J. Nonlinear Funct. Anal. 2013 (2013), Article ID 3

MATHRES
Journal of Nonlinear Functional Analysis
Available online at http://jnfa.mathres.org

WIENER-HOPF EQUATIONS METHODS FOR GENERALIZED VARIATIONAL INEQUALITIES

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Abstract. In this paper, the solvability of generalized variational inequalities is investigated in the framework of Hilbert spaces. The convergence criteria of iterative methods under some mild conditions are obtained. Our results include the previous results as special cases and can be considered as an improvement and refinement of the previously known results.

Keywords. Variational inequality; Hilbert space; Nonexpansive mapping; Algorithm.

1. Introduction

Variational inequalities and hemi-variational inequalities have significant applications in various fields of mathematics, physics, economics, and engineering sciences. The solvability of variational inequalities based on iterative methods has been extensively investigated; see [1-19] and the references therein. Recently, relaxed monotone operators have applications to constrained hemi-variational inequalities. Since in the study of constrained problems in reflexive Banach spaces E the set of all admissible elements is non-convex but star-shaped, corresponding variational formulations are no longer variational inequalities. Using hemi-variational inequalities, one can prove the existence of solutions to the following type of non-convex constrained problems (P): find u in C such that

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Received August 25, 2013

$$\langle Au - g, v \rangle \geq 0, \quad \forall v \in T_C(u), \tag{1.1}$$

where the admissible set $C \subset E$ is a star-shaped set with respect to a certain ball $B_E(u_0, \rho)$ and $T_C(u)$ denotes Clarke's tangent cone of C at u in C. It is easily seen that when C is convex, (1.1) reduces to the variational inequality of finding u in C such that

$$\langle Au - g, v \rangle \geq 0, \quad \forall v \in C. \tag{1.2}$$

Example. [8] Let $A : E \to E^*$ be a maximal monotone operator from a reflexive Banach s-pace E into E^* with strong monotonicity and let $C \subset E$ be star-shaped with respect to a ball $B_E(u_0, \rho)$. Suppose that $Au_0 - g \neq 0$ and that distance function d_C satisfies the condition of relaxed monotonicity

$$\langle u^* - v^*, u - v \rangle \geq -c\|u - v\|^2, \quad \forall u, v \in E,$$

and for any $u^* \in \partial d_C(u)$ and v^* in $\partial d_C(v)$ with c satisfying $0 < c < \frac{4a^2\rho}{\|Au_0 - g\|^2}$, where a is the constant for strong monotonicity of A. Here ∂d_C is a relaxed monotone operator. Then the problem (P) has at least one solution.

It is well known that the variational inequalities are equivalent to the fixed point problems. This alternative equivalent formulation is very important from the numerical analysis point of view and has played a significant part in several numerical methods for solving variational inequalities and complementarity. In particular, the solution of the variational inequalities can be computed using the iterative projection methods. Now we have a variety of techniques to suggest and analyze various numerical methods including projection technique and its variant form, auxiliary principle and Wiener-Hopf equations for solving variational inequalities and related optimization problems. In this papers, using the projection technique, one can show that variational inequalities are equivalent to the Wiener-Hopf equations, the origin of which can be traced back of Shi [9]. It has been shown that the Wiener-Hopf equations are more flexible and general than the projection methods. Noor [1] and Wu and Li [10] has used the Wiener-Hopf equations technique to study the sensitivity analysis, dynamical systems as well as to suggest and analyze several iterative methods for solving variational inequalities. In this article, we consider, based on the Wiener-Hopf equations method, the problem of finding a common element of fixed points of nonexpansive mappings and the set of solution of the variational inequalities. Results proved in this paper may be viewed as significant and improvement of previously known results.

Let P_C be the projection of a separable real Hilbert space H onto the nonempty closed convex subset C. We consider the variational inequality problem which denoted by $VI(C,A)$: find $u \in C$ such that

$$\langle Au+w, v-u\rangle \geq 0, \quad \forall v \in C, w \in Tu, \tag{1.3}$$

where A and T are two nonlinear mappings. Related to the variational inequalities, we have the problems of solving the Wiener-Hopf equations. To be more precise, Let $Q_C = I - SP_C$, where P_C is the projection of H onto the closed convex set C, I is the identity operator, S is a nonexpansive mapping. We consider the problem of finding $z \in H$ such that

$$ASP_Cz + w + \rho^{-1}Q_Cz = 0, \quad \forall w \in TSP_Cz, \tag{1.4}$$

which is called the generalized Wiener-Hopf equation. Next we denote the solution of (1.4) by WHE(H,A,S), which includes the original Wiener-Hopf equation, introduced by Shi [9] as a special case. It is well know that the variational inequalities and Wiener-Hopf equations are equivalent. This equivalent has played a fundamental and basic role in developing some efficient and robust methods for solving variational inequalities and related optimization problems. Recall the following definitions. A mapping A of C into H is called monotone if

$$\langle Au - Av, u - v\rangle \geq 0, \quad \forall u, v \in C.$$

A is called r-strongly monotone if there exists a constant $r > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq r\|x-y\|^2, \quad \forall x, y \in C.$$

This implies that $\|Ax - Ay\| \geq r\|x-y\|, \quad \forall x, y \in C$, that is, A is r-expansive and, when $r = 1$, it is expansive. A is said to be μ-cocoercive if there exists a constant $\mu > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \mu\|Ax-Ay\|^2, \quad \forall x, y \in C.$$

Clearly, every μ-cocoercive mapping A is $\frac{1}{\mu}$-Lipschitz continuous. A is called relaxed γ-cocoerceive if there exists a constant $\gamma > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq (-\gamma)\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

A is said to be relaxed (γ, r)-cocoercive if there exist two constants $u, v > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq (-\gamma)\|Ax - Ay\|^2 + r\|x - y\|^2, \quad \forall x, y \in C.$$

For $\gamma = 0$, A is r-strongly monotone. This class of mappings is more general that the class of strongly monotone mappings. It is easy to see that we have the following implication: r-strongly monotonicity $\Rightarrow$ relaxed (γ, r)-cocoercivity.

$T : H \to 2^H$ is said to be a relaxed monotone operator if there exists a constant $k > 0$ such that

$$\langle w_1 - w_2, u - v \rangle \geq -k\|u - v\|^2,$$

where $w_1 \in Tu$ and $w_2 \in Tv$.

A multivalued operator T is Lipschitz continuous if there exists a constant $\lambda > 0$ such that

$$\|w_1 - w_2\| \leq \lambda \|u - v\|,$$

where $w_1 \in Tu$ and $w_2 \in Tv$.

$S : C \to C$ is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

$S : C \to C$ is said to be strictly pseudocontractive if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C.$$

It is clear that strictly pseudocontractive mappings include nonexpansive mapping as a special case. Next, we shall denote the set of fixed points of S by $F(S)$.

In order to prove our main results, we need the following lemmas and definitions.

Lemma 1.1. The element of $u \in C$ is a common element of $VI(C, A) \cap F(S)$ if and only if $z \in H$ satisfies the Wiener-Hopf equation (1.2), where $u = SP_C z$ and $z = u - \rho(Au + w)$, where $\rho > 0$ is a constant.

Proof. Since $u \in VI(C,A)$, from (1.5), we have

$$u = P_C[u - \rho(Au + w)].$$

Since u is also a fixed point of nonexpansive mapping S, we have

$$u = Su = SP_C[u - \rho(Au + w)].$$

Let $z = u - \rho(Au + w)$. That is,

$$\begin{cases} u = SP_C z, \\ z = u - \rho(Au + w), \end{cases}$$

from which it follows that $z = SP_C z - \rho(ASP_C z + w)$, which is exactly the Wiener-Hopf equation (1.4). This completes the proof.

Lemma 1.2 [18] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \lambda_n)a_n + b_n, \quad \forall n \geq n_0,$$

where n_0 is some nonnegative integer, $\{\lambda_n\}$ is a sequence in $[0,1]$ with $\sum_{n=1}^{\infty} \lambda_n = \infty$, $b_n = o(\lambda_n)$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.3. *For any $z \in H$, $u \in C$ satisfies the inequality:*

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in C,$$

if and only if $u = P_C z$.

Lemma 1.4. [10] $u \in C$ is a solution of the $VI(C,A)$ if and only if u satisfies

$$u = P_C[u - \rho(Au + w)], \tag{1.5}$$

where w is in Tu and $\rho > 0$ is a constant.

2. Algorithms

Algorithm 2.1. For any $z_0 \in H$, compute the sequence $\{z_n\}$ by the iterative processes:

$$\begin{cases} u_n = (\alpha I + (1-\alpha)T)P_C z_n, \\ z_{n+1} = (1-\alpha_n)z_n + \alpha_n[u_n - \rho(Au_n + w_n)], \quad n \geq 0, \end{cases} \tag{2.1}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ for all $n \geq 0$ and S is a strictly contractive mapping.

(I) If $S = I$, the identity mapping, in Algorithm 2.1, then we have the following algorithm:

Algorithm 2.2. For any $z_0 \in H$, compute the sequence $\{z_n\}$ by the iterative processes:

$$\begin{cases} u_n = P_C z_n, \\ z_{n+1} = (1-\alpha_n)z_n + \alpha_n[u_n - \rho(Au_n + w_n)], \quad n \geq 0, \end{cases} \tag{2.2}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ for all $n \geq 0$.

(II) If $S = I$, the identity mapping, and $\{\alpha_n\} = 1$ for all $n \geq 0$, in Algorithm 2.1, then we have the following:

Algorithm 2.3. For any $z_0 \in C$, compute the sequence $\{z_n\}$ by the iterative processes:

$$\begin{cases} u_n = P_C z_n, \\ z_{n+1} = u_n - \rho(Au_n + w_n), \quad n \geq 0, \end{cases} \tag{2.3}$$

which was basically considered by Verma [7].

3. Convergence Analysis

Theorem 3.1. *Let C be a closed convex subset of a separable real Hilbert space H. Let $A : C \to H$ be a relaxed (γ, r)-cocoerceive and μ-Lipschitz continuous mapping and let S be a κ-strictly psuedocontractive mapping from C into itself such that $F(S) \cap VI(C,A) \neq \emptyset$. Let $T : H \to 2^H$ be a multi-valued relaxed monotone and Lipschitz continuous operator with corresponding constant $k > 0$ and $m > 0$. Let $\{z_n\}$ and $\{u_n\}$ be sequences generated by Algorithm 2.1. $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\alpha \in [\kappa, 1)$;

(iii) $0 < \rho < \frac{2(r-\gamma\mu-k)}{(\mu+m)^2}, \quad r > \gamma\mu + k,$

then the sequences $\{u_n\}$ *and* $\{z_n\}$ *converge strongly to* $u^* \in F(S) \cap VI(C,A)$ *and* $z^* \in WHE(H,A,S)$*, respectively.*

Proof. Let $R : \alpha I + (1-\alpha)S$. In view of the restriction (ii), we find that R is nonexpansive with $F(R) = F(S)$. Letting $u \in C$ be the common elements of $F(S) \cap VI(C,A)$, we have

$$\begin{cases} u^* = RP_C z^*, \\ z^* = (1-\alpha_n)z^* + \alpha_n[u^* - \rho(Au^* + w^*)], \end{cases}$$

where $w^* \in Tu^*$ and $z^* \in WHE(H,A,S)$. Observing (2.1), we obtain

$$\begin{aligned} &\|z_{n+1} - z^*\| \\ &= \|(1-\alpha_n)z_n + \alpha_n[u_n - \rho(Au_n + w_n)] - z^*\| \\ &= \|(1-\alpha_n)z_n + \alpha_n[u_n - \rho(Au_n + w_n)] \\ &\quad - (1-\alpha_n)z^* + \alpha_n[u^* - \rho(Au^* + w^*)]\| \\ &\le (1-\alpha_n)\|z_n - z^*\| + \alpha_n\|u_n - u^* - \rho[(Au_n + w_n) - (Au^* + w^*)]\|. \end{aligned} \tag{3.1}$$

Now, we consider the second term of rightside of (3.1). By the assumption that A is relaxed (γ, r)-cocoercive, μ-Lipschitz continuous and T is relaxed monotone, m-Lipschitz continuous, we obtain

$$\begin{aligned} &\|u_n - u^* - \rho[(Au_n + w_n) - (Au^* + w^*)]\|^2 \\ &= \|u_n - u^*\|^2 - 2\rho\langle(Au_n + w_n) - (Au^* + w^*), u_n - u^*\rangle \\ &\quad + \rho^2\|(Au_n + w_n) - (Au^* + w^*)\|^2 \\ &= \|u_n - u^*\|^2 - 2\rho\langle Au_n - Au^*, u_n - u^*\rangle - 2\rho\langle w_n - w^*, u_n - u^*\rangle \\ &\quad + \rho^2\|(Au_n + w_n) - (Au^* + w^*)\|^2 \\ &\le \|u_n - u^*\|^2 - 2\rho(-\gamma\|Au_n - Au^*\| + r\|u_n - u^*\|) + 2\rho k\|u_n - u^*\| \\ &\quad + \rho^2\|(Au_n + w_n) - (Au^* + w^*)\|^2 \\ &\le \|u_n - u^*\|^2 + 2\rho(\gamma\mu - r + k)\|u_n - u^*\| + \rho^2\|(Au_n + w_n) - (Au^* + w^*)\|^2. \end{aligned} \tag{3.2}$$

Next, we consider the third term of right-side of (3.2).

$$\begin{aligned} &\|(Au_n+w_n)-(Au^*+w^*)\| = \|(Au_n-Au^*)+(w_n-w^*)\| \\ &\leq \|Au_n-Au^*\|+\|w_n-w^*\| \leq (\mu+m)\|u_n-u^*\|. \end{aligned} \tag{3.3}$$

Substitute (3.3) into (3.2) yields that

$$\begin{aligned} &\|u_n-u^*-\rho[(Au_n+w_n)-(Au^*+w^*)]\|^2 \\ &\leq \|u_n-u^*\|^2+2\rho(\gamma\mu-r+k)\|u_n-u^*\|+\rho^2(\mu+m)^2\|u_n-u^*\|^2 \\ &= [1+2\rho(\gamma\mu-r+k)+\rho^2(\mu+m)^2]\|u_n-u^*\|^2 \\ &= \theta^2\|u_n-u^*\|^2, \end{aligned} \tag{3.4}$$

where $\theta = \sqrt{1+2\rho(\gamma\mu-r+k)+\rho^2(\mu+m)^2}$. From condition (ii), we have $\theta < 1$. Substituting (3.4) into (3.1), we have

$$\|z_{n+1}-z^*\| \leq (1-\alpha_n)\|z_n-z^*\|+\alpha_n\theta\|u_n-u^*\|. \tag{3.5}$$

Since R is nonexpansive, we find that

$$\|u_n-u^*\| = \|RP_Cz_n-RP_Cz^*\| \leq \|z_n-z^*\|. \tag{3.6}$$

Substituting (3.6) into (3.5), we arrive at

$$\begin{aligned} \|z_{n+1}-z^*\| &\leq (1-\alpha_n)\|z_n-z^*\|+\alpha_n\theta\|z_n-z^*\| \\ &\leq [1-\alpha_n(1-\theta)]\|z_n-z^*\|. \end{aligned} \tag{3.7}$$

Observing condition (i) and applying Lemma 2.1 into (3.7), we can obtain $\lim_{n\to\infty}\|z_n-z^*\| = 0$. On the other hand, observing (3.6), we have $\lim_{n\to\infty}\|u_n-u^*\| = 0$. This completes the proof.

From Theorem 3.1, the following results are not hard to derive.

Corollary 3.2. *Let C be a closed convex subset of a separable real Hilbert space H. Let $A : C \to H$ be a relaxed (γ, r)-cocoerceive and μ-Lipschitz continuous mapping and let S be a nonexpanive mapping from C into itself such that $F(S) \cap VI(C,A) \neq \emptyset$. Let $T : H \to 2^H$ be a multi-valued relaxed monotone and Lipschitz continuous operator with corresponding constant $k > 0$ and $m > 0$. Let $\{z_n\}$ and $\{u_n\}$ be sequences generated by Algorithm 2.1. $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \rho < \frac{2(r-\gamma\mu-k)}{(\mu+m)^2}, \quad r > \gamma\mu + k,$

then the sequences $\{u_n\}$ *and* $\{z_n\}$ *converge strongly to* $u^* \in F(S) \cap VI(C,A)$ *and* $z^* \in WHE(H,A,S)$, *respectively.*

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