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COEFFICIENT ESTIMATES FOR BI-UNIVALENT FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

GAGANDEEP SINGH

Department of Mathematics, DIPS College(Co-Educational), Dhilwan (Kapurthala), Punjab, India

Abstract. In this paper, we introduce certain subclasses of bi-univalent functions with respect to symmetric points in the unit disc $E = \{z :| z |< 1\}$ and obtain coefficient estimates for the functions in these classes.

Keywords: Starlike functions with respect to symmetric points, convex functions with respect to symmetric points, univalent functions, bi-univalent functions, coefficient estimate, subordination.

1. Introduction and preliminaries

Let U be the class of bounded functions of the form

$$u(z) = \sum_{k=1}^{\infty} c_k z^k$$

which are analytic in the unit disc $E = \{z :| z |< 1\}$ and satisfying the conditions $u(0) = 0$ and $| u(z) |< 1$.

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Let A be the class of analytic functions $f(z)$ in E of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

Further, let S be the class of functions $f(z) \in A$ and univalent in E.

It is well known that every function $f \in S$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z(z \in E)$$

and

$$f(f^{-1}(w)) = w\left(\mid w \mid < r_0(f) : r_0(f) \geq \frac{1}{4}\right)$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + ...$$

Let f and g be two analytic functions in E. Then f is said to be subordinate to g (symbolically $f \prec g$) if there exists a bounded function $u(z) \in U$ such that $f(z) = g(u(z))$. This result is known as principle of subordination.

A function $f \in A$ is said to be bi-univalent in E if both f and f^{-1} are univalent in E.

Let Σ denote the class of bi-univalent functions in E given by (1).

Lewin [6] investigated the class Σ of bi-univalent functions and obtained the bound for the second coefficient. Brannan and Taha [2] considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced bi-starlike functins and bi-convex functions and obtained estimates on the initial coefficients. Bounds for the initial coefficients of several sub-classes of bi-univalent functions were also investigated in [3,5-10].

In this paper, estimates on the initial coefficients for bi-univalent functions with respect to symmetric points are obtained. This class was motivated by Rosihan M. Ali et al. [1].

In the sequel, to avoid repetition, we assume that $0 < \alpha \leq 1$, $\lambda \geq 0$, $-1 \leq B < A \leq 1$, $z \in E$, $w \in E$ and $g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots$.

Definition 1.1. A function $f(z) \in A$ is said to be in the class $M_\Sigma^s(\alpha, \lambda; A, B)$ if the following conditions are satisfied:

$$f \in \Sigma \text{ and } (1-\lambda)\frac{2zf'(z)}{f(z)-f(-z)} + \lambda\frac{2(zf'(z))'}{(f(z)-f(-z))'} \prec \left(\frac{1+Az}{1+Bz}\right)^\alpha$$

and

$$(1-\lambda)\frac{2wg'(w)}{g(w)-g(-w)} + \lambda\frac{2(wg'(w))'}{(g(w)-g(-w))'} \prec \left(\frac{1+Aw}{1+Bw}\right)^\alpha.$$

In particular, $M_\Sigma^s(\alpha, \lambda; 1, -1) \equiv M_\Sigma^s(\alpha, \lambda)$ and $M_\Sigma^s(1, \lambda; 1-2\beta, -1) \equiv M_\Sigma^s(\beta, \lambda)$.

Definition 1.2. A function $f(z) \in A$ is said to be in the class $N_\Sigma^s(\alpha, \lambda; A, B)$ if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \frac{2zf'(z) + \lambda 2z^2 f''(z)}{(1-\lambda)(f(z)-f(-z)) + \lambda z(f(z)-f(-z))'} \prec \left(\frac{1+Az}{1+Bz}\right)^\alpha$$

and

$$\frac{2wg'(w) + \lambda 2w^2 g''(w)}{(1-\lambda)(g(w)-g(-w)) + \lambda w(g(w)-g(-w))'} \prec \left(\frac{1+Aw}{1+Bw}\right)^\alpha.$$

Particularly, $N_\Sigma^s(\alpha, \lambda; 1, -1) \equiv N_\Sigma^s(\alpha, \lambda)$ and $N_\Sigma^s(1, \lambda; 1-2\beta, -1) \equiv N_\Sigma^s(\beta, \lambda)$.

Definition 1.3. A function $f(z) \in A$ is said to be in the class $P_{\Sigma}^{s}(\alpha, \lambda; A, B)$ if the following conditions are satisfied:

$$f \in \Sigma \text{ and } \left(\frac{2zf'(z)}{f(z) - f(-z)}\right)^{(1-\lambda)} + \left(\frac{2(zf'(z))'}{(f(z) - f(-z))'}\right)^{\lambda} \prec \left(\frac{1 + Az}{1 + Bz}\right)^{\alpha}$$

and

$$\left(\frac{2wg'(w)}{g(w) - g(-w)}\right)^{(1-\lambda)} + \left(\frac{2(wg'(w))'}{(g(w) - g(-w))'}\right)^{\lambda} \prec \left(\frac{1 + Aw}{1 + Bw}\right)^{\alpha}.$$

Obviously, $P_{\Sigma}^{s}(\alpha, \lambda; 1, -1) \equiv P_{\Sigma}^{s}(\alpha, \lambda)$ and $P_{\Sigma}^{s}(1, \lambda; 1 - 2\beta, -1) \equiv P_{\Sigma}^{s}(\beta, \lambda)$.

For deriving our main results, we need to the following lemma:

Lemma 1.1. If $p(z) = \dfrac{1 + Au(z)}{1 + Bu(z)} = 1 + \sum_{k=1}^{\infty} p_k z^k$, $u(z) \in U$,

then $|p_n| \leq (A - B), n \geq 1$.

This result is due to Goel and Mehrok [4].

2. Coefficient bounds for the function class $M_{\Sigma}^{s}(\alpha, \lambda; A, B)$

Theorem 2.1. If $f \in M_{\Sigma}^{s}(\alpha, \lambda; A, B)$, then

$$|a_2| \leq \frac{\alpha\sqrt{(A - B)}}{\sqrt{2[(1 + \lambda)^2 - \alpha\lambda^2]}} \tag{2}$$

and

$$|a_3| \leq \frac{\alpha^2(A - B)^2}{4(1 + \lambda)^2} + \frac{\alpha(A - B)}{2(1 + 2\lambda)}. \tag{3}$$

Proof. From definition 1.1, by principle of subordination, we have

$$(1 - \lambda)\frac{2zf'(z)}{f(z) - f(-z)} + \lambda\frac{2(zf'(z))'}{(f(z) - f(-z))'} = \left(\frac{1 + Au(z)}{1 + Bu(z)}\right)^{\alpha} = [p(z)]^{\alpha}, u \in U \tag{4}$$

and

$$(1-\lambda)\frac{2wg'(w)}{g(w)-g(-w)}+\lambda\frac{2(wg'(w))'}{(g(w)-g(-w))'}=\left(\frac{1+Av(w)}{1+Bv(w)}\right)^{\alpha}=[q(w)]^{\alpha}, v\in U, \tag{5}$$

where $p(z)=1+p_1z+p_2z^2+...$ and $q(w)=1+q_1w+q_2w^2+....$

On expanding and equating the coefficients of z and z^2 in (4) and of w and w^2 in (5), we obtain

$$2(1+\lambda)a_2=\alpha p_1, \tag{6}$$

$$2(1+2\lambda)a_3=\alpha p_2+\frac{\alpha(\alpha-1)p_1^2}{2} \tag{7}$$

and

$$-2(1+\lambda)a_2=\alpha q_1, \tag{8}$$

$$2(1+2\lambda)(2a_2^2-a_3)=\alpha q_2+\frac{\alpha(\alpha-1)q_1^2}{2}. \tag{9}$$

(6) and (8) together gives

$$p_1=-q_1 \tag{10}$$

and

$$8(1+\lambda)^2a_2^2=\alpha^2(p_1^2+q_1^2). \tag{11}$$

Adding (7) and (9) and using (11), it yields

$$4(1+2\lambda)a_2^2=\alpha(p_2+q_2)+\frac{4(\alpha-1)(1+\lambda)^2a_2^2}{\alpha}. \tag{12}$$

(12) gives

$$a_2^2=\frac{\alpha^2(p_2+q_2)}{4\alpha(1+2\lambda)+4(1-\alpha)(1+\lambda)^2}. \tag{13}$$

On applying Lemma 1.1 to the coefficients p_2 and q_2, we can easily obtain (2).
Now subtracting (9) from (7), we get

$$4(1+2\lambda)a_3-4(1+2\lambda)a_2^2=\alpha(p_2-q_2)+\frac{\alpha(\alpha-1)(p_1^2-q_1^2)}{2}. \tag{14}$$

Using (10) and (11), (14) yields

$$a_3 = \frac{\alpha^2 p_1^2}{4(1+\lambda)^2} + \frac{\alpha(p_2 - q_2)}{4(1+2\lambda)}. \tag{15}$$

Applying Lemma 1.1 to the coefficients p_2,q_2 and p_1 in (15), (3) is obvious.

For $A = 1, B = -1$, Theorem 2.1 gives the following result:

Corollary 2.1. If $f(z) \in M_{\Sigma}^{s}(\alpha, \lambda)$, then

$$|a_2| \leq \frac{\alpha}{\sqrt{(1+\lambda)^2 - \alpha\lambda^2}}$$

and

$$|a_3| \leq \frac{\alpha^2}{(1+\lambda)^2} + \frac{\alpha}{1+2\lambda}.$$

Putting $A = 1-2\beta, B = -1$ and $\alpha = 1$ in Theorem 2.1, we obtain the following result:

Corollary 2.2. If $f(z) \in M_{\Sigma}^{s}(\beta, \lambda)$, then

$$|a_2| \leq \sqrt{\frac{1-\beta}{1+2\lambda}}$$

and

$$|a_3| \leq \frac{(1-\beta)^2}{(1+\lambda)^2} + \frac{(1-\beta)}{1+2\lambda}.$$

3. Coefficient bounds for the function class $N_{\Sigma}^{s}(\alpha, \lambda; A, B)$

Theorem 3.1. If $f \in N_{\Sigma}^{s}(\alpha, \lambda; A, B)$, then

$$|a_2| \leq \frac{\alpha\sqrt{(A-B)}}{\sqrt{2[(1+\lambda)^2 - \alpha\lambda^2]}}$$

and

$$|a_3| \leq \frac{\alpha^2(A-B)^2}{4(1+\lambda)^2} + \frac{\alpha(A-B)}{2(1+2\lambda)}.$$

The proof of this theorem is similar to as in Theorem 2.1.

4. Coefficient bounds for the function class $P_{\Sigma}^{s}(\alpha, \lambda\, ; A, B)$

Theorem 4.1. If $f \in P_{\Sigma}^{s}(\alpha, \lambda; A, B)$, then

$$|a_2| \leq \frac{\alpha\sqrt{(A-B)}}{\sqrt{2[(1+\lambda)^2 - \alpha\lambda]}} \tag{16}$$

and

$$|a_3| \leq \frac{\alpha^2(A-B)^2}{4(1+\lambda)^2} + \frac{\alpha(A-B)}{2(1+2\lambda)}. \tag{17}$$

Proof. From definition 1.3, by principle of subordination, we have

$$\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)^{(1-\lambda)} + \left(\frac{2(zf'(z))'}{(f(z)-f(-z))'}\right)^{\lambda} = \left(\frac{1+Au(z)}{1+Bu(z)}\right)^{\alpha} = [p(z)]^{\alpha}, u \in U \tag{18}$$

and

$$\left(\frac{2wg'(w)}{g(w)-g(-w)}\right)^{(1-\lambda)} + \left(\frac{2(wg'(w))'}{(g(w)-g(-w))'}\right)^{\lambda} = \left(\frac{1+Av(w)}{1+Bv(w)}\right)^{\alpha} = [q(w)]^{\alpha}, v \in U, \tag{19}$$

where $p(z) = 1 + p_1 z + p_2 z^2 + ...$ and $q(w) = 1 + q_1 w + q_2 w^2 + ...$

On expanding and equating the coefficients of z and z^2 in (18) and of w and w^2 in (19), we obtain

$$2(1+\lambda)a_2 = \alpha p_1, \tag{20}$$

$$2(1+2\lambda)a_3 - 2\lambda(1-\lambda)a_2^2 = \alpha p_2 + \frac{\alpha(\alpha-1)p_1^2}{2} \tag{21}$$

and

$$-2(1+\lambda)a_2 = \alpha q_1, \tag{22}$$

$$2(1+2\lambda)(2a_2^2 - a_3) - 2\lambda(1-\lambda)a_2^2 = \alpha q_2 + \frac{\alpha(\alpha-1)q_1^2}{2}. \tag{23}$$

(20) and (22) together gives

$$p_1 = -q_1 \tag{24}$$

and

$$8(1+\lambda)^2 a_2^2 = \alpha^2(p_1^2 + q_1^2). \tag{25}$$

Adding (21) and (23) and using (25), it yields

$$a_2^2 = \frac{\alpha^2(p_2+q_2)}{4[(1+\lambda)^2-\alpha\lambda]}. \tag{26}$$

On applying Lemma 1.1 to the coefficients p_2 and q_2, we can easily obtain (16).

Now subtracting (23) from (21), we get

$$4(1+2\lambda)a_3 - 4(1+2\lambda)a_2^2 = \alpha(p_2-q_2) + \frac{\alpha(\alpha-1)(p_1^2-q_1^2)}{2}. \tag{27}$$

Using (24) and (25), (27) yields

$$a_3 = \frac{\alpha^2 p_1^2}{4(1+\lambda)^2} + \frac{\alpha(p_2-q_2)}{4(1+2\lambda)}. \tag{28}$$

Applying Lemma 1.1 to the coefficients p_2,q_2 and p_1 in (28), (17) is obvious.

For $A=1, B=-1$, Theorem 4.1 gives the following result:

Corollary 4.1. If $f(z) \in P_\Sigma^s(\alpha,\lambda)$, then

$$|a_2| \leq \frac{\alpha}{\sqrt{(1+\lambda)^2-\alpha\lambda}}$$

and

$$|a_3| \leq \frac{\alpha^2}{(1+\lambda)^2} + \frac{\alpha}{1+2\lambda}.$$

Putting $A=1-2\beta, B=-1$ and $\alpha=1$ in Theorem 4.1, we obtain the following result:

Corollary 4.2. If $f(z) \in M_\Sigma^s(\beta,\lambda)$, then

$$|a_2| \leq \sqrt{\frac{1-\beta}{1+2\lambda}}$$

and

$$|a_3| \leq \frac{(1-\beta)^2}{(1+\lambda)^2} + \frac{(1-\beta)}{1+2\lambda}.$$

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