 + \frac{\alpha(\alpha - 1)q_1^2}{2}.
\end{equation}

(20) and (22) together gives

\begin{equation}
p_1 = -q_1
\end{equation}

and

\begin{equation}8(1 + \lambda)^2 a_2^2 = \alpha^2 (p_1^2 + q_1^2).
\end{equation}
Adding (21) and (23) and using (25), it yields

\[ a_2^2 = \frac{\alpha^2(p_2 + q_2)}{4[(1 + \lambda)^2 - \alpha \lambda]} \]  

(26)

On applying Lemma 1.1 to the coefficients \( p_2 \) and \( q_2 \), we can easily obtain (16).

Now subtracting (23) from (21), we get

\[ 4(1 + 2\lambda)a_3 - 4(1 + 2\lambda)a_2^2 = \alpha(p_2 - q_2) + \frac{\alpha(\alpha - 1)(p_1^2 - q_1^2)}{2} \]  

(27)

Using (24) and (25), (27) yields

\[ a_3 = \frac{\alpha^2 p_1^2}{4(1 + \lambda)^2} + \frac{\alpha(p_2 - q_2)}{4(1 + 2\lambda)}. \]  

(28)

Applying Lemma 1.1 to the coefficients \( p_2, q_2 \) and \( p_1 \) in (28), (17) is obvious.

For \( A = 1, B = -1 \), Theorem 4.1 gives the following result:

**Corollary 4.1.** If \( f(z) \in P^\alpha_\Sigma(\alpha, \lambda) \), then

\[ |a_2| \leq \frac{\alpha}{\sqrt{(1 + \lambda)^2 - \alpha \lambda}} \]

and

\[ |a_3| \leq \frac{\alpha^2}{(1 + \lambda)^2} + \frac{\alpha}{1 + 2\lambda}. \]

Putting \( A = 1 - 2\beta, B = -1 \) and \( \alpha = 1 \) in Theorem 4.1, we obtain the following result:

**Corollary 4.2.** If \( f(z) \in M^\beta_\Sigma(\beta, \lambda) \), then

\[ |a_2| \leq \sqrt{\frac{1 - \beta}{1 + 2\lambda}} \]

and

\[ |a_3| \leq \frac{(1 - \beta)^2}{(1 + \lambda)^2} + \frac{(1 - \beta)}{1 + 2\lambda}. \]
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REFERENCES


