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# ZERO THEOREMS OF ACCRETIVE OPERATORS IN REFLEXIVE BANACH SPACES

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**Abstract.** In this article, an Ishikawa-type iterative algorithm for approximating a common zero of a finite family of $m$-accretive mappings in reflexive Banach spaces is investigated. Strong convergence theorems under some mild conditions imposed on parameters are obtained.

**Keywords.** Accretive mapping; Pseudo-contractive mapping; Banach space; Weakly continuous duality mapping.

## 1. Introduction

Common zero (fixed) points of a family of nonlinear mappings has been considered by many authors. The well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings. The problem of finding an optimal point that minimizes a given cost function over common set of fixed points of a family of nonlinear mappings is of wide interdisciplinary interest and practical importance. A simple algorithmic solution to the problem of minimizing a quadratic function over common set of fixed points of a family of nonlinear mappings is of extreme value in many applications including set theoretic signal estimation.

Throughout this paper, we denote by $E$ and $E$ a real Banach space and dual space of $E$, respectively. Let $C$ be a nonempty subset of $E$ and $T : C \to C$ be a mapping. Let $J$ denote the

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normalized duality mapping from $E$ into $2^{E^*}$ given by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that, if $E^*$ is strictly convex, then $J$ is single-valued.

Recall that T is said to nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Recall that $T$ is said to be pseudo-contractive if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \leq ||x-y||^2, \quad \forall x, y \in C.$$

Clearly, the class of nonexpansive mappings is a subset of the class of pseudo-contractive mappings. Closely related to the class of pseudo-contractive mappings is the class of accretive mappings. Recall that a (possibly multi-valued) operator $A$ with domain $D(A)$ and range $R(A)$ in $E$ is accretive, if for each $x_i \in D(A)$ and $y_i \in Ax_i (i = 1, 2)$, there exists a $j(x_2 - x_1) \in J(x_2 - x_1)$ such that

$$\langle y_2 - y_1, j(x_2 - x_1)\rangle \geq 0.$$

An accretive operator $A$ is $m$-accretive if $R(I + rA) = E$ for each $r > 0$. The set of zeros of $A$ is denoted by $N(A)$. Hence, $N(A) = \{z \in D(A) : 0 \in A(z)\} = A^{-1}(0)$. For each $r > 0$, we denote by $J_r$ the resolvent of $A$, i.e., $J_r = (I + rA)^{-1}$. Note that if $A$ is $m$-accretive, then $J_r : E \to E$ is nonexpansive and $F(J_r) = F$ for all $r > 0$. We also denote by $A_r$ the Yosida approximation of $A$, i.e., $A_r = \frac{1}{r}(I - J_r)$. It is known that $J_r$ is a nonexpansive mapping from $E$ to $C := \overline{D(A)}$ which will be assumed convex.

We observe that $p$ is a zero of the accretive mapping $A$ if and only if it is a fixed point of the pseudo-contractive mapping $T := I - A$. It is now well known [1] that, if $A$ is accretive, then the solutions of the equation $Ax = 0$ correspond with the equilibrium points of some evolution systems. Consequently, considerable research works, especially, for the past 15 years or more, have been devoted to the iterative methods for approximating the zeros of an accretive mapping $A$; see [2-11].

One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping. More precisely, take $t \in (0,1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1-t)Tx, \quad x \in C, \tag{1.1}$$

where $u \in C$ is a fixed point. Banach's Contraction Mapping Principle guarantees that $T_t$ has a unique fixed point $x_t$ in $C$. Browder [12] proved that if $X$ is a Hilbert space, then $x_t$ converges strongly to a fixed point of $T$ that is nearest to $u$. Reich [13] extended Broweder's result to the setting of Banach spaces and proved that if $X$ is a uniformly smooth Banach space, then $x_t$ converges strongly to a fixed point of $T$ and the limit defines the (unique) sunny nonexpansive retraction from $C$ onto $F(T)$. Recently, Xu [14] proved Reich's results hold in reflexive Banach spaces which have a weakly continuous duality mapping.

Benavides et al. [15] studied the sequence $\{x_n\}$ generated by the following algorithm:

$$x_{n+1} = \alpha_n u + (1-\alpha_n) J_{r_n} x_n, \quad n \geq 0 \tag{1.2}$$

and proved strongly convergence of iterative scheme (1.2) in uniformly smooth Banach spaces which have a weakly continuous duality mapping.

Recall that the normal Manns iterative process was introduced by Mann [16] in 1953. Since then, construction of fixed points for nonexpansive mappings via the normal Manns iterative process has been extensively investigated by many authors. The normal Mann's iterative process generates a sequence $\{x_n\}$ in the following manner:

$$\forall x_1 \in C, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad \forall n \geq 1, \tag{1.3}$$

where the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval (0,1).

If $T$ is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n (1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by normal Mann's iterative process (1.2) converges weakly to a fixed point of $T$. Therefore, many authors try to modify normal Mann's iteration process to have strong convergence for nonexpansive mappings and other extensions.

Recently, Qin and Su [17] modified normal Mann' iterative process for $m$-accretive mappings to have strong convergence. To be more precisely, They proved the following theorem:

**Theorem QS.** *Assume that $E$ is reflexive and has a weakly continuous duality map $J_\varphi$ with gauge $\varphi$. Suppose that $A$ is an m-accretive operator in $E$ such that $C = D(\bar{A})$ is convex and $A^{-1}(0) \neq \emptyset$. Given a point $u \in C$ and $\{\alpha_n\}_{n=0}^\infty$ in (0,1), $\{\beta_n\}_{n=0}^\infty$ in [0,1]. Suppose that the sequences $\{\alpha_n\}_{n=0}^\infty$, $\{\beta_n\}_{n=0}^\infty$ and $\{r_n\}_{n=0}^\infty$ satisfy the conditions:*

(i) $\sum_{n=0}^\infty \alpha_n = \infty, \alpha_n \to 0$;

(ii) $r_n \geq \varepsilon$ *for all $n$ and* $\beta_n \in [0,a)$ *for some* $a \in (0,1)$*;*

(iii) $\sum_{n=0}^\infty |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=0}^\infty |\beta_{n+1} - \beta_n| < \infty$ *and* $\sum_{n=0}^\infty |r_n - r_{n-1}| < \infty$.

*Let $\{x_n\}_{n=1}^\infty$ be the composite iterative process defined by*

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n) J_{r_n} x_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_n. \end{cases} \tag{1.4}$$

*Then $\{x_n\}_{n=1}^\infty$ converges strongly to a zero point of $A$.*

Very recently, Zegeye and Shahzed [18] studied the convergence problem of a family of $m$-accretive mappings. To be more precisely, they proved the following result.

**Theorem ZS.** *Let $E$ be a strictly convex and reflexive Banach space with a uniformly Gâteaux differentiable norm, $K$ be a nonempty closed convex subset of $E$ and $A_i : K \to E$ $(i = 1, 2, \cdots, r)$ be a family of m-accretive mappings with $\bigcap_{i=1}^r N(A_i) \neq \emptyset$. For any $u, x_0 \in K$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} = \alpha_n u + (1-\alpha_n) S_r x_n, \quad n \geq 0, \tag{1.5}$$

*where $\{\alpha_n\}$ be a real sequence which satisfies the following conditions: $\lim_{n\to\infty} \alpha_n = 0$; $\sum_{n=0}^\infty \alpha_n = \infty$; $\sum_{n=0}^\infty |\alpha_n - \alpha_{n-1}| < \infty$ or $\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$ and $S_r := a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_r J_{A_r}$ with $J_{A_i} := (I + A_i)^{-1}$ for $0 < a_i < 1$ for $i = 0, 1, 2, \cdots, r$ and $\sum_{i=0}^r a_i = 1$. If every nonempty closed bounded convex subset of $E$ has the fixed point property for a nonexpansive mapping, then $\{x_n\}$ converges strongly to a common solution of the equations $A_i x = 0$ for $i = 1, 2, \cdots, r$.*

This paper, motivated by the above results, introduces a Ishikawa-like iterative algorithm as follows.

$$\begin{cases} x_0 \in C, \ chosen\ arbitrarily \\ y_n = \beta_n x_n + (1-\beta_n) S_r x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n, \quad n \geq 0, \end{cases} \tag{1.6}$$

where $f : C \to C$ be contractive mapping, $S_r := a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_r J_{A_r}$ with $0 < a_i < 1$ for $i = 0, 1, 2, \cdots, r$, $\sum_{i=0}^{r} a_i = 1$ and $\{\alpha_n\}$, $\{\beta_n\}$ are two real sequences in $(0,1)$. We prove strong convergence of a finite family of $m$-accretive mappings in reflexive Banach spaces which have a weakly continuous duality mapping. Our results improve the recent ones announced by many authors

## 2. Preliminaries

The norm of $E$ is said to be Gâteaux differentiable (and $E$ is said to be smooth) if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y$ in its unit sphere $U = \{x \in E : \|x\| = 1\}$. It is said to be uniformly Fréchet differentiable (and $E$ is said to be uniformly smooth ) if the limit in (2.1) is attained uniformly for $(x, y) \in U \times U$.

A Banach space $E$ is said to be strictly convex if, for $a_i \in (0,1)$, $i = 1, 2, \cdots, r$, such that $\sum_{i=1}^{r} a_i = 1$,

$$||a_1 x_1 + a_2 x_2 + \cdots + a_r x_r|| < 1, \quad \forall x_i \in E, i = 1, 2, \cdots, r,$$

with $||x_i|| = 1$, $i = 1, 2, \cdots, r$, and $x_i \neq x_j$ for some $i \neq j$. In a strictly convex Banach space $E$, we have that, if

$$||x_1|| = ||x_2|| = \cdots = ||x_r|| = ||a_1 x_1 + a_2 x_2 + \cdots + a_r x_r||$$

for $x_i \in E$, $a_i \in (0,1)$, $i = 1, 2, \cdots, r$, where $\sum_{i=1}^{r} a_i = 1$, then $x_1 = x_2 = \cdots = x_r$.

Recall that a gauge is a continuous strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Associated to a gauge $\varphi$ is the duality map $J_\varphi : E \to E^*$

defined by

$$J_\varphi(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\varphi(\|x\|), \|x^*\| = \varphi(\|x\|)\},\ x \in E.$$

Following Recall that a Banach space $E$ has a weakly continuous duality map if there exists a gauge $\varphi$ for which the duality map $J_\varphi(x)$ is single-valued and *weak*-to-*weak*$^*$ sequentially continuous (i.e., if $\{x_n\}$ is a sequence in $E$ weakly convergent to a point $x$, then the sequence $J_\varphi(x_n)$ converges weak$^*$ly to $J_\varphi$). It is known that $l^p$ has a weakly continuous duality map for all $1 < p < \infty$.

**Lemma 2.1.** *Let $E$ be a real Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping. Then, for any $x, y \in E$, the following holds:*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y).$$

**Lemma 2.2** [19] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $E$ and let $\beta_n$ be a sequence in [0,1] with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \ge 0$ and*

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

*Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

**Lemma 2.3** [18] *Let $C$ be a nonempty closed convex subset of a strictly convex Banach space $E$. Let $A_i : C \to E$, $i = 1, 2, \cdots, r$, be a family of m-accretive mappings such that $\bigcap_{i=1}^r N(A_i) \ne \emptyset$. Let $a_0, a_1, a_2, \cdots, a_r$ be real numbers in $(0,1)$ such that $\sum_{i=0}^r a_i = 1$ and $S_r := a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_r J_{A_r}$, where $J_{A_i} := (I + A_i)^{-1}$. Then $S_r$ is nonexpansive and $F(S_r) = \bigcap_{i=1}^r N(A_i)$.*

Recall that if $C$ and $D$ are nonempty subsets of a Banach space $E$ such that $C$ is nonempty closed convex and $D \subset C$, then a map $Q : C \to D$ is sunny provided $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in C$ and $t \ge 0$ whenever $x + t(x - Q(x)) \in C$. A sunny nonexpansive retraction is a sunny retraction, which is also nonexpansive. Sunny nonexpansive retractions play an important role in our argument. They are characterized as follows: if $E$ is a smooth Banach space, then $Q : C \to D$ is a sunny nonexpansive retraction if and only if there holds the inequality

$$\langle x - Qx, J(y - Qx)\rangle \le 0 \quad \text{for all } x \in C \text{ and } y \in D.$$

Xu [14] showed that if $E$ is a reflexive Banach and has a weakly continuous duality then there is a sunny nonexpansive retraction from $C$ onto $F(T)$ and it can be constructed as follows.

**Lemma 2.4** [14] *Let $E$ be a reflexive Banach space and has a weakly continuous duality map $J_\varphi(x)$ with gauge $\varphi$. Let $C$ be closed convex subset of $E$ and let $T : C \to C$ be a nonexpansive mapping. Fix $u \in C$ and $t \in (0,1)$. Let $x_t \in C$ be the unique solution in $C$ to Eq.(1.1). Then $T$ has a fixed point if and only if $x_t$ remains bounded as $t \to 0^+$, and in this case, $x_t$ converges as $t \to 0^+$ strongly to a fixed point of $T$.*

Under the condition of Lemma 2.4, we define a map $Q : C \to F(T)$ by $Q(u) := \lim_{t\to 0} x_t, \quad u \in C$. from [14 Theorem 3.2] we know $Q$ is the sunny nonexpansive retraction from $C$ onto $F(T)$.

**Lemma 2.5** [20]. *Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the condition*

$$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \gamma_n\sigma_n, \quad n \ge 0,$$

*where $\{\gamma_n\}_{n=0}^\infty \subset (0,1)$ and $\{\sigma_n\}_{n=0}^\infty$ such that*

(i) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=0}^\infty \gamma_n = \infty$;

(ii) *either* $\limsup_{n\to\infty} \sigma_n \le 0$ *or* $\sum_{n=0}^\infty |\gamma_n\sigma_n| < \infty$.

*Then $\{\alpha_n\}_{n=0}^\infty$ converges to zero.*

## 3. Main results

**Theorem 3.1.** *Let $E$ be a strictly convex and reflexive Banach space which has a weakly continuous duality map $J_\varphi(x)$ with the gauge $\varphi$. Let $C$ be a nonempty closed convex subset of $E$. Let $A_i : C \to E$ $i = 1,2,\cdots,r$ be a family of m-accretive mappings with $\bigcap_{i=1}^r N(A_i) \ne \emptyset$. Let $J_{A_i} := (I + A_i)^{-1}$ for $i = 1,2,\cdots,r$. Let $\{x_n\}$ be generated by*

$$\begin{cases} x_0 \in C, \text{ chosen arbitrarily} \\ y_n = \beta_n x_n + (1-\beta_n)S_r x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n, \quad n \ge 0, \end{cases}$$

*where $f:C\to C$ be a contractive mapping, $S_r:=a_0I+a_1J_{A_1}+a_2J_{A_2}+\cdots+a_rJ_{A_r}$ with $0<a_i<1$ for $i=0,1,2,\cdots,r$, $\sum_{i=0}^{r}a_i=1$ and $\{\alpha_n\}$, $\{\beta_n\}$ be two real sequences in $(0,1)$ which satisfy the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=0}^{\infty}\alpha_n=\infty$;
(ii) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$.

*If every nonempty closed bounded convex subset of $E$ has the fixed point property for a nonexpansive mapping, then $\{x_n\}$ converges strongly to a common solution of the equations $A_ix=0$ for $i=1,2,\cdots,r$.*

**Proof.** In view of Lemma 2.3, we have $S_r$ is a nonexpansive mapping and $F(S_r)=\bigcap_{i=1}^{r}N(A_i)\neq\emptyset$. We observe that $\{x_n\}_{n=0}^{\infty}$ is bounded. Indeed, take a fixed point $p$ of $S_r$ and notice that

$$\begin{aligned}\|y_n-p\|&=\|\beta_nx_n+(1-\beta_n)S_rx_n-p\|\\&\le\beta_n\|x_n-p\|+(1-\beta_n)\|S_rx_n-p\|\\&\le\beta_n\|x_n-p\|+(1-\beta_n)\|x_n-p\|\\&=\|x_n-p\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\|&=\|\alpha_n(f(x_n)-p)+(1-\alpha_n)(y_n-p)||\\&\le\alpha_n\|f(x_n)-p||+(1-\alpha_n)\|y_n-p\|\\&\le(1-\alpha_n(1-\alpha))\|u-p||+(1-\alpha_n)\|x_n-p\|.\end{aligned}$$

By simple inductions, we find the sequence $\{x_n\}$ is bounded, so is $\{y_n\}$.

Next, we claim that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.1}$$

Put $l_n = \frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$. That is, $x_{n+1} = (1-\beta_n)l_n + \beta_n x_n$. Now, we compute $l_{n+1} - l_n$. In view of

$$\begin{aligned} l_{n+1} - l_n =& \frac{\alpha_{n+1} f(x_{n+1}) + (1-\alpha_{n+1})y_{n+1} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} \\ & - \frac{\alpha_n f(x_n) + (1-\alpha_n)y_n - \beta_n x_n}{1-\beta_n} \\ =& \frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1}) - y_{n+1}) + \frac{\alpha_n}{1-\beta_n}(y_n - f(x_n)) \\ & + S_r x_{n+1} - S_r x_n, \end{aligned}$$

we find

$$\begin{aligned} \|l_{n+1} - l_n\| \leq & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1}) - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|y_n - f(x_n)\| \\ & + \|S_r x_{n+1} - S_r x_n\| \\ \leq & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1}) - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|y_n - f(x_n)\| \\ & + \|x_{n+1} - x_n\|. \end{aligned}$$

It follows that

$$\begin{aligned} & \|l_{n+1} - l_n\| - \|x_n - x_{n-1}\| \\ \leq & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1}) - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|y_n - f(x_n)\|. \end{aligned}$$

Observe conditions (i) and (ii) and take the limits as $n \to \infty$ gets

$$\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

It follows that $\lim_{n\to\infty}\|l_n - x_n\| = 0$. Since $x_{n+1} - x_n = (1-\beta_n)(l_n - x_n)$, we have that (3.1) holds. It follows that

$$\lim_{n\to\infty}\|x_{n+1} - y_n\| = 0. \tag{3.2}$$

Notice that $\|x_n - y_n\| \leq \|x_{n+1} - x_n\| + \|x_{n+1} - y_n\|$. It follows from (3.1) and (3.2) that

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.3}$$

On the other hand, we have

$$(1-\beta_n)\|S_r x_n - x_n\| \leq \|x_n - y_n\|.$$

In view of the condition (ii), we have

$$\lim_{n\to\infty}\|S_r x_n - x_n\| = 0. \tag{3.4}$$

Next we claim

$$\limsup_{n\to\infty}\langle f(q)-q, J(x_n-q)\rangle \le 0, \tag{3.5}$$

where $q = Q(f(q)) = \lim_{t\to 0} x_t$ with $x_t$ being the fixed point of the contraction $x_t \mapsto tf(t)+(1-t)S_r x_t$ by Lemma 2.4. From $x_t$ solves the fixed point equation $x_t = tu+(1-t)S_r x_t$, we have

$$\begin{aligned}
\|x_t-x_n\|^2 &\le (1-t)^2\|S_r x_t - x_n\|^2 + 2t\langle f(x_t)-x_n, J(x_t-x_n)\rangle \\
&\le (1-t)^2(\|S_r x_t - S_r x_n\| + \|S_r x_n - x_n\|)^2 + 2t\langle x_t - x_n, J(x_t-x_n)\rangle \\
&\quad + 2t\langle f(x_t)-x_t, J(x_t-x_n)\rangle \\
&\le (1-t)^2(\|x_t-x_n\| + \|S_r x_n - x_n\|)^2 + 2t\|x_t-x_n\|^2 \\
&\quad + 2t\langle f(x_t)-x_t, J(x_t-x_n)\rangle \\
&\le (1+t^2)\|x_t-x_n\|^2 + (2t\|x_t-x_n\| + \|S_r x_n - x_n\|)\|S_r x_n - x_n\| \\
&\quad + 2t\langle f(x_t)-x_t, J(x_t-x_n)\rangle,
\end{aligned}$$

from which it follows that

$$\langle f(x_t)-x_t, J(x_n-x_t)\rangle \le \frac{t}{2}\|x_t-x_n\|^2 + \frac{\|S_r x_n - x_n\|}{2}(2t\|x_t-x_n\| + \|S_r x_n - x_n\|).$$

Noticing (3.4) and letting $n\to\infty$, we get

$$\limsup_{n\to\infty}\langle f(x_t)-x_t, j(x_n-x_t)\rangle \le \frac{t}{2}M_1, \tag{3.6}$$

where $M_1$ is a constant such that $\|x_t - x_n\| \le M_1$ for all $t\in(0,1)$ and $n\ge 1$. Since $x_t \to q = Q(f(q))$ and the duality mapping $j$ is norm-to-weak$^*$ uniformly continuous on bounded subsets of $E$, it follows that (3.5) holds.

Finally, we show $\{x_n\}$ converges strongly to $q = Q(f)$. It follows from Lemma 2.1 that

$$\begin{aligned}
&\|x_{n+1}-q\|^2 \\
&= \|\alpha_n(f(x_n)-q) + (1-\alpha_n)(y_n-q)\|^2 \\
&\le (1-\alpha_n)^2\|y_n-q\|^2 + 2\alpha_n\langle u-q, J(x_{n+1}-q)\rangle \\
&\le (1-\alpha_n)^2(\beta_n\|x_n-q\| + (1-\beta_n)\|S_r x_n - q\|)^2 + 2\alpha_n\langle f(q)-q, J(x_{n+1}-q)\rangle \\
&\le (1-\alpha_n(1-\alpha))\|x_n-q\|^2 + 2\alpha_n\langle f(q)-q, J(x_{n+1}-q)\rangle.
\end{aligned}$$

Now we apply Lemma 2.5 to see that $x_n\to q$ as $n\to\infty$. This completes the proof.

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