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ITERATIONS FOR EQUILIBRIUM AND FIXED POINT PROBLEMS

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Abstract. In this paper, the problem of finding a common element in the set of solutions to equilibrium problems and the set of fixed points of strict pseudocontractions is investigated. Strong convergence theorems of common elements are established in the framework of Hilbert spaces.

Keywords. Equilibrium problem; Nonexpansive mapping; Fixed point; Hilbert space.

1. Introduction-Preliminaries

Throughout this paper, we assume that H is a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. Let C be a nonempty closed and convex subset of H.

Let $A : C \to H$ be a nonlinear mapping. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. We consider the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x,y) + \langle Ax, y-x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

In this paper, the set of such $x \in C$ is denoted by $EP(F,A)$, i.e.,

$$EP(F,A) = \{x \in C : F(x,y) + \langle Ax, y-x\rangle \geq 0, \quad \forall y \in C\}.$$

If $A = 0$, then we have the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x,y) \geq 0, \quad \forall y \in C.$$

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$$EP(F) = \{x \in C : F(x,y) \geq 0, \quad \forall y \in C\}.$$

The problem (1.1) was studied by many authors, see, for example, [1-13] and the reference therein. Let $T : C \to H$ be a nonlinear mapping. Putting $F(x,y) = \langle Tx, y-x\rangle$, $\forall x,y \in C$, we see that $z \in FP(F)$ if and only if $\langle Tz, y-z\rangle \geq 0$, $\forall y \in C$. That is, z is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of the problem (1.3). To study the problem (1.3), we may assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x,y \in C$;

(A3) for each $x,y,z \in C$, $\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$;

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semi-continuous.

Recall that the classical variational inequality problem is to find $u \in C$ such that

$$\langle Au, v-u\rangle \geq 0, \quad \forall v \in C. \tag{1.2}$$

For given $z \in H$ and $u \in C$, the following inequality holds $\langle u-z, v-u\rangle \geq 0$, $\forall v \in C$, if and only if $u = P_C z$, where P_C denotes the metric projection from H onto C. From above, we see that $u \in C$ is a solution to the problem (1.1) if and only if u is a fixed point of the following mapping $P_C(I - \rho A)u$, where $\rho > 0$ is a constant and I is identity mapping. This implies that the variational inequality problem (1.1) is equivalent to the fixed point problem. This alternative formula is very important form the numerical analysis point of view. Recently, many authors studied the problem (1.1) by iterative methods provided that A has some monotonicity. Let $S : C \to H$ be a mapping. In this paper, we use $F(S)$ to denote the set of fixed points of S. Recall that the mapping S is said to be contractive if there exits a constant $\alpha \in (0,1)$ such that $\|Sx - Sy\| \leq \alpha\|x-y\|$, $\forall x,y \in C$. S is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x-y\|, \quad \forall x,y \in C.$$

S is said to be strictly pseudocontractive with the coefficient $k \in [0,1)$ if

$$\|Sx - Sy\|^2 \leq \|x-y\|^2 + k\|(I-S)x - (I-S)y\|^2, \quad \forall x,y \in C.$$

For such a case, S is also said to be a k-strict pseudocontraction.

The class of strict pseudocontractions is one of the most important classes of mappings among nonlinear mappings. Within the past several decades, many authors have been devoting to the studies on the existence and convergence of fixed points for strict pseudocontractions. Recently, Zhou [14] considered a convex combination method to study strict pseudocontractions. More precisely, take $t \in (0,1)$ and define a mapping S_t by $S_t x = tx + (1-t)Tx$, where T is a strict pseudocontraction. Under appropriate restrictions on t, it is proved the mapping S_t is nonexpansive; see [14] for more details. Therefore, the techniques of studying nonexpansive mappings can be applied to study strict pseudocontractions.

Recently, Takahashi and Takahashi [15] studied the problem (1.1) by considering an iterative method. To be more precise, they proved the following Theorem.

Theorem TT. *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let S be a nonexpansive mapping of C into H such that $F(S) \cap EP(F) \neq \emptyset$. Let f be a contraction of H into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{cases} F(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geq 0, \quad \forall u \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S y_n, \quad n \geq 1 \end{cases}$$

where $\{\alpha_n\} \in [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty, \ \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

$$\liminf_{n\to\infty} r_n > 0, \ \textit{and} \ \sum_{n=1}^{\infty} |r_{n+1} - r_n| < 0.$$

Then $\{x_n\}$ and $\{y_n\}$ converge strongly to $z \in F(S) \cap EP(F)$, where $z = P_{F(S)\cap EP(F)} f(z)$.

In this paper, motivated by Takahashi and Takahashi [15], we introduce a general iterative algorithm to study the equilibrium problem and the fixed point problem. In order to prove our main results, we need the following lemmas.

Lemma 2.7 [17] *Let $A : C \to H$ be an α-inverse-strongly monotone mapping and f be a bifunction satisfying the conditions (A1)-(A4). Let $r > 0$ be any given number and $x \in H$ define a* *mapping* $K_r : C \to C$ *as follows: for any* $x \in C$,

$$K_r x = \{p \in C : F(p,q) + \langle Ap, q-p\rangle + \frac{1}{r}\langle q-p, Jp - Jx\rangle \geq 0, \quad \forall q \in C.$$

Then the following conclusions hold:

(1) K_r *is single-valued;*

(2) K_r *is a firmly nonexpansive-type mapping;*

(3) $F(K_r) = EP(F,A)$ *is closed and convex.*

Lemma 1.2 [16] *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* E *and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose that* $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.3. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* T_1 *and* T_2 *be two nonexpansive mappings from* H *into* C *with a common fixed point. Define a mapping* $S : H \to C$ *by*

$$Sx = \lambda T_1 x + (1-\lambda)T_2 x, \quad \forall x \in C,$$

where λ *is a constant in* $(0,1)$. *Then* S *is nonexpansive and* $F(S) = F(T_1) \cap F(T_2)$.

Lemma 1.4 *Let* E *be a real Hilbert space,* C *a nonempty closed convex subset of* E *and* $S : H \to C$ *a nonexpansive mapping. Then* $I - S$ *is demiclosed at zero.*

Lemma 1.5 *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 1.6 (Zhou [14]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to H$ a k-strict pseudocontraction with a fixed point. Then $F(T) = F(P_C T)$ is closed and convex. Define $S : C \to H$ by $Sx = kx + (1-k)Tx$ for each $x \in C$. Then S is nonexpansive such that $F(S) = F(T)$.*

2. Main results

Theorem 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F_1 and F_2 be two bifunctions from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), respectively. Let A_1 and A_2 are two inverse-strongly monotone mapping from C to H. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be sequences in $(0,1)$. Let f be a contraction with the coefficient $\alpha \in (0,1)$ from H into itself and $T : C \to H$ a nonexpansive mapping with a fixed point. Assume that $\mathscr{F} = EP(F_1, A_1) \cap EP(F_2, A_2) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in H$ and*

$$\begin{cases} F_1(u_n, u) + \langle A_1 u_n, u - u_n\rangle + \frac{1}{r}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in C, \\ F_2(v_n, v) + \langle A_2 v_n, u - v_n\rangle + \frac{1}{s}\langle v - v_n, v_n - x_n\rangle \geq 0, \quad \forall v \in C, \\ y_n = \delta_n u_n + (1-\delta_n) v_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S y_n, \quad \forall n \geq 1, \end{cases}$$

where r, s are two positive constants. Assume that the following restrictions are satisfied

- (R1) $\alpha_n + \beta_n + \gamma_n = 1$, $\forall n \geq 1$;
- (R2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (R3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
- (R4) $\lim_{n\to\infty} \delta_n = \delta \in (0,1)$.

Then the sequence $\{x_n\}$ converges strongly to $z \in F$, where $z = P_{\mathscr{F}} f(z)$.

Proof. From Lemma 1.1, we see that u_n and v_n can be rewritten as $T_r x_n$ and $T_s x_n$, where K_r and K_s are defined as (1.4). It also from Lemma 1.1 that K_r and K_s are single-valued, firmly nonexpansive, $EF(F_1, A_1) = F(K_r)$ and $EF(F_2, A_2) = F(K_s)$. Fix $p \in \mathscr{F}$. From the algorithm

(Υ) and the nonexpansivity of K_r and K_s, we obtain that

$$\begin{aligned}\|y_n - p\| &\leq \delta_n\|u_n - p\| + (1-\delta_n)\|v_n - p\| \\ &= \delta_n\|T_r x_n - T_r p\| + (1-\delta_n)\|T_s x_n - T_s p\| \\ &\leq \delta_n\|x_n - p\| + (1-\delta_n)\|x_n - p\| \\ &= \|x_n - p\|.\end{aligned}$$

From Lemma 1.6, we have that S is nonexpansive with $F(S) = F(T)$. It follows that

$$\begin{aligned}\|x_{n+1} - p\| &\leq \alpha_n\|f(x_n) - f(p)\| + \alpha_n\|f(p) - p\| + \beta_n\|x_n - p\| + \gamma_n\|Sy_n - p\| \\ &\leq \alpha_n\|f(x_n) - f(p)\| + \alpha_n\|f(p) - p\| + \beta_n\|x_n - p\| + \gamma_n\|y_n - p\| \\ &\leq \alpha_n\alpha\|x_n - p\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\|x_n - p\| \\ &= [1 - \alpha_n(1-\alpha)]\|x_n - p\| + \alpha_n\|f(p) - p\| \\ &\leq \max\{\|x_n - p\|, \frac{\|f(p) - p\|}{1-\alpha}\}.\end{aligned}$$

By mathematical induction, we arrive at $\|x_{n+1} - p\| \leq \max\{\|x_1 - p\|, \frac{\|f(p)-p\|}{1-\alpha}\}$. This shows that the sequence $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{u_n\}$ and $\{v_n\}$. On the other hand, we have

$$\begin{aligned}y_{n+1} - y_n &= \delta_{n+1}u_{n+1} + (1-\delta_{n+1})v_{n+1} - [\delta_n u_n + (1-\delta_n)v_n] \\ &= \delta_{n+1}(u_{n+1} - u_n) + (\delta_{n+1} - \delta_n)u_n + (1-\delta_{n+1})(v_{n+1} - v_n) + (\delta_n - \delta_{n+1})v_n.\end{aligned}$$

This implies that

$$\begin{aligned}&\|y_{n+1} - y_n\| \\ &\leq \delta_{n+1}\|u_{n+1} - u_n\| + (1-\delta_{n+1})\|v_{n+1} - v_n\| + |\delta_{n+1} - \delta_n|(\|u_n\| + \|v_n\|) \\ &= \delta_{n+1}\|T_r x_{n+1} - T_r x_n\| + (1-\delta_{n+1})\|T_s x_{n+1} - T_s x_n\| + |\delta_{n+1} - \delta_n|(\|u_n\| + \|v_n\|) \\ &\leq \|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|M,\end{aligned} \tag{2.1}$$

where M is an appropriate constant such that $M \geq \sup_{n\geq 1}\{\|u_n\| + \|v_n\|\}$. Putting $e_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}$, we have

$$x_{n+1} = (1-\beta_n)e_n + \beta_n x_n, \quad \forall n \geq 1. \tag{2.2}$$

Now, we compute $\|e_{n+1}-e_n\|$. we have

$$\|e_{n+1}-e_n\| \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-Sy_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|Sy_n - f(x_n)\| + \|y_{n+1}-y_n\|. \tag{2.3}$$

Substituting (2.1) into (2.3), we arrive at

$$\|e_{n+1}-e_n\| - \|x_{n+1}-x_n\| \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|y_n - f(x_n)\| + |\delta_{n+1}-\delta_n|M.$$

From (R2), (R3) and (R4), we obtain that

$$\limsup_{n\to\infty}(\|e_{n+1}-e_n\| - \|x_{n+1}-x_n\|) \le 0.$$

It follows from Lemma 1.2 that $\lim_{n\to\infty}\|e_n - x_n\| = 0$. In view of (2.2), we see that $x_{n+1}-x_n = (1-\beta_n)(e_n - x_n)$. It follows from (R3) that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{2.4}$$

On the other hand, we have $(1-\beta_n)\|x_n - Sy_n\| \le \|x_n - x_{n+1}\| + \alpha_n\|f(x_n)-Sy_n\|$.
It follows from (R2), (R3) and (2.4) that

$$\lim_{n\to\infty}\|x_n - Sy_n\| = 0. \tag{2.5}$$

On the other hand, for each $p \in \mathscr{F}$, we have

$$\begin{aligned}\|u_n - p\|^2 &= \|T_r x_n - T_r p\|^2 \\ &\le \langle T_r x_n - T_r p, x_n - p\rangle \\ &= \langle u_n - p, x_n - p\rangle \\ &= \frac{1}{2}(\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2).\end{aligned}$$

It follows that

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{2.6}$$

In a similar way, we can prove

$$\|v_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - v_n\|^2. \tag{2.7}$$

In view of (2.6) and (2.7), we see that

$$\begin{aligned}\|y_n - p\|^2 &= \|\delta_n u_n + (1-\delta_n)v_n - p\|^2 \\ &\le \delta_n\|u_n - p\|^2 + (1-\delta_n)\|v_n - p\|^2 \\ &\le \|x_n - p\|^2 - \delta_n\|x_n - u_n\|^2 - (1-\delta_n)\|x_n - v_n\|^2.\end{aligned} \tag{2.8}$$

This implies that

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n S y_n - p\|^2 \\ &\le \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|Sy_n - p\|^2 \\ &\le \alpha_n\|f(x_n) - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2 \\ &\le \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 - \delta_n\gamma_n\|x_n - u_n\|^2 - (1-\delta_n)\gamma_n\|x_n - v_n\|^2.\end{aligned} \tag{2.9}$$

It follows that

$$\begin{aligned}\delta_n\gamma_n\|x_n - u_n\|^2 &\le \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\le \alpha_n\|f(x_n) - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\|.\end{aligned} \tag{2.10}$$

From (R2), (R3) and (R4), we arrive at

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{2.11}$$

Thanks to (2.9), we also have

$$\begin{aligned}(1-\delta_n)\gamma_n\|x_n - v_n\|^2 &\le \alpha_n\|f(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\le \alpha_n\|f(x_n) - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\|.\end{aligned}$$

From (R2), (R3) and (R4), we arrive at

$$\lim_{n\to\infty}\|x_n - v_n\| = 0. \tag{2.12}$$

On the other hand, we have

$$\|y_n - x_n\| = \|\delta_n u_n + (1-\delta_n)v_n - x_n\| \le \delta_n\|u_n - x_n\| + (1-\delta_n)\|v_n - x_n\|.$$

This shows that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{2.13}$$

Note that $\|x_n - Sx_n\| \le \|x_n - Sy_n\| + \|Sy_n - Sx_n\| \le \|x_n - Sy_n\| + \|y_n - x_n\|$. Combining (2.5) with (2.13), we arrive at

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{2.14}$$

Define a mapping $Q : H \to C$ by $Qx = \delta T_r x + (1-\delta)T_s x, \quad \forall x \in H$, where $\lim_{n\to\infty} \delta_n = \delta \in (0,1)$. From Lemma 1.3, we have that Q is nonexpansive with

$$F(Q) = F(K_r) \cap F(K_s) = EP(F_1, A_1) \cap FP(F_2, A_2). \tag{2.15}$$

Next, we show that

$$\lim_{n\to\infty} \|x_n - Qx_n\| = 0. \tag{2.16}$$

Notice that

$$\begin{aligned} \|x_n - Qx_n\| &\le \|x_n - y_n\| + \|y_n - Qx_n\| \\ &= \|x_n - y_n\| + \|\delta_n K_r x_n + (1-\delta_n)K_s x_n - [\delta K_r x_n + (1-\delta)K_s x_n]\| \\ &= \|x_n - y_n\| + |\delta_n - \delta| M, \end{aligned}$$

From (R4) and (2.13), we see that (2.16) holds. Since $P_{\mathscr{F}} f$ is a contraction with the coefficient $\alpha \in (0,1)$, we have that there exists a unique fixed point. We use z to denote the unique fixed point of the mapping $P_{\mathscr{F}} f$. That is, $z = P_{\mathscr{F}} f(z)$. Next, we show that

$$\limsup_{n\to\infty} \langle f(z) - z, x_n - z\rangle \le 0. \tag{2.17}$$

To show (2.17), we may choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle f(z) - z, x_n - z\rangle = \lim_{i\to\infty} \langle f(z) - z, x_{n_i} - z\rangle. \tag{2.18}$$

Since $\{x_{n_i}\}$ is bounded, we can choose a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converges weakly to q. We may, assume without loss of generality, that $x_{n_i} \rightharpoonup q$. It follows from (2.16) that

$$\lim_{i\to\infty} \|x_{n_i} - Qx_{n_i}\| = 0.$$

In view of Lemma 1.4, we obtain that $q \in F(Q)$. Next, we show that $q \in F(S)$. Assume that $q \notin F(S)$. From (2.14) and Opial's condition, we have

$$\begin{aligned}\liminf_{i\to\infty}\|x_{n_i}-q\| &< \liminf_{i\to\infty}\|x_{n_i}-Sq\|\\ &\le \liminf_{i\to\infty}(\|x_{n_i}-Sx_{n_i}\|+\|Sx_{n_i}-Sq\|)\\ &\le \liminf_{i\to\infty}\|x_{n_i}-q\|,\end{aligned}$$

which derives a contradiction. This implies that $q \in F(S)$. That is, $q \in F(Q)\cap F(S) = EP(F_1,A_1)\cap EP(F_2,A_2)\cap F(T)$. From (2.18), we have $\limsup_{n\to\infty}\langle f(z)-z, x_n-z\rangle = \langle f(z)-z, q-z\rangle \le 0$. That is,

$$\limsup_{n\to\infty}\langle f(z)-z, x_{n+1}-z\rangle \le 0. \tag{2.19}$$

Finally, we prove that $x_n \to z$ as $n \to \infty$. Notice that

$$\begin{aligned}&\|x_{n+1}-z\|^2\\ &\le \alpha_n\|f(x_n)-f(z)\|\|x_{n+1}-z\| + \alpha_n\langle f(z)-z, x_{n+1}-z\rangle\\ &\quad + \beta_n\|x_n-z\|\|x_{n+1}-z\| + \gamma_n\|y_n-z\|\|x_{n+1}-z\|\\ &\le \alpha_n\alpha\|x_n-z\|\|x_{n+1}-z\| + \alpha_n\langle f(z)-z, x_{n+1}-z\rangle\\ &\quad + \beta_n\|x_n-z\|\|x_{n+1}-z\| + \gamma_n\|x_n-z\|\|x_{n+1}-z\|\\ &= [1-\alpha_n(1-\alpha)]\|x_n-z\|\|x_{n+1}-z\| + \alpha_n\langle f(z)-z, x_{n+1}-z\rangle\\ &\le \frac{1-\alpha_n(1-\alpha)}{2}(\|x_n-z\|^2+\|x_{n+1}-z\|^2) + \alpha_n\langle f(z)-z, x_{n+1}-z\rangle\\ &\le \frac{1-\alpha_n(1-\alpha)}{2}\|x_n-z\|^2 + \frac{1}{2}\|x_{n+1}-z\|^2 + \alpha_n\langle f(z)-z, x_{n+1}-z\rangle.\end{aligned}$$

It follows that

$$\|x_{n+1}-z\|^2 \le [1-\alpha_n(1-\alpha)]\|x_n-z\|^2 + 2\alpha_n\langle f(z)-z, x_{n+1}-z\rangle.$$

From (2.19) and (R2), we can conclude the desired conclusion easily by Lemma 1.5. This completes the proof.

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