J. Nonlinear Funct. Anal. 2013 (2013), Article ID 5

Journal of Nonlinear Functional Analysis
Available online at http://jnfa.mathres.org

SOME RESULTS ON NONEXPANSIVE AND RELAXED COCOERCIVE MAPPINGS

YUAN QING

Department of Mathematics, Hangzhou Normal University, Hangzhou, China

Abstract. In this paper, we instigate a viscosity approximation method for nonexpansive and relaxed cocoercive mappings. A strong convergence theorem is established in the framework of Hilbert spaces.

Keywords. Relaxed cocoercive mapping; Variational inequality; Nonexpansive mapping; Fixed point.

1. Introduction and preliminaries

Throughout this paper, we assume that H is a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$. Let C be a nonempty closed and convex subset of H and let P_C denote the metric projection from H onto C.

Consider the following generalized variational inequality problem. Give nonlinear mappings $T_1 : C \to H$ and $T_2 : C \to H$, find an $u \in C$ such that

$$\langle u - T_1 u + \lambda T_2 u, v - u \rangle \geq 0, \quad \forall v \in C, \tag{1.1}$$

where λ is a constant. In this paper, we use $VI(T_1, T_2)$ to denote the set of solutions of variational inequality problem (1.1).

It is easy to see that an element $u \in C$ is a solution to problem (1.1) if and only if $u \in C$ is a fixed point of mapping $P_C(T_1 - \lambda T_2)$.

E-mail address: yuanqingbuaa@hotmail.com

Received January 17, 2013

If $T_1 = I$, the identity mapping, then problem (1.1) is reduced to the following. Find $u \in C$ such that

$$\langle \lambda T_2 u, v - u \rangle \geq 0, \quad \forall v \in C. \tag{1.2}$$

Variational inequality problem (1.2) was introduced by Stampacchia [1] in 1964. Problem (1.2) has emerged as a fascinating and interesting branch of mathematical and engineering sciences with a wide range of applications in industry, finance, economics, social, ecology, regional, pure and applied sciences. In this paper, we use $VI(C, T_2)$ to denote the set of solutions of variational inequality problem (1.2). Recently, gradient methods have been extensively investigated for solving problems (1.1) and (1.2); see [2-9] and the references therein.

Let $A : C \to H$ be a mapping. Recall that A is said to be monotone iff

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

A is said to be relaxed η-cocoercive if there exists a positive real number $\eta > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq (-\eta)\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

A is said to be ρ-strongly monotone if there exists a positive real number $\rho > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \rho\|x - y\|^2, \quad \forall x, y \in C.$$

A is said to be relaxed (η, ρ)-cocoercive if there exist positive real numbers $\eta, \rho > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq (-\eta)\|Ax - Ay\|^2 + \rho\|x - y\|^2, \quad \forall x, y \in C.$$

Let $S : C \to C$ be a mapping. Recall that S is said to be contractive if there exits a constant $0 \leq \alpha < 1$ such that

$$\|Sx - Sy\| \leq \alpha\|x - y\|, \quad \forall x, y \in C.$$

S is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

In this paper, we use $F(S)$ to denote the set of fixed points of the mapping S. It is know if C is weakly compact, then $F(S)$ is not empty.

Recently, many authors studied the problem of finding a common element of the set of solutions of variational inequality problem (1.2) and of the set of fixed points of a nonexpansive mapping; see, for example, [9-14] and the references therein.

In this paper, Motivated by the research work going on in this direction, we consider variational inequality problem (1.2) and a fixed point problem of nonexpansive mappings by viscosity iterative method. Strong convergence theorems of common elements are established in the framework of Hilbert spaces.

Lemma 1.1. [15] *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $S_1 : C \to C$ and $S_2 : C \to C$ be nonexpansive mappings on C. Suppose that $F(S_1) \cap F(S_2)$ is nonempty. Define a mapping $S : C \to C$ by*

$$Sx = aS_1x + (1-a)S_2x, \quad \forall x \in C.$$

Then S is nonexpansive with $F(S) = F(S_1) \cap F(S_2)$.

Lemma 1.2 [16] *Let C be a nonempty closed and convex subset of a real Hilbert space H and $S : C \to C$ a nonexpansive mapping. Then $I - S$ is demi-closed at zero.*

Lemma 1.3. [17] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Hilbert space H and let $\{\beta_n\}$ be a sequence in $(0,1)$ with*

$$0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \ge 0$ and

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.4. [18] *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(a) $\lim_{n\to\infty} \gamma_n = 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(b) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty}\alpha_n = 0$.

Now, we are in a position to show our main results.

2. Main results

Theorem 2.1. *Let H be a real Hilbert space and let C be a nonempty closed and convex subset of H. Let $T_1 : C \to H$ be a relaxed (η_1,ρ_1)-cocoercive and μ_1-Lipschitz continuous mapping and let $T_2 : C \to H$ a relaxed (η_2,ρ_2)-cocoercive and μ_2-Lipschitz continuous mapping, respectively. Let $f : C \to C$ be a contractive mapping and let $S : C \to C$ be a nonexpansive mapping with a fixed point. Assume that $F(S)\cap VI(T_1,T_2) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner: $x_1 \in C$, chosen arbitrarily, $y_n = \delta_n Sx_n + (1-\delta_n)P_C(T_1x_n - \lambda T_2x_n)$, $x_{n+1} = \alpha_n f(x_n)+\beta_n x_n+\gamma_n y_n, \quad n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $(0,1)$ satisfying the following restrictions: $\alpha_n+\beta_n+\gamma_n = 1, \quad \forall n \geq 1$; $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$; $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha = \infty$; $\lim_{n\to\infty}\delta_n = \delta \in (0,1)$, and λ is a constant such that $\sqrt{1-2\rho_1+\mu_1^2+2\eta_1\mu_1^2}+\sqrt{1-2\lambda\rho_2+\lambda^2\mu_2^2+2\lambda\eta_2\mu_2^2} \leq 1$. Then the sequence $\{x_n\}$ generated by the algorithm (Υ) converges strongly to a common element $\bar{x} \in F(S)\cap VI(T_1,T_2)$, which uniquely solves the following variational inequality*

$$\langle f(\bar{x})-\bar{x},\bar{x}-x^*\rangle \geq 0, \quad \forall x^* \in F(S)\cap VI(T_1,T_2).$$

Proof. The proof is split into 5 steps.

Step 1. Show that $\{x_n\}$ is bounded.

Note that $P_C(T_1-\lambda T_2)$ is nonexpansive. Indeed, for each $x,y \in C$, we have

$$\begin{aligned}
&\|P_C(T_1-\lambda T_2)x - P_C(T_1-\lambda T_2)y\| \\
&\leq \|(T_1-\lambda T_2)x-(T_1-\lambda T_2)y\| \qquad (2.1)\\
&\leq \|(x-y)-(T_1x-T_1y)\|+\|(x-y)-\lambda(T_2x-T_2y)\|.
\end{aligned}$$

It follows from the assumption that T_1 is relaxed (η_1,ρ_1)-cocoercive and μ_1-Lipschitz continuous that

$$\begin{aligned}\|x-y-(T_1x-T_1y)\|^2 &= \|x-y\|^2-2\langle T_1x-T_1y,x-y\rangle+\|T_1x-T_1y\|^2\\ &\leq \|x-y\|^2-2[(-\eta_1)\|T_1x-T_1y\|^2+\rho_1\|x-y\|^2]+\mu_1^2\|x-y\|^2\\ &= (1-2\rho_1+\mu_1^2)\|x-y\|^2+2\eta_1\|T_1x-T_1y\|^2\\ &\leq \theta_1^2\|x-y\|^2,\end{aligned}$$

where $\theta_1=\sqrt{1-2\rho_1+\mu_1^2+2\eta_1\mu_1^2}$. That is,

$$\|x-y-(T_1x-T_1y)\|\leq \theta_1\|x-y\|. \tag{2.2}$$

On the other hand, by the the assumption that T_2 is relaxed (η_2,ρ_2)-cocoercive and μ_2-Lipschitz continuous, we arrive at

$$\begin{aligned}&\|x-y-\lambda(T_2x-T_2y)\|^2\\ &= \|x-y\|^2-2\lambda\langle T_2x-T_2y,x-y\rangle+\lambda^2\|T_2x-T_2y\|^2\\ &\leq \|x-y\|^2-2\lambda[(-\eta_2)\|T_2x-T_2y\|^2+\rho_2\|x-y\|^2]+\lambda^2\mu_2^2\|x-y\|^2\\ &= (1-2\lambda\rho_2+\lambda^2\mu_2^2)\|x-y\|^2+2\lambda\eta_2\|T_3x-T_3y\|^2\\ &\leq \theta_2^2\|x-y\|^2,\end{aligned}$$

where $\theta_2=\sqrt{1-2\lambda\rho_2+\lambda^2\mu_2^2+2\lambda\eta_2\mu_2^2}$. This implies that

$$\|x-y-\lambda(T_2x-T_2y)\|\leq \theta_2\|x-y\|. \tag{2.3}$$

From the assumption (e), we see that $\theta_1+\theta_2\leq 1$. Substituting (2.2) and (2.3) into (2.1), we see that

$$\|P_C(T_1-\lambda T_2)x-P_C(T_1-\lambda T_2)y\|\leq\|x-y\|.$$

This shows that $P_C(T_1-\lambda T_2)$ is nonexpansive. Fixing $p\in VI(T_1,T_2)\cap F(S)$, we see that $p=P_C(T_1-\lambda T_2)p$ and $p=Sp$. Put $z_n=P_C(T_1x_n-\lambda T_2x_n)$. It follows that

$$\begin{aligned}\|y_n-p\| &\leq \delta_n\|Sx_n-Sp\|+(1-\delta_n)\|P_C(T_1x_n-\lambda T_2x_n)-P_C(T_1p-\lambda T_2p)\|\\ &\leq \|x_n-p\|.\end{aligned} \tag{2.4}$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\| &\le \alpha_n\|f(x_n)-p\|+\beta_n\|x_n-p\|+\gamma_n\|Sy_n-p\|\\ &\le \alpha_n\|f(p)-p\|+\alpha_n\alpha\|x_n-p\|+\beta_n\|x_n-p\|+\gamma_n\|y_n-p\|\\ &\le \alpha_n(1-\alpha)\frac{\|f(p)-p\|}{1-\alpha}+(1-\alpha_n(1-\alpha))\|x_n-p\|.\end{aligned}$$

By mathematical inductions, we see that $\{x_n\}$ is bounded.

Step 2. Show that $x_{n+1}-x_n\to 0$ as $n\to\infty$.

Since the mapping $P_C(T_1-\lambda T_2)$ is nonexpansive, we see that

$$\|z_{n+1}-z_n\|=\|P_C(T_1-\lambda T_2)x_{n+1}-P_C(T_1-\lambda T_2)x_n\|\le\|x_{n+1}-x_n\|. \tag{2.5}$$

It follows that

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|\delta_{n+1}Sx_{n+1}+(1-\delta_{n+1})z_{n+1}-\delta_nSx_n-(1-\delta_n)z_n\|\\ &\le \delta_{n+1}\|Sx_{n+1}-Sx_n\|+(1-\delta_{n+1})\|z_{n+1}-z_n\|+|\delta_{n+1}-\delta_n|\|Sx_n-z_n\|\\ &\le \delta_{n+1}\|x_{n+1}-x_n\|+(1-\delta_{n+1})\|x_{n+1}-x_n\|+|\delta_{n+1}-\delta_n|\|Sx_n-z_n\|\\ &\le \|x_{n+1}-x_n\|+|\delta_{n+1}-\delta_n|M,\end{aligned} \tag{2.6}$$

where M is an appropriate constant such that $M\ge\sup_{n\ge1}\{\|Sx_n-z_n\|\}$. Put $l_n=\frac{x_{n+1}-\beta_nx_n}{1-\beta_n}$, for all $n\ge1$. That is,

$$x_{n+1}=(1-\beta_n)l_n+\beta_nx_n,\quad \forall n\ge1. \tag{2.7}$$

Now, we estimate $\|l_{n+1}-l_n\|$. From

$$\begin{aligned}l_{n+1}-l_n &= \frac{\alpha_{n+1}f(x_{n+1})+\gamma_{n+1}y_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_nf(x_n)+\gamma_ny_n}{1-\beta_n}\\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}f(x_{n+1})+\frac{1-\beta_{n+1}-\alpha_{n+1}}{1-\beta_{n+1}}y_{n+1}-\frac{\alpha_n}{1-\beta_n}f(x_n)-\frac{1-\beta_n-\alpha_n}{1-\beta_n}y_n\\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(f(x_{n+1})-y_{n+1})+\frac{\alpha_n}{1-\beta_n}(y_n-f(x_n))+y_{n+1}-y_n,\end{aligned}$$

we have

$$\|l_{n+1}-l_n\|\le\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-y_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|y_n-f(x_n)\|+\|y_{n+1}-y_n\|. \tag{2.8}$$

From (2.6), we obtain

$$\|l_{n+1}-l_n\|-\|x_{n+1}-x_n\|\le\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|f(x_{n+1})-Sy_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|Sy_n-f(x_n)\|+|\delta_{n+1}-\delta_n|M.$$

It follows that $\lim_{n\to\infty}\|l_{n+1}-l_n\|-\|x_{n+1}-x_{n+1}\|<0$. Using Lemma 1.3, we have

$$\lim_{n\to\infty}\|l_n-x_n\|=0. \tag{2.9}$$

By (2.7), we see that $x_{n+1}-x_n=(1-\beta_n)(l_n-x_n)$. Therefore, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{2.10}$$

Note that

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{2.11}$$

Define a mapping $R:C\to C$ by

$$Rx=\delta Sx+(1-\delta)P_C(T_1-\lambda T_2)x,\quad \forall x\in C,$$

where $\delta=\lim_{n\to\infty}\delta_n$. From Lemma 1.1, we see that R is nonexpansive with $F(R)=F(P_C(T_1-\lambda T_2))\cap F(S)=VI(T_1,T_2)\cap F(S)$.

Step 3. Show that $Rx_n-x_n\to 0$ as $n\to\infty$. Note that

$$\begin{aligned}\|Rx_n-x_n\|&\le\|Rx_n-y_n\|+\|y_n-x_n\|\\&\le\|\delta Sx_n+(1-\delta)P_C(T_1-\lambda T_2)x_n-\delta_nSx_n-(1-\delta_n)P_C(T_1-\lambda T_2)x_n\|+\|y_n-x_n\|\\&\le|\delta-\delta_n|M+\|y_n-x_n\|.\end{aligned}$$

Using (2.11), we obtain

$$\lim_{n\to\infty}\|Rx_n-x_n\|=0. \tag{2.12}$$

Step 4. Show that $\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x},x_n-\bar{x}\rangle\le 0$.

To show it, we can choose a sequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x},x_n-\bar{x}\rangle=\lim_{i\to\infty}\langle f(\bar{x})-\bar{x},x_{n_i}-\bar{x}\rangle. \tag{2.13}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to b. Without loss of generality, we may assume that $x_{n_i}\rightharpoonup b$. From Lemma 1.2, we see that

$$b\in F(R)=F(P_C(T_1-\lambda T_2))\cap F(S)=VI(T_1,T_2)\cap F(S).$$

It follows from (2.13) that

$$\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x},x_n-\bar{x}\rangle=\lim_{i\to\infty}\langle f(\bar{x})-\bar{x},x_{n_i}-\bar{x}\rangle=\langle f(\bar{x})-\bar{x},b-\bar{x}\rangle\le 0. \tag{2.14}$$

Step 5. Show that $x_n \to \bar{x}$ as $n \to \infty$. Note that

$$\begin{aligned}\|x_{n+1}-\bar{x}\|^2 &= \alpha_n\langle f(x_n)-\bar{x}, x_{n+1}-\bar{x}\rangle + \beta_n\langle x_n-\bar{x}, x_{n+1}-\bar{x}\rangle + \gamma_n\langle y_n-\bar{x}, x_{n+1}-\bar{x}\rangle \\ &\le \alpha_n\langle f(x_n)-\bar{x}, x_{n+1}-\bar{x}\rangle + \beta_n\|x_n-\bar{x}\|\|x_{n+1}-\bar{x}\| + \gamma_n\|y_n-\bar{x}\|\|x_{n+1}-\bar{x}\| \\ &\le \alpha_n\langle f(\bar{x})-\bar{x}, x_{n+1}-\bar{x}\rangle + \big(1-\alpha_n(1-\alpha)\big)\|x_n-\bar{x}\|\|x_{n+1}-\bar{x}\|.\end{aligned}$$

It follows that

$$\|x_{n+1}-\bar{x}\|^2 \le 2\alpha_n\langle f(\bar{x})-\bar{x}, x_{n+1}-\bar{x}\rangle + \big(1-\alpha_n(1-\alpha)\big)\|x_n-\bar{x}\|^2.$$

From Lemma 1.4, we have $\lim_{n\to\infty}\|x_n-\bar{x}\| = 0$. This completes the proof.

Finally, we give a convergence theorem on problem 1.2 as a sub-result of Theorem 2.1.

Corollary 2.2. *Let H be a real Hilbert space and let C be a nonempty closed and convex subset of H. Let $T_2 : C \to H$ a relaxed (η_2, ρ_2)-cocoercive and μ_2-Lipschitz continuous mapping, respectively. Let $f : C \to C$ be a contractive mapping and let $S : C \to C$ be a nonexpansive mapping with a fixed point. Assume that $F(S) \cap VI(C, T_2) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner: $x_1 \in C$, chosen arbitrarily, $y_n = \delta_n Sx_n + (1-\delta_n)P_C(x_n - \lambda T_2 x_n)$, $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \quad n \ge 1$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are sequences in $(0,1)$ satisfying the following restrictions: $\alpha_n + \beta_n + \gamma_n = 1, \quad \forall n \ge 1$; $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$; $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha = \infty$; $\lim_{n\to\infty}\delta_n = \delta \in (0,1)$, and λ is a constant such that $0 < \lambda < \frac{2(\rho-\eta\mu^2)}{\mu^2}$. Then the sequence $\{x_n\}$ converges strongly to a element $\bar{x} \in VI(C, T)$, which uniquely solves the following variational inequality*

$$\langle f(\bar{x})-\bar{x}, \bar{x}-x^*\rangle \ge 0, \quad \forall x^* \in VI(C, T).$$

REFERENCES

[1] G. Stampacchia, Formes bilineaires coercivities sur les ensembles convexes, C.R. Acad. Sci. Paris 258 (1964), 4413-4416.

[2] M.A. Noor, Extended general variational inequalities, Appl. Math. Lett. 22 (2009), 182-186.

[3] M.A. Noor, Iterative methods for generalized variational inequality, Appl. Math. Lett. 15 (2002), 77-82.

[4] M.A. Noor, Predictor-corrector algorithm for general variational inequalities, Appl. Math. Lett. 14 (2001), 53-58.

[5] H. Zegeye, N. Shahzad, Strong convergence theorem for a common point of solution of variational inequality and fixed point problem, Adv. Fixed Point Theory, 2 (2012), 374-397.

[6] J. Shen, L.P. Pang, An approximate bundle method for solving variational inequalities, Commn. Optim. Theory, 1 (2012), 1-18.

[7] R.U. Verma, Generalized variational inequalities involving multivalued relaxed monotone operators, Appl. Math. Lett. 10 (1997) 107-109.

[8] Y.J. Cho, X. Qin, M. Shang, Y. Su, Generalized nonlinear variational inclusions involving (A,η)-monotone mappings in Hilbert Spaces, J. Inequal. Appl. 2007 (2007), Article ID 29653.

[9] L.C. Ceng, J.C. Yao, An extragradient-like approximation method for variational inequality problems and fixed point problems, Appl. Math. Comput. 190 (2007), 205-215.

[10] H. Iiduka, W. Takahashi, Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings, Nonlinear Anal. 61 (2005), 341-350.

[11] W. Takahashi, M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings J. Optim. Theory Appl. 118 (2003) 417-428.

[12] Y.J. Cho, X. Qin, Systems of generalized nonlinear variational inequalities and its projection methods, Nonlinear Anal. 69 (2008), 4443-4451.

[13] Y. Yao, J.C. Yao, On modified iterative method for nonexpansive mappings and monotone mappings, Appl. Math. Comput. 186 (2007), 1551-1558.

[14] X. Qin, M. Shang. H. Zhou, Strong convergence of a general iterative method for variational inequality problems and fixed point problems in Hilbert spaces, Appl. Math. Comput. 200 (2008), 242-253.

[15] R.E. Bruck, Properties of fixed point sets of nonexpansive mappings in Banach spaces, Tras. Amer. Math. Soc. 179 (1973) 251-262.

[16] F.E. Browder, Nonlinear operators and nonlinear equations of evolution in Banach spaces, Proc. Symp. Pure. Math. 18 (1976) 78-81.

[17] T. Suzuki, Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochne integrals, J. Math. Anal. Appl. 305 (2005) 227-239.

[18] H.K. Xu, Iterative algorithms for nonlinear operators, J. London Math. Soc. 66 (2002) 240-256.