SOME RESULTS ON NONEXPANSIVE AND RELAXED COCOERCIVE MAPPINGS

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Abstract. In this paper, we instigate a viscosity approximation method for nonexpansive and relaxed cocoercive mappings. A strong convergence theorem is established in the framework of Hilbert spaces.

Keywords. Relaxed cocoercive mapping; Variational inequality; Nonexpansive mapping; Fixed point.

1. Introduction and preliminaries

Throughout this paper, we assume that $H$ is a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. Let $C$ be a nonempty closed and convex subset of $H$ and let $P_C$ denote the metric projection from $H$ onto $C$.

Consider the following generalized variational inequality problem. Give nonlinear mappings $T_1 : C \to H$ and $T_2 : C \to H$, find an $u \in C$ such that

$$\langle u - T_1 u + \lambda T_2 u, v - u \rangle \geq 0, \quad \forall v \in C, \quad (1.1)$$

where $\lambda$ is a constant. In this paper, we use $VI(T_1, T_2)$ to denote the set of solutions of variational inequality problem (1.1).

It is easy to see that an element $u \in C$ is a solution to problem (1.1) if and only if $u \in C$ is a fixed point of mapping $P_C(T_1 - \lambda T_2)$.

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If $T_1 = I$, the identity mapping, then problem (1.1) is reduced to the following. Find $u \in C$ such that
\[
\langle \lambda T_2 u, v - u \rangle \geq 0, \quad \forall v \in C. \tag{1.2}
\]

Variational inequality problem (1.2) was introduced by Stampacchia [1] in 1964. Problem (1.2) has emerged as a fascinating and interesting branch of mathematical and engineering sciences with a wide range of applications in industry, finance, economics, social, ecology, regional, pure and applied sciences. In this paper, we use $VI(C, T_2)$ to denote the set of solutions of variational inequality problem (1.2). Recently, gradient methods have been extensively investigated for solving problems (1.1) and (1.2); see [2-9] and the references therein.

Let $A : C \to H$ be a mapping. Recall that $A$ is said to be monotone iff
\[
\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.
\]

$A$ is said to be relaxed $\eta$-cocoercive if there exists a positive real number $\eta > 0$ such that
\[
\langle Ax - Ay, x - y \rangle \geq (-\eta)\|Ax - Ay\|^2, \quad \forall x, y \in C.
\]

$A$ is said to be $\rho$-strongly monotone if there exists a positive real number $\rho > 0$ such that
\[
\langle Ax - Ay, x - y \rangle \geq \rho\|x - y\|^2, \quad \forall x, y \in C.
\]

$A$ is said to be relaxed $(\eta, \rho)$-cocoercive if there exist positive real numbers $\eta, \rho > 0$ such that
\[
\langle Ax - Ay, x - y \rangle \geq (-\eta)\|Ax - Ay\|^2 + \rho\|x - y\|^2, \quad \forall x, y \in C.
\]

Let $S : C \to C$ be a mapping. Recall that $S$ is said to be contractive if there exits a constant $0 \leq \alpha < 1$ such that
\[
\|Sx - Sy\| \leq \alpha\|x - y\|, \quad \forall x, y \in C.
\]

$S$ is said to be nonexpansive if
\[
\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.
\]

In this paper, we use $F(S)$ to denote the set of fixed points of the mapping $S$. It is know if $C$ is weakly compact, then $F(S)$ is not empty.
Recently, many authors studied the problem of finding a common element of the set of solutions of variational inequality problem (1.2) and of the set of fixed points of a nonexpansive mapping; see, for example, [9-14] and the references therein.

In this paper, Motivated by the research work going on in this direction, we consider variational inequality problem (1.2) and a fixed point problem of nonexpansive mappings by viscosity iterative method. Strong convergence theorems of common elements are established in the framework of Hilbert spaces.

**Lemma 1.1.** [15] Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $S_1 : C \to C$ and $S_2 : C \to C$ be nonexpansive mappings on C. Suppose that $F(S_1) \cap F(S_2)$ is nonempty. Define a mapping $S : C \to C$ by

$$Sx = aS_1x + (1 - a)S_2x, \quad \forall x \in C.$$ 

Then S is nonexpansive with $F(S) = F(S_1) \cap F(S_2)$.

**Lemma 1.2** [16] Let C be a nonempty closed and convex subset of a real Hilbert space H and $S : C \to C$ a nonexpansive mapping. Then $I - S$ is demi-closed at zero.

**Lemma 1.3.** [17] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Hilbert space H and let $\{\beta_n\}$ be a sequence in $(0, 1)$ with

$$0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1.$$ 

Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and

$$\limsup_{n \to \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$ 

Then $\lim_{n \to \infty} \|y_n - x_n\| = 0$.

**Lemma 1.4.** [18] Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(a) $\lim_{n \to \infty} \gamma_n = 0, \sum_{n=1}^{\infty} \gamma_n = \infty$;

(b) $\limsup_{n \to \infty} \delta_n / \gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$. 

Then \( \lim_{n \to \infty} \alpha_n = 0 \).

Now, we are in a position to show our main results.

2. Main results

**Theorem 2.1.** Let \( H \) be a real Hilbert space and let \( C \) be a nonempty closed and convex subset of \( H \). Let \( T_1 : C \to H \) be a relaxed \((\eta_1, \rho_1)\)-cocoercive and \( \mu_1 \)-Lipschitz continuous mapping and let \( T_2 : C \to H \) a relaxed \((\eta_2, \rho_2)\)-cocoercive and \( \mu_2 \)-Lipschitz continuous mapping, respectively. Let \( f : C \to C \) be a contractive mapping and let \( S : C \to C \) be a nonexpansive mapping with a fixed point. Assume that \( F(S) \cap \text{VI}(T_1, T_2) \neq \emptyset \). Let \( \{x_n\} \) be a sequence generated by the following manner: \( x_1 \in C \), chosen arbitrarily, \( y_n = \delta_n Sx_n + (1 - \delta_n)P_C(T_1x_n - \lambda T_2x_n) \), \( x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n \), \( n \geq 1 \), where \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) and \( \{\delta_n\} \) are sequences in \((0, 1)\) satisfying the following restrictions: \( \alpha_n + \beta_n + \gamma_n = 1 \), \( \forall n \geq 1; 0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1 \); \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \); \( \lim_{n \to \infty} \delta_n = \delta \in (0, 1) \), and \( \lambda \) is a constant such that \( \sqrt{1 - 2\rho_1 + \mu_1^2 + 2\eta_1 \mu_1^2 + \sqrt{1 - 2\rho_2 + \lambda^2 \mu_2^2 + 2\lambda \eta_2 \mu_2^2}} \leq 1 \). Then the sequence \( \{x_n\} \) generated by the algorithm \((\Upsilon)\) converges strongly to a common element \( \bar{x} \in F(S) \cap \text{VI}(T_1, T_2) \), which uniquely solves the following variational inequality

\[
\langle f(\bar{x}) - \bar{x}, \bar{x} - x^* \rangle \geq 0, \quad \forall x^* \in F(S) \cap \text{VI}(T_1, T_2).
\]

**Proof.** The proof is split into 5 steps.

Step 1. Show that \( \{x_n\} \) is bounded.

Note that \( P_C(T_1 - \lambda T_2) \) is nonexpansive. Indeed, for each \( x, y \in C \), we have

\[
\begin{align*}
\|P_C(T_1 - \lambda T_2)x - P_C(T_1 - \lambda T_2)y\| & \leq \|(T_1 - \lambda T_2)x - (T_1 - \lambda T_2)y\| \\
& \leq \|(x - y) - (T_1 x - T_1 y)\| + \|(x - y) - \lambda (T_2 x - T_2 y)\|. \tag{2.1}
\end{align*}
\]
It follows from the assumption that \( T_1 \) is relaxed \((\eta_1, \rho_1)\)-cocoercive and \( \mu_1 \)-Lipschitz continuous that

\[
\|x - y - (T_1 x - T_1 y)\|^2 = \|x - y\|^2 - 2\langle T_1 x - T_1 y, x - y \rangle + \|T_1 x - T_1 y\|^2
\]

\[
\leq \|x - y\|^2 - 2((-\eta_1)\|T_1 x - T_1 y\|^2 + \rho_1\|x - y\|^2) + \mu_1^2\|x - y\|^2
\]

\[
= (1 - 2\rho_1 + \mu_1^2)\|x - y\|^2 + 2\eta_1\|T_1 x - T_1 y\|^2
\]

\[
\leq \theta_1^2\|x - y\|^2,
\]

where \( \theta_1 = \sqrt{1 - 2\rho_1 + \mu_1^2 + 2\eta_1\mu_1^2} \). That is,

\[
\|x - y - (T_1 x - T_1 y)\| \leq \theta_1\|x - y\|.
\] (2.2)

On the other hand, by the assumption that \( T_2 \) is relaxed \((\eta_2, \rho_2)\)-cocoercive and \( \mu_2 \)-Lipschitz continuous, we arrive at

\[
\|x - y - \lambda (T_2 x - T_2 y)\|^2
\]

\[
= \|x - y\|^2 - 2\lambda \langle T_2 x - T_2 y, x - y \rangle + \lambda^2\|T_2 x - T_2 y\|^2
\]

\[
\leq \|x - y\|^2 - 2\lambda [(-\eta_2)\|T_2 x - T_2 y\|^2 + \rho_2\|x - y\|^2] + \lambda^2\mu_2^2\|x - y\|^2
\]

\[
= (1 - 2\lambda\rho_2 + \lambda^2\mu_2^2)\|x - y\|^2 + 2\lambda\eta_2\|T_3 x - T_3 y\|^2
\]

\[
\leq \theta_2^2\|x - y\|^2,
\]

where \( \theta_2 = \sqrt{1 - 2\lambda\rho_2 + \lambda^2\mu_2^2 + 2\lambda\eta_2\mu_2^2} \). This implies that

\[
\|x - y - \lambda (T_2 x - T_2 y)\| \leq \theta_2\|x - y\|.
\] (2.3)

From the assumption (e), we see that \( \theta_1 + \theta_2 \leq 1 \). Substituting (2.2) and (2.3) into (2.1), we see that

\[
\|PC(T_1 - \lambda T_2)x - PC(T_1 - \lambda T_2)y\| \leq \|x - y\|.
\]

This shows that \( PC(T_1 - \lambda T_2) \) is nonexpansive. Fixing \( p \in VI(T_1, T_2) \cap F(S) \), we see that \( p = PC(T_1 - \lambda T_2)p \) and \( p = Sp \). Put \( z_n = PC(T_1 x_n - \lambda T_2 x_n) \). It follows that

\[
\|y_n - p\| \leq \delta_n\|Sx_n - Sp\| + (1 - \delta_n)\|PC(T_1 x_n - \lambda T_2 x_n) - PC(T_1 p - \lambda T_2 p)\|
\]

\[
\leq \|x_n - p\|.
\] (2.4)
It follows that
\[
\|x_{n+1} - p\| \leq \alpha_n\|f(x_n) - p\| + \beta_n\|x_n - p\| + \gamma_n\|Sy_n - p\|
\]
\[
\leq \alpha_n\|f(p) - p\| + \alpha_n\alpha\|x_n - p\| + \beta_n\|x_n - p\| + \gamma_n\|y_n - p\|
\]
\[
\leq \alpha_n(1 - \alpha)\|f(p) - p\| + (1 - \alpha_n(1 - \alpha))\|x_n - p\|.
\]

By mathematical inductions, we see that \(\{x_n\}\) is bounded.

Step 2. Show that \(x_{n+1} - x_n \rightarrow 0\) as \(n \rightarrow \infty\).

Since the mapping \(P_C(T_1 - \lambda T_2)\) is nonexpansive, we see that
\[
\|z_{n+1} - z_n\| = \|P_C(T_1 - \lambda T_2)x_{n+1} - P_C(T_1 - \lambda T_2)x_n\| \leq \|x_{n+1} - x_n\|. \quad (2.5)
\]

It follows that
\[
\|y_{n+1} - y_n\| = \|\delta_{n+1}Sx_{n+1} + (1 - \delta_{n+1})z_{n+1} - \delta_nSx_n - (1 - \delta_n)z_n\|
\]
\[
\leq \delta_{n+1}\|Sx_{n+1} - Sx_n\| + (1 - \delta_{n+1})\|z_{n+1} - z_n\| + |\delta_{n+1} - \delta_n|\|Sx_n - z_n\|
\]
\[
\leq \delta_{n+1}\|x_{n+1} - x_n\| + (1 - \delta_{n+1})\|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|\|Sx_n - z_n\|
\]
\[
\leq \|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|M,
\]

where \(M\) is an appropriate constant such that \(M \geq \sup_{n \geq 1}\{\|Sx_n - z_n\|\}\). Put \(l_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}\), for all \(n \geq 1\). That is,
\[
x_{n+1} = (1 - \beta_n)l_n + \beta_n x_n, \quad \forall n \geq 1. \quad (2.7)
\]

Now, we estimate \(\|l_{n+1} - l_n\|\). From
\[
l_{n+1} - l_n = \frac{\alpha_{n+1}f(x_{n+1}) + \gamma_{n+1}y_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_nf(x_n) + \gamma_n y_n}{1 - \beta_n}
\]
\[
= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}f(x_{n+1}) + \frac{1 - \beta_{n+1} - \alpha_{n+1}}{1 - \beta_{n+1}}y_{n+1} - \frac{\alpha_n}{1 - \beta_n}f(x_n) - \frac{1 - \beta_n - \alpha_n}{1 - \beta_n}y_n
\]
\[
= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(f(x_{n+1}) - y_{n+1}) + \frac{\alpha_n}{1 - \beta_n}(y_n - f(x_n)) + y_{n+1} - y_n,
\]
we have
\[
\|l_{n+1} - l_n\| \leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|f(x_{n+1}) - y_{n+1}\| + \frac{\alpha_n}{1 - \beta_n}\|y_n - f(x_n)\| + \|y_{n+1} - y_n\|. \quad (2.8)
\]

From (2.6), we obtain
\[
\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|f(x_{n+1}) - Sy_{n+1}\| + \frac{\alpha_n}{1 - \beta_n}\|Sy_n - f(x_n)\| + |\delta_{n+1} - \delta_n|M.
\]
It follows that \( \lim_{n \to \infty} \| l_{n+1} - l_n \| - \| x_{n+1} - x_{n+1} \| < 0 \). Using Lemma 1.3, we have

\[
\lim_{n \to \infty} \| l_n - x_n \| = 0. \tag{2.9}
\]

By (2.7), we see that \( x_{n+1} - x_n = (1 - \beta_n)(l_n - x_n) \). Therefore, we have

\[
\lim_{n \to \infty} \| x_{n+1} - x_n \| = 0. \tag{2.10}
\]

Note that

\[
\lim_{n \to \infty} \| y_n - x_n \| = 0. \tag{2.11}
\]

Define a mapping \( R : C \to C \) by

\[
Rx = \delta Sx + (1 - \delta)PC(T_1 - \lambda T_2)x, \quad \forall x \in C,
\]

where \( \delta = \lim_{n \to \infty} \delta_n \). From Lemma 1.1, we see that \( R \) is nonexpansive with \( F(R) = F(\{PC(T_1 - \lambda T_2)\}) \cap F(S) = VI(T_1, T_2) \cap F(S) \).

Step 3. Show that \( Rx_n - x_n \to 0 \) as \( n \to \infty \). Note that

\[
\| Rx_n - x_n \| \leq \| Rx_n - y_n \| + \| y_n - x_n \|
\]

\[
\leq \| \delta Sx_n + (1 - \delta)PC(T_1 - \lambda T_2)x_n - \delta_n Sx_n - (1 - \delta_n)PC(T_1 - \lambda T_2)x_n \| + \| y_n - x_n \|
\]

\[
\leq \| \delta - \delta_n \| M + \| y_n - x_n \|.
\]

Using (2.11), we obtain

\[
\lim_{n \to \infty} \| Rx_n - x_n \| = 0. \tag{2.12}
\]

Step 4. Show that \( \limsup_{n \to \infty} \langle f(\bar{x}) - \bar{x}, x_n - \bar{x} \rangle \leq 0 \).

To show it, we can choose a sequence \( \{x_{n_i}\} \) of \( \{x_n\} \) such that

\[
\lim_{n \to \infty} \langle f(\bar{x}) - \bar{x}, x_n - \bar{x} \rangle = \lim_{i \to \infty} \langle f(\bar{x}) - \bar{x}, x_{n_i} - \bar{x} \rangle. \tag{2.13}
\]

Since \( \{x_{n_i}\} \) is bounded, there exists a subsequence \( \{x_{n_{i_j}}\} \) of \( \{x_{n_i}\} \) which converges weakly to \( b \). Without loss of generality, we may assume that \( x_{n_i} \to b \). From Lemma 1.2, we see that

\[
b \in F(R) = F(\{PC(T_1 - \lambda T_2)\}) \cap F(S) = VI(T_1, T_2) \cap F(S).
\]

It follows from (2.13) that

\[
\limsup_{n \to \infty} \langle f(\bar{x}) - \bar{x}, x_n - \bar{x} \rangle = \lim_{i \to \infty} \langle f(\bar{x}) - \bar{x}, x_{n_i} - \bar{x} \rangle = \langle f(\bar{x}) - \bar{x}, b - \bar{x} \rangle \leq 0. \tag{2.14}
\]
Step 5. Show that $x_n \to \bar{x}$ as $n \to \infty$. Note that
\[
\|x_{n+1} - \bar{x}\|^2 = \alpha_n \langle f(x_n) - \bar{x}, x_{n+1} - \bar{x} \rangle + \beta_n \langle x_n - \bar{x}, x_{n+1} - \bar{x} \rangle + \gamma_n \langle y_n - \bar{x}, x_{n+1} - \bar{x} \rangle
\]
\[
\leq \alpha_n \langle f(x_n) - \bar{x}, x_{n+1} - \bar{x} \rangle + \beta_n \|x_n - \bar{x}\| \|x_{n+1} - \bar{x}\| + \gamma_n \|y_n - \bar{x}\| \|x_{n+1} - \bar{x}\|
\]
\[
\leq \alpha_n \langle f(\bar{x}) - \bar{x}, x_{n+1} - \bar{x} \rangle + (1 - \alpha_n (1 - \alpha)) \|x_n - \bar{x}\| \|x_{n+1} - \bar{x}\|.
\]
It follows that
\[
\|x_{n+1} - \bar{x}\|^2 \leq 2\alpha_n \langle f(\bar{x}) - \bar{x}, x_{n+1} - \bar{x} \rangle + (1 - \alpha_n (1 - \alpha)) \|x_n - \bar{x}\|^2.
\]
From Lemma 1.4, we have $\lim_{n \to \infty} \|x_n - \bar{x}\| = 0$. This completes the proof.

Finally, we give a convergence theorem on problem 1.2 as a sub-result of Theorem 2.1.

**Corollary 2.2.** Let $H$ be a real Hilbert space and let $C$ be a nonempty closed and convex subset of $H$. Let $T_2 : C \to H$ a relaxed $(\eta_2, \rho_2)$-cocoercive and $\mu_2$-Lipschitz continuous mapping, respectively. Let $f : C \to C$ be a contractive mapping and let $S : C \to C$ be a nonexpansive mapping with a fixed point. Assume that $F(S) \cap VI(C, T_2) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner: $x_1 \in C$, chosen arbitrarily, $y_n = \delta_n Sx_n + (1 - \delta_n)PC(x_n - \lambda T_2 x_n)$, $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n$, $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $(0,1)$ satisfying the following restrictions: $\alpha_n + \beta_n + \gamma_n = 1$, $\forall n \geq 1$; $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$; $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha = \infty$; $\lim_{n \to \infty} \delta_n = \delta \in (0,1)$, and $\lambda$ is a constant such that $0 < \lambda < \frac{2(\rho_2 - \eta_2 \mu_2)}{\mu_2^2}$. Then the sequence $\{x_n\}$ converges strongly to an element $\bar{x} \in VI(C,T)$, which uniquely solves the following variational inequality
\[
\langle f(\bar{x}) - \bar{x}, \bar{x} - x^* \rangle \geq 0, \quad \forall x^* \in VI(C,T).
\]

**References**


