SOME PROPERTIES FOR A NEW CERTAIN SUBCLASS OF STARLIKE FUNCTIONS

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Abstract. In this paper, we obtained some properties for class of functions related to the class of starlike functions using a linear multiplier operator \(D^{\alpha,q,s}_{\lambda,\ell} f(z)\) \((n \in \mathbb{N}_0, \ \lambda \geq 0, \ \ell \geq 0)\), such as partial sums, integral means, square root and integral transform for these class are discussed.

Keywords. Analytic functions; Starlike functions; Convex functions; Partial sums; Integral means.

1. Introduction

Let \(\mathcal{A}\) denote the class of functions of the form
\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k,
\]
which are analytic in the open unit disc \(U = \{z : |z| < 1\}\). Further, by \(S\) we shall denote the class of all functions in \(\mathcal{A}\) which are univalent and normalized by \(f(0) = 0 = f'(0) - 1\). We denote by \(S^*(\alpha)\) and \(K(\alpha)\) the subclasses of \(S\) consisting of all functions which are, respectively, starlike and convex of order \(\alpha\) \((0 \leq \alpha < 1)\). Thus,
\[
S^*(\alpha) = \left\{ f \in S : \Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha \ (0 \leq \alpha < 1; z \in U) \right\}
\]
and
\[
K(\alpha) = \left\{ f \in S : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \ (0 \leq \alpha < 1; z \in U) \right\}.
\]

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The classes $S^*(\alpha)$ and $K(\alpha)$ were introduced by Robertson [21]. From (1.2) and (1.3) it follows that

\begin{equation}
(1.4)
\end{equation}

\[
f(z) \in K(\alpha) \leftrightarrow zf'(z) \in S^*(\alpha).
\]

We note that:

\[
S^*(0) = S^*,\; K(0) = K.
\]

Let $f \in \mathcal{A}$ be given by (1.1) and $g \in \mathcal{A}$ given by

\begin{equation}
(1.5)
g(z) = z + \sum_{k=2}^{\infty} b_k z^k \quad (b_k > 0).
\end{equation}

We define the Hadamard product (or convolution) of $f$ and $g$ is defined by

\begin{equation}
(1.6)
(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).
\end{equation}

Also denote by $T$ the subclass of $S$ consisting of functions of the form

\begin{equation}
(1.7)
f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0; z \in U).
\end{equation}

For positive real values of $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ ($\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, \ldots\}; j = 1, 2, \ldots, s$), we now define the generalized hypergeometric function $\eta F_1(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ by (see, for example, [32])

\begin{equation}
(1.8)
\eta F_1(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_q)_k}{(\beta_1)_k \cdots (\beta_s)_k} \frac{z^k}{k!}
\end{equation}

\[ (q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \ldots\}; z \in U), \]

where $(a)_m$ is the Pochhammer symbol defined by

\begin{equation}
(1.9)
(a)_m = \frac{\Gamma(a + m)}{\Gamma(a)} = \begin{cases} 1 & (m = 0), \\ a(a + 1) \cdots (a + m - 1) & (m \in \mathbb{N}). \end{cases}
\end{equation}

Corresponding to the function $h(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ defined by

\begin{equation}
(1.10)
h(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = z \eta F_1(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z),
\end{equation}

we consider a linear operator $H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) : \mathcal{A} \rightarrow \mathcal{A}$ which is defined by the following Hadamard product (or convolution):

\begin{equation}
H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s)f(z) = h(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) * f(z).
\end{equation}

We observe that for function $f(z)$ of the form (1.1) we have

\begin{equation}
H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s)f(z) = z + \sum_{k=2}^{\infty} \Gamma_k(\alpha_1) a_k z^k,
\end{equation}

where

\begin{equation}
\Gamma_k(\alpha_1) = \frac{(\alpha_1)_{k-1} \cdots (\alpha_q)_{k-1}}{(\beta_1)_{k-1} \cdots (\beta_s)_{k-1}} \frac{1}{(k-1)!} \quad (k \geq 2).
\end{equation}
For convenience, we write

\[(1.14) \quad H_{q,s}[\alpha_1] = H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s).\]

The linear operator \(H_{q,s}[\alpha_1]\) was introduced and studied by Dziok and Srivastava [8]. This operator contains in turn many interesting operators such as the Hohlov operator [11], Carlson-Shaffer operator [6], Ruscheweyh derivative operator [22], and the Srivastava-Owa fractional derivative operator [19].

EL-Ashwah et al. [9] were defined the linear multiplier Dziok-Srivastava operator \(D_{\lambda,\ell}^{n,\alpha,\gamma} f(z)\) as follows:

\[(1.15) \quad D_{\lambda,\ell}^{n,\alpha,\gamma} f(z) = z + \sum_{k=2}^{\infty} \Phi_{k,n}(\alpha_1, \lambda, \ell) a_k z^k \quad (n \in \mathbb{N}),\]

where

\[(1.16) \quad \Phi_{k,n}(\alpha_1, \lambda, \ell) = \left[\frac{\ell + 1 + \lambda (k - 1)}{\ell + 1} \Gamma_k(\alpha_1)\right]^n.\]

The operator \(D_{\lambda,\ell}^{n,\alpha,\gamma} f(z)\), can be written in terms of convolution as

\[(1.17) \quad D_{\lambda,\ell}^{n,\alpha,\gamma} f(z) = [(h(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) \ast g_{\lambda,\ell}(z)) \ast \ldots \ast (h(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) \ast g_{\lambda,\ell}(z))] \ast f(z),\]

\[n - \text{times}\]

where

\[g_{\lambda,\ell}(z) = \frac{(\ell + 1)z - (\ell + 1 - \lambda)z^2}{(\ell + 1)(1 - z)^2} = \frac{z - (1 - \lambda/(\ell + 1))z^2}{(1 - z)^2}.\]

By specializing the parameters \(q, s, \alpha_1, \beta_1, \lambda \) and \(\ell\), we obtain the following operators studied by various authors:

(i) For \(q = 2, s = 1, \alpha_1 = 2, \alpha_2 = 1, \beta_1 = 2 - \alpha (\alpha \neq 2, \ldots) \) and \(\ell = 0\), we have \(D_{\lambda,0}^{n,1,1} f(z) = D_{\lambda}^{n,1} f(z)\) (see Al-Oboudi and Al-Amoudi [2] and Aouf and Mostafa [3]);

(ii) For \(q = 2, s = 1, \alpha_1 = a (a > 0), \alpha_2 = 1, \beta_1 = c (c > 0) \) and \(\ell = 0\), we have \(D_{\lambda,0}^{n,1,2} f(z) = I_{a,c,\lambda} f(z)\) (see Prajapat and Raina [20]);

(iii) For \(q = 2, s = 1 \) and \(\alpha_1 = \alpha_2 = \beta_1 = 1\), we have \(D_{\lambda,0}^{n,1,1} f(z) = F(\lambda, \ell) f(z)\) (see Catas [7]);

(iv) For \(q = 2, s = 1, \alpha_1 = \alpha_2 = \beta_1 = 1 \) and \(\ell = 0\), we have \(D_{\lambda,0}^{n,1,2} f(z) = D_{\lambda}^{n} f(z)\) (see Al-Oboudi [1]) and \(D_{0,0}^{n,1,2} f(z) = D_{0}^{n} f(z)\) (see Salagean [23]);

(v) \(D_{0,0}^{1,1,1} f(z) = H_{q,s}[\alpha_1]\) (see Dziok and Srivastava [8]).

For \(0 \leq \delta \leq 1, 0 < \beta \leq 1, -1 \leq A < B \leq 1, 0 \leq \gamma \leq 1\), we let the class \(TF_{\lambda,\ell}^{n,\alpha,\gamma,\delta}(\alpha, \beta; \gamma, A, B)\) denote the subclass of \(T\) consisting of functions of the form (1.7) and satisfying the condition:

\[(1.18) \quad \left|\frac{z^\gamma \left(\frac{z^\delta}{F(z)} - 1\right)}{(B - A) \gamma \left(\frac{z^\delta}{F(z)} - \alpha\right) - B \left(\frac{z^\delta}{F(z)} - 1\right)}\right| < \beta\]

\[0 \leq \alpha < 1; 0 < \beta \leq 1; z \in U),\]
where

\[
\frac{zF_\delta'(z)}{F_\delta(z)} = \frac{z(D_{n,\delta}^{\alpha,\beta} f(z))' + \delta z^2(D_{n,\delta}^{\alpha,\beta} f(z))''}{(1 - \delta)D_{n,\delta}^{\alpha,\beta} f(z) + \delta z(D_{n,\delta}^{\alpha,\beta} f(z))'} \quad (0 \leq \delta \leq 1).
\]

We note that:

(i) For \( n = 1, \lambda = 0 \) and \( \ell = 0 \), we have \( TF_{0,0}^{1,q,s,\delta}(\alpha, \beta, \gamma, A, B) = HF_\delta(\alpha, \beta, A, B) \) (see Murugusundaramoorthy et al. [17], also see Vijaya and Deepa [33]);

(ii) For \( q = 2, s = 1, \alpha_1 = \alpha_2 = \beta_1 = \gamma = B = \beta_1 = 1, \lambda = \ell = 0 \) and \( A = -1 \), we have \( TF_{0,0}^{0,2,1,0}(\alpha, \beta, 1, -1, 1) = Q(j, \delta, \alpha, n) \) (see Aouf and Silverman [4], with \( j = 1 \));

(iii) For \( q = 2, s = 1, \alpha_1 = \alpha_2 = \beta_1 = \gamma = B = 1, \lambda = \delta = \ell = 0 \) and \( A = -1 \), we have \( TF_{0,0}^{n,2,1,0}(\alpha, \beta, 1, -1, 1) = S_\delta(\alpha, \beta) \) (see Salagean [25] and [26]);

(iv) For \( q = 2, s = 1, \alpha_1 = \alpha_2 = \beta_1 = \gamma = B = 1, n = \lambda = \delta = 0, \ell = 0 \) and \( A = -1 \), we have \( TF_{0,0}^{0,2,1,0}(\alpha, \beta, 1, -1, 1) = S^*(\alpha, \beta) \) (see Gupta and Jain [10], also see Salagean [24]);

(v) For \( q = 2, s = 1, \alpha_1 = \alpha_2 = \beta_1 = \gamma = B = 1, n = \lambda = \ell = 0 \) and \( A = -1 \), we have \( TF_{0,0}^{0,2,1,0}(\alpha, 1, -1, 1) = P(n, \delta, \alpha) \) (see Kim et al. [12], with \( n = 1 \));

(vi) For \( q = 2, s = 1, \alpha_1 = \alpha_2 = \beta_1 = \gamma = B = 1, n = \lambda = \ell = 0 \) and \( A = -1 \), we have \( T^{n,2,1,0}(\alpha, 1, 1, -1, 1) = T^*(\alpha) \) and \( TF_{0,0}^{1,q,s,\delta}(\alpha, 1, 1, -1, 1) = \mathcal{C}(\alpha) \) (see Silverman [27]).

(vii) For \( q = 2, s = 1, \alpha_1 = \alpha_2 = \beta_1 = \gamma = B = \beta_1 = 1, \lambda = \delta = \ell = 0 \) and \( A = -1 \), we have \( TF_{0,0}^{n,2,1,0}(\alpha, \beta, -1, 1, 1) = S_\delta(\alpha, \beta) \) (see Salagean [23]);

Also, we note that:

(i) For \( n = 0 \), we have

\[
TF_{\lambda,\ell}^{0,q,s}(\alpha, \beta, \gamma, A, B) = \left\{ f \in T : \left| \frac{zF_\delta'(z)}{F_\delta(z)} \right| < \beta \right\},
\]

where

\[
\frac{zF_\delta(z)}{F_\delta(z)} = \frac{zf'(z) + \delta z^2 f''(z)}{(1 - \delta)f(z) + \delta zf'(z)} \quad (0 \leq \delta \leq 1);
\]

(ii) For \( q = 2, s = 1, \alpha_1 = 2, \alpha_2 = 1, \beta_1 = 2 - \alpha \) \((\alpha \neq 2, 3, \ldots)\) and \( \ell = 0 \), we have

\[
TF_{\lambda,0}^{n,2,1}(\alpha, \beta, \gamma, A, B) = \left\{ f \in T : \left| \frac{zF_\delta'(z)}{F_\delta(z)} \right| < \beta \right\},
\]

where

\[
\frac{zF_\delta(z)}{F_\delta(z)} = \frac{zd\alpha f(z))' + \delta z^2(D^{n,\alpha}_\delta f(z))''}{(1 - \delta)D^{n,\alpha}_\delta f(z) + \delta z(D^{n,\alpha}_\delta f(z))'} \quad (0 \leq \delta \leq 1);
\]
In this paper, we obtain partial sums, square root, integral means and integral transform for the class $w$.

(iii) For $q = 2$, $s = 1$, $\alpha_1 = a$ ($a > 0$), $\alpha_2 = 1$, $\beta_1 = c$ ($c > 0$) and $\ell = 0$, we have

$$TF_{\lambda, 0}^{n, 2, 1}(\alpha, \beta, \gamma, A, B)$$

$$= IF_{a, c, \lambda}^{n}(\alpha, \beta, \gamma, A, B) = \left\{ f \in T : \left| \frac{z F_{\delta}^{\ell}(z)}{F_{\delta}(z)} \right| < \beta \ (z \in U) \right\},$$

where

$$\frac{z F_{\delta}^{\ell}(z)}{F_{\delta}(z)} = \frac{z f^\prime(z)^{n}(\lambda, \ell, f(z))\prime + \delta z^{2}(\lambda, \ell, f(z))''}{(1 - \delta)\lambda f(z) + \delta z f(z)^{\prime}} \quad (0 \leq \delta \leq 1);$$

(iv) For $q = 2$, $s = 1$ and $\alpha_1 = \alpha_2 = \beta_1 = 1$, we have

$$TF_{\lambda, \ell}^{n, 2, 1}(\alpha, \beta, \gamma, A, B)$$

$$= IF_{\lambda, \ell}^{n}(\alpha, \beta, \gamma, A, B) = \left\{ f \in T : \left| \frac{z F_{\delta}^{\ell}(z)}{F_{\delta}(z)} \right| < \beta \ (z \in U) \right\},$$

where

$$\frac{z F_{\delta}^{\ell}(z)}{F_{\delta}(z)} = \frac{z f^\prime(z)^{n}(\lambda, \ell, f(z))\prime + \delta z^{2}(\lambda, \ell, f(z))''}{(1 - \delta)\lambda f(z) + \delta z f(z)^{\prime}} \quad (0 \leq \delta \leq 1);$$

(v) For $q = 2$, $s = 1$, $\alpha_1 = \alpha_2 = \beta_1 = 1$ and $\ell = 0$, we have

$$TF_{\lambda, 0}^{n, 2, 1}(\alpha, \beta, \gamma, A, B)$$

$$= TF_{\lambda, 0}^{n}(\alpha, \beta, \gamma, A, B) = \left\{ f \in T : \left| \frac{z F_{\delta}^{\ell}(z)}{F_{\delta}(z)} \right| < \beta \ (z \in U) \right\},$$

where

$$\frac{z F_{\delta}^{\ell}(z)}{F_{\delta}(z)} = \frac{z f^\prime(z)^{n}(\lambda, f(z))\prime + \delta z^{2}(\lambda, f(z))''}{(1 - \delta)\lambda f(z) + \delta z f(z)^{\prime}} \quad (0 \leq \delta \leq 1);$$

(vi) For $q = 2$, $s = 1$, $\alpha_1 = \alpha_2 = \beta_1 = 1$, $\lambda = 0$ and $\ell = 0$, we have

$$TF_{0, 0}^{n, 2, 1}(\alpha, \beta, \gamma, A, B)$$

$$= TF_{0, 0}^{n}(\alpha, \beta, \gamma, A, B) = \left\{ f \in T : \left| \frac{z F_{\delta}^{\ell}(z)}{F_{\delta}(z)} \right| < \beta \ (z \in U) \right\},$$

where

$$\frac{z F_{\delta}^{\ell}(z)}{F_{\delta}(z)} = \frac{z f^\prime(z)^{n}(f(z))\prime + \delta z^{2}(f(z))''}{(1 - \delta)\lambda f(z) + \delta z f(z)^{\prime}} \quad (0 \leq \delta \leq 1).$$

**Definition 1** [13]. (Subordination) For analytic functions $f$ and $g$ with

$f(0) = g(0)$, $f$ is said to be subordinate to $g$, denote by $f \prec g$, if there exists an analytic function $w$ such that

$w(0) = 0$, $|w(z)| < 1$ and $f(z) = g(w(z))$ ($z \in U$). If $g(z)$ is univalent function, then $f \prec g$, if and only if

$f(0) = g(0)$ and $f(U) \subset g(U)$.

In this paper, we obtain partial sums, square root, integral means and integral transform for the class $TF_{\lambda, \ell}^{n, 2, 1}(\alpha, \beta, \gamma, A, B)$. 
2. Coefficient estimates

Unless otherwise mentioned, we assume in the reminder of this paper that $0 \leq \delta \leq 1$, $-1 \leq A < B \leq 1$, $q \leq s + 1$, $q, s \in \mathbb{N}_0$, $\lambda \geq 0$, $\ell \geq 0$, $n \in \mathbb{N}_0$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 < \gamma \leq 1$ and $z \in U$.

**Theorem 1.** Let the function $f(z)$ be defined by (1.7). Then $f(z)$ is in the class $TF^{n,q,s}_{\alpha,\beta}(\alpha, \beta, \gamma, A, B)$ if and only if

$$
\sum_{k=2}^{\infty} \psi^A_k(k, \delta, \alpha, \beta, \gamma) a_k \leq 1,
$$

where

$$
\psi^A_k(k, \delta, \alpha, \beta, \gamma) = \frac{(1+k\delta-\delta)(1-\alpha B)(k-1)+(B-A)\beta \gamma (k-1)}{(B-A)\beta \gamma (1-\alpha)} \Phi_{k,n}(\alpha_1, \lambda, \ell)
$$

and $\Phi_{k,n}(\alpha_1, \lambda, \ell)$ is given by (1.16).

**Proof.** Assume that the inequality (2.1) holds true, we find from (1.7) and (1.22) that

\[
\left| zF'_{\delta}(z) - F_{\delta}(z) \right| - \beta \left| (B-A) \gamma \left( zF'_{\delta}(z) - \alpha F_{\delta}(z) \right) - B \left( zF'_{\delta}(z) - F_{\delta}(z) \right) \right|
\]

\[
= \sum_{k=2}^{\infty} (1+k\delta-\delta)(k-1) \Phi_{k,n}(\alpha_1, \lambda, \ell) a_k z^k - \beta \left| (B-A) \gamma (1-\alpha) \right| z
\]

\[
+ \sum_{k=2}^{\infty} \left| [(1+k\delta-\delta)(B-A)\gamma (k-\alpha) - B(k-1)] \Phi_{k,n}(\alpha_1, \lambda, \ell) a_k z^k \right|
\]

\[
\leq \sum_{k=2}^{\infty} (1+k\delta-\delta)(k-1) \Phi_{k,n}(\alpha_1, \lambda, \ell) a_k z^k - (B-A) \beta \gamma (1-\alpha) r
\]

\[
+ \sum_{k=2}^{\infty} \beta \left| [(1+k\delta-\delta)(B-A)\gamma (k-\alpha) - B(k-1)] \Phi_{k,n}(\alpha_1, \lambda, \ell) a_k z^k \right|
\]

\[
\leq \sum_{k=2}^{\infty} (1+k\delta-\delta)[(1-\beta B)(k-1) + (B-A)\beta \gamma (k-\alpha)] \Phi_{k,n}(\alpha_1, \lambda, \ell) a_k
\]

\[- (B-A) \beta \gamma (1-\alpha) \leq 0 \ (z \in U).
\]

Hence, by the maximum modulus theorem, we have $f(z) \in TF^{n,q,s}_{\alpha,\beta}(\alpha, \beta, \gamma, A, B)$.

Conversely, Let

\[
\left| \frac{zF'_{\delta}(z)}{F_{\delta}(z)} - 1 \right| - \beta \left| (B-A) \gamma \left( \frac{zF'_{\delta}(z)}{F_{\delta}(z)} - \alpha \right) - B \left( \frac{zF'_{\delta}(z)}{F_{\delta}(z)} - 1 \right) \right|
\]

\[
= \left| \sum_{k=2}^{\infty} (1+k\delta-\delta)(k-1) \Phi_{k,n}(\alpha_1, \lambda, \ell) a_k z^k \right|
\]

\[
\frac{(B-A)\gamma (1-\alpha) z - \sum_{k=2}^{\infty} (1+k\delta-\delta)(k-1) \Phi_{k,n}(\alpha_1, \lambda, \ell) a_k z^k}{(B-A)\gamma (1-\alpha) z - \sum_{k=2}^{\infty} (1+k\delta-\delta)(k-1) \Phi_{k,n}(\alpha_1, \lambda, \ell) a_k z^k} \leq \beta \ (z \in U).
\]

Now since $\Re \{z\} \leq |z|$ for all $z$, we have

$$
\Re \left\{ \frac{\sum_{k=2}^{\infty} (1+k\delta-\delta)(k-1) \Phi_{k,n}(\alpha_1, \lambda, \ell) a_k z^{k-1}}{(B-A)\gamma (1-\alpha) z - \sum_{k=2}^{\infty} (1+k\delta-\delta)(k-1) \Phi_{k,n}(\alpha_1, \lambda, \ell) a_k z^k} \right\} < \beta.
$$
Choos values of \( z \) on the real axis so that \( f'(z) \) is real. Then upon clearing the denominator in (2.3) and letting \( z \to 1 \), through real values, we have
\[
\sum_{k=2}^{\infty} (1 + k\delta - \delta)[(1 - \beta B)(k - 1) + (B - A)\beta\gamma(k - \alpha)]\Phi_{k,n}(\alpha_1, \lambda, \ell, a_k) - (B - A)\beta\gamma(1 - \alpha) \leq 0.
\]
This completes the proof of Theorem 1.

3. Partial sums

Following the earlier works by Silverman [30] and Silva [27] on partial sums of analytic functions, we consider in this section partial sums of functions in the class \( TF_{\lambda, \ell}^{n,q,s,\delta}(\alpha, \beta, \gamma, A, B) \) and obtain sharp lower bounds for the ratios of real part of \( f(z) \) to \( f_m(z) \), \( f_m(z) \) to \( f(z) \), \( f'(z) \) to \( f_m'(z) \) and \( f_m'(z) \) to \( f'(z) \), respectively.

**Theorem 2.** Let the function \( f(z) \) of the form (1.7) be in the class \( TF_{\lambda, \ell}^{n,q,s,\delta}(\alpha, \beta, \gamma, A, B) \). Define the partial sums \( f_1(z) \) and \( f_m(z) \), by
\[
(3.1) \quad f_1(z) = z \text{ and } f_m(z) = z - \sum_{k=2}^{m} a_k z^k \quad (m \in \mathbb{N}/\{1\}).
\]
Then
\[
(3.2) \quad \Re \left\{ \frac{f(z)}{f_m(z)} \right\} > \frac{\psi_B^A(m + 1, \delta, \alpha, \beta, \gamma) - 1}{\psi_B^A(m + 1, \delta, \alpha, \beta, \gamma)} \quad (z \in U; n \in \mathbb{N}),
\]
and
\[
(3.3) \quad \Re \left\{ \frac{f_m(z)}{f(z)} \right\} > \frac{\psi_B^A(m + 1, \delta, \alpha, \beta, \gamma)}{1 + \psi_B^A(m + 1, \delta, \alpha, \beta, \gamma)}.
\]
The result is sharp for the extremal function is given by
\[
(3.4) \quad f(z) = z + \frac{z^{m+1}}{\psi_B^A(m + 1, \delta, \alpha, \beta, \gamma)}.
\]

**Proof.** For \( \psi_B^A(k, \delta, \alpha, \beta, \gamma) \) given by (2.2) it is easily to show that
\[
\psi_B^A(k + 1, \delta, \alpha, \beta, \gamma) > \psi_B^A(k, \delta, \alpha, \beta, \gamma) > 1 \quad (k \geq 2).
\]
Therefore we have
\[
(3.5) \quad \sum_{k=2}^{m} a_k + \psi_B^A(m + 1, \delta, \alpha, \beta, \gamma) \sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty} \psi_B^A(k, \delta, \alpha, \beta, \gamma) a_k \leq 1.
\]
By setting
\[
g_1(z) = \psi_B^A(m + 1, \delta, \alpha, \beta, \gamma) \left\{ \frac{f(z)}{f_m(z)} - \left( \frac{\psi_B^A(m + 1, \delta, \alpha, \beta, \gamma) - 1}{\psi_B^A(m + 1, \delta, \alpha, \beta, \gamma)} \right) \right\}
\]
\[
= 1 - \frac{\psi_B^A(m + 1, \delta, \alpha, \beta, \gamma) \sum_{k=m+1}^{\infty} a_k z^{k-1}}{1 - \sum_{k=2}^{m} a_k z^{k-1}},
\]
we see from (3.5) that

\begin{equation}
\frac{g_1(z) - 1}{g_1(z) + 1} \leq \frac{\psi_B^\alpha(m + 1, \delta, \alpha, \beta, \gamma) \sum_{k=m+1}^{\infty} a_k}{2 - 2 \sum_{k=2}^{\infty} a_k - \psi_B^\alpha(m + 1, \delta, \alpha, \beta, \gamma) \sum_{k=m+1}^{\infty} a_k} \leq 1 \quad (z \in U),
\end{equation}

which yields the assertion (3.2) of Theorem 2. For \( z = re^{\pi i} \) that

\begin{equation}
\frac{f(z)}{f_m(z)} = 1 + \frac{z^m}{\psi_B^\alpha(m + 1, \delta, \alpha, \beta, \gamma)} \to 1 - \frac{1}{\psi_B^\alpha(m + 1, \delta, \alpha, \beta, \gamma)},
\end{equation}
as \( r \to 1^- \). Similarly, if we take

\begin{equation}
g_2(z) = (1 + \psi_B^\alpha(m + 1, \delta, \alpha, \beta, \gamma)) \left\{ \frac{f_m(z)}{f(z)} - \frac{\psi_B^\alpha(m + 1, \delta, \alpha, \beta, \gamma)}{1 + \psi_B^\alpha(m + 1, \delta, \alpha, \beta, \gamma)} \right\}
\end{equation}

\begin{equation}
= 1 + \frac{(1 + \psi_B^\alpha(m + 1, \delta, \alpha, \beta, \gamma)) \sum_{k=m+1}^{\infty} a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} a_k z^{k-1}},
\end{equation}
and using (3.5), we have

\begin{equation}
\frac{g_2(z) - 1}{g_2(z) + 1} \leq \frac{(1 + \psi_B^\alpha(m + 1, \delta, \alpha, \beta, \gamma)) \sum_{k=m+1}^{\infty} a_k}{2 - 2 \sum_{k=2}^{\infty} a_k - (1 + \psi_B^\alpha(m + 1, \delta, \alpha, \beta, \gamma)) \sum_{k=m+1}^{\infty} a_k},
\end{equation}
which leads us immediately to the assertion (3.3). This completes the proof of Theorem 2.

**Theorem 3.** Let the function \( f(z) \) defined by (1.1) be in the class \( \mathcal{T}_A^{\alpha, \beta, \gamma} \) \((\alpha, \beta, \gamma, A, B)\) satisfies the condition (2.1), then

\begin{equation}
\text{Re} \left\{ \frac{f'(z)}{f_m(z)} \right\} \geq \frac{\psi_B^\alpha(m + 1, \delta, \alpha, \beta, \gamma) - (m + 1)}{\psi_B^\alpha(m + 1, \delta, \alpha, \beta, \gamma)},
\end{equation}

and

\begin{equation}
\text{Re} \left\{ \frac{f_m'(z)}{f'(z)} \right\} \geq 1 - \frac{\psi_B^\alpha(m + 1, \delta, \alpha, \beta, \gamma)}{m + 1 + \psi_B^\alpha(m + 1, \delta, \alpha, \beta, \gamma)}.
\end{equation}
The result is sharp for the extremal function is given by (3.4).

**Proof.** Let

\begin{equation}
g_3(z) = \psi_B^\alpha(m + 1, \delta, \alpha, \beta, \gamma) \left\{ \frac{f'(z)}{f_m'(z)} - \left( \frac{\psi_B^\alpha(m + 1, \delta, \alpha, \beta, \gamma) - (m + 1)}{\psi_B^\alpha(m + 1, \delta, \alpha, \beta, \gamma)} \right) \right\}
\end{equation}

\begin{equation}
= 1 - \frac{(\frac{\psi_B^\alpha(m + 1, \delta, \alpha, \beta, \gamma)}{m + 1}) \sum_{k=m+1}^{\infty} k a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} k a_k z^{k-1}}.
\end{equation}

Using (3.5), we have

\begin{equation}
\frac{g_3(z) - 1}{g_3(z) + 1} \leq \frac{(\frac{\psi_B^\alpha(m + 1, \delta, \alpha, \beta, \gamma)}{m + 1}) \sum_{k=m+1}^{\infty} k a_k}{2 - 2 \sum_{k=2}^{\infty} k a_k - (\frac{\psi_B^\alpha(m + 1, \delta, \alpha, \beta, \gamma)}{m + 1}) \sum_{k=m+1}^{\infty} k a_k} \leq 1,
\end{equation}
if

\begin{equation}
\sum_{k=2}^{m} k a_k + (\frac{\psi_B^\alpha(m + 1, \delta, \alpha, \beta, \gamma)}{m + 1}) \left\{ \sum_{k=m+1}^{\infty} k a_k \right\} \leq 1.
\end{equation}
Since the left hand of (3.13) is bounded above by \( \sum_{k=2}^{m} \psi_B^k(k, \delta, \alpha, \beta, \gamma) a_k \) if
\[
(3.14) \quad \sum_{k=2}^{m} \psi_B^k(k, \delta, \alpha, \beta, \gamma) - k a_k + \sum_{k=m+1}^{\infty} \psi_B^k(k, \delta, \alpha, \beta, \gamma) - \frac{k \psi_B^k(m+1, \delta, \alpha, \beta, \gamma)}{m+1} a_k \geq 0,
\]
which proves the assertion (3.10) of Theorem 3. The proof of the assertion (3.11) is similar, thus, we omit it.

### 4. Integral means

In this section integral means for functions belonging to the class \( T_{n,q,t}^{\alpha,\beta,\gamma}(\alpha, \beta, \gamma, A, B) \) are obtained.

In [28], Silverman found that the function \( f_2(z) = z - \frac{z^2}{2} \) is often extremal over the family \( T \). He applied this function to resolve his integral means inequality, conjectured in [29] and settled in [31], that
\[
\int_0^{2\pi} |g(re^{\theta})|^\eta d\phi \leq \int_0^{2\pi} |f(re^{\theta})|^\eta d\phi,
\]
for all \( f \in T, \eta > 0 \) and \( 0 < r < 1 \). In [31], he also proved his conjecture for the subclasses \( T^*(\alpha) \) and \( C(\alpha) \) of \( T \).

In 1925, Littlewood [14] proved the following subordination theorem.

**Lemma 1.** If the functions \( f \) and \( g \) are analytic in \( U \) with \( g \prec f \), then for \( \eta > 0 \), and \( 0 < r < 1 \),
\[
\int_0^{2\pi} |g(re^{\theta})|^\eta d\phi \leq \int_0^{2\pi} |f(re^{\theta})|^\eta d\phi.
\]

Using Theorem 1 and Lemma 1, we prove the following result.

**Theorem 4.** Let \( f(z) \in T_{n,q,t}^{\alpha,\beta,\gamma}(\alpha, \beta, \gamma, A, B) \), \( \eta > 0 \), \( 0 \leq \alpha < 1 \), \( 0 \leq \gamma < 1 \), \( n \geq 0 \) and \( f_2(z) \) is given by
\[
f_2(z) = z - \psi_B^k(2, \delta, \alpha, \beta, \gamma) z^2,
\]
where
\[
\psi_B^k(2, \delta, \alpha, \beta, \gamma) = \frac{(1 + \delta)(1 - \beta B) + (B - A) \beta \gamma(2 - \alpha)}{(B - A) \beta \gamma(1 - \alpha)} \Phi_{2,n}(\alpha_1, \lambda, \ell).
\]

Then for \( z = re^{\theta}, 0 < r < 1 \), we have
\[
\int_0^{2\pi} |f(z)|^\eta d\phi \leq \int_0^{2\pi} |f_2(z)|^\eta d\phi.
\]

**Proof.** For \( f(z) \) is given by (1.7), (4.3) is equivalent to prove that
\[
\int_0^{2\pi} \left| 1 - \sum_{k=2}^{\infty} |a_k| z^{k-1} \right|^{\eta} d\phi \leq \int_0^{2\pi} \left| 1 - \psi_B^k(2, \delta, \alpha, \beta, \gamma) \right|^\eta d\phi.
\]
By Lemma 2, it suffices to show that
\[ 1 - \sum_{k=2}^{\infty} |a_k| z^{k-1} < 1 - \psi_B^4(2, \delta, \alpha, \beta, \gamma)z. \]

Setting
\[ (4.4) 1 - \sum_{k=2}^{\infty} a_k z^{k-1} = 1 - \psi_B^4(2, \delta, \alpha, \beta, \gamma)w(z), \]
we see from (4.4) and (2.1) that
\[ |w(z)| = \left| \sum_{k=2}^{\infty} \psi_B^4(k, \delta, \alpha, \beta, \gamma)a_k \right| \leq |z| \sum_{k=2}^{\infty} \psi_B^4(k, \delta, \alpha, \beta, \gamma)a_k \leq |z|. \]

This completes the proof of Theorem 4.

5. Square root transformation

Definition 2. Let \( f(z) \in S \) and \( h(z) = \sqrt{f(z^2)} \), then \( h(z) \in S \) and \( h(z) = z + \sum_{k=2}^{\infty} c_{2k-1} z^{2k-1} \) for \( |z| < 1 \), the function \( h \) is called a square root transformation of \( f(z) \).

Theorem 5. Let the function \( f(z) \) defined by (1.7) be in the class \( T_{\alpha, \beta, \gamma, A, B}^{n, q, s} \) and \( h(z) \) be the square root transformation of \( f(z) \), then
\[ (5.1) r \sqrt{1 - \psi_B^4(2, \delta, \alpha, \beta, \gamma)r^2} \leq |h(z)| \leq r \sqrt{1 + \psi_B^4(2, \delta, \alpha, \beta, \gamma)r^2}, \]
where
\[ (5.2) f(z) = z - \psi_B^4(2, \delta, \alpha, \beta, \gamma)z^2 (|z| = \pm r) \]
and \( \psi_B^4(2, \delta, \alpha, \beta, \gamma) \) is given by (4.2).

Proof. We have
\[ (5.3) r^2 - \psi_B^4(2, \delta, \alpha, \beta, \gamma)r^4 \leq |f(z^2)| \leq r^2 + \psi_B^4(2, \delta, \alpha, \beta, \gamma)r^4. \]

Using (5.3) in the definition 2 we find
\[ |h(z)| = \sqrt{|f(z^2)|} \leq \sqrt{r^2 + \psi_B^4(2, \delta, \alpha, \beta, \gamma)r^4} \]
\[ (5.4) = r \sqrt{1 + \psi_B^4(2, \delta, \alpha, \beta, \gamma)r^2}. \]
Since, $1 \leq \psi_B^1(2, \delta, \alpha, \beta, \gamma)$ and $r = |z| < 1$, we have

$$1 + \psi_B^1(2, \delta, \alpha, \beta, \gamma) r^2 \geq 1 - \psi_B^1(2, \delta, \alpha, \beta, \gamma) r^2,$$

and hence,

$$|h(z)| = \sqrt{|f(z^2)|} \geq \sqrt{r^2 - \psi_B^1(2, \delta, \alpha, \beta, \gamma) r^4}$$

$$= r \sqrt{1 - \psi_B^1(2, \delta, \alpha, \beta, \gamma) r^2}.$$

This completes the proof of Theorem 5.

5. Integral transform

For $f \in S$ we define the integral transform

$$V_\mu(f(z)) = \int_0^1 \mu(t) \frac{f(tz)}{t} dt,$$

where $\mu(t)$ is a real valued, non-negative weight function normalized so that $\int_0^1 \mu(t) dt = 1$. Since special cases of $\mu(t)$ are particularly interesting such as $\mu(t) = (1 + c)t^c$, $c > -1$, for which $V_\mu$ is known as the Bernardi operator [5], and $\mu(t) = \left(\frac{c + 1}{\Gamma(\eta)}\right) t^c \left(\log \frac{1}{t}\right)^{-1}$ $(c > -1; \eta \geq 0)$, which gives the Komatu operator [13], see also [18].

Now we show that the class $TF^{n,q,s,\delta}_{\lambda, \ell}(\alpha, \beta, \gamma, A, B)$ is closed under $V_\mu(f)$.

**Theorem 6.** Let the function $f(z)$ defined by (1.7) be in the class $TF^{n,q,s,\delta}_{\lambda, \ell}(\alpha, \beta, \gamma, A, B)$, then

$$V_\mu(f(z)) \in TF^{n,q,s,\delta}_{\lambda, \ell}(\alpha, \beta, \gamma, A, B).$$

**Proof.** From (6.1), we have

$$V_\mu(f(z)) = \left(\frac{c + 1}{\Gamma(\eta)}\right) \lim_{r \to 0^+} \left\{ \int_r^1 t^c (\log t)^{-1} \left( z - \sum_{k=2}^\infty a_k z^k t^{k-1} \right) dt \right\}$$

$$= \left(\frac{c + 1}{\Gamma(\eta)}\right) \lim_{r \to 0^+} \left\{ \int_r^1 t^c (\log t)^{-1} \left( z - \sum_{k=2}^\infty a_k z^k t^{k-1} \right) dt \right\}$$

$$= z - \sum_{k=2}^\infty \left(\frac{c + 1}{c + k}\right)^{\eta} a_k z^k.$$
We need to prove that
\[
\sum_{k=2}^{\infty} (1 + k \delta - \delta)[(1 - \beta B)(k - 1) + (B - A)\beta \gamma(1 - \alpha)]\Phi_{k,n}(\alpha_1, \lambda, \ell) (c + 1)^{\eta} a_k \leq 1.
\]
(6.3)

On the other hand by Theorem 1, \( f(z) \) ∈ \( TF_{\delta, \ell}^{\eta, q, s, \delta}(\alpha, \beta, \gamma, A, B) \) if and only if
\[
\sum_{k=2}^{\infty} \psi_{B}^{A}(k, \delta, \alpha, \beta, \gamma) a_k \leq 1.
\]
where \( \psi_{B}^{A}(2, \delta, \alpha, \beta, \gamma) \) is given by (2.2). Since \( \frac{c+1}{c+k} < 1 \) (\( k \geq 2 \)), therefore (6.3) holds and the proof of Theorem 6 is completed.

**Theorem 7.** Let the function \( f(z) \) defined by (1.7) be in the class \( TF_{\delta, \ell}^{\eta, q, s, \delta}(\alpha, \beta, \gamma, A, B) \). Then \( V_\mu(f(z)) \) is starlike of order \( \xi \) (\( 0 \leq \xi < 1 \)) in the disc \( |z| < r_1 \), where
\[
r_1 = \inf_{k \geq 2} \left[ \left( \frac{c+k}{c+1} \right)^{\eta} \left( \frac{1-\xi}{k-\xi} \right) \right]^{\frac{1}{\eta}},
\]
(6.4)

where \( \psi_{B}^{A}(2, \delta, \alpha, \beta, \gamma) \) is given by (2.2).

**Proof.** It is sufficient to show that
\[
\left| \frac{z[V_\mu(f(z))]'}{V_\mu(f(z))} - 1 \right| \leq 1 - \xi \quad \text{for } |z| < r_1,
\]
(6.5)

where \( r_1 \) is given by (6.4). We have
\[
\left| \frac{z[V_\mu(f(z))]'}{V_\mu(f(z))} - 1 \right| \leq \sum_{k=2}^{\infty} \frac{(k-1)(\frac{c+1}{c+k})^{\eta} a_k |z|^{k-1}}{1 - \sum_{k=2}^{\infty} (\frac{c+1}{c+k})^{\eta} a_k |z|^{k-1}}.
\]
Thus
\[
\left| \frac{z[V_\mu(f(z))]'}{V_\mu(f(z))} - 1 \right| \leq 1 - \xi,
\]
if
\[
\sum_{k=2}^{\infty} \left( \frac{c+1}{c+k} \right)^{\eta} \left( \frac{k-\xi}{1-\xi} \right) a_k |z|^{k-1} \leq 1.
\]
(6.6)

But, by Theorem 1, (6.6) will be true if
\[
\left( \frac{c+1}{c+k} \right)^{\eta} \left( \frac{k-\xi}{1-\xi} \right) |z|^{k-1} \leq \psi_{B}^{A}(k, \delta, \alpha, \beta, \gamma),
\]
that is, if
\[
r_1 = |z| \leq \left[ \left( \frac{c+k}{c+1} \right)^{\eta} \frac{(1-\xi)}{k-\xi} \right]^{\frac{1}{\eta}}.
\]
(6.7)

Theorem 7 follows easily from (6.7).

Using arguments similar to the proof of Theorem 6, we obtain the following theorems.

**Theorem 8.** (i) Let the function \( f(z) \) is starlike of order \( \rho \), then \( V_\mu(f(z)) \) is also starlike of order \( \alpha \).
(ii) Let the function $f(z)$ be convex of order $\rho$, then $V_\mu(f(z))$ is also convex of order $\alpha$.

**Theorem 9.** Let the function $f(z)$ defined by (1.7) be in the class $TF^{n,q,s}_\lambda(\alpha,\beta,\gamma,A,B)$. Then $V_\mu(f(z))$ is convex of order $\xi$ ($0 \leq \xi < 1$) in the disc $|z| < r_2$, where

\begin{equation}
(6.8) \quad r_2 = \inf_{k \geq 2} \left( \frac{c+k}{c+1} \right)^{\frac{\eta}{(1-\xi)}} \frac{(1-\xi)\psi_\beta^1(k,\delta,\alpha,\beta,\gamma)}{k(k-\xi)} \right)^{1-\gamma},
\end{equation}

where $\psi_\beta^1(k,\delta,\alpha,\beta,\gamma)$ is given by (2.2).

**Remarks.**

(i) Putting $n = \delta = 0$ and $(B-A) = 2(B-A)$ ($-1 \leq A < B \leq 1$) in the above results, we obtain the corresponding results obtained by Magesh et al. [15, with $m = 0$ and $\alpha = \frac{1}{2}$];

(ii) Putting $n = 1, \lambda = 0$ and $\ell = 0$, in the above results, we obtain the corresponding results obtained by Vijaya and Deepa [33];

(iii) Putting $n = 1, \lambda = 0$ and $\ell = 0$, in Theorems 4, 5, 6 and 7, respectively we obtain the corresponding results obtained by Murugusundaramoorthy et al. [17, Theorems 4.1, 4.2, 4.3 and 4.4] respectively;

(iv) Specializing the parameters $n, q, s, \alpha_1, \beta_1, \lambda$ and $\ell$, we obtain results corresponding to the classes $TF^{q,s}_\lambda(\alpha,\beta,\gamma,A,B)$, $TF_\lambda^{n,\alpha}(\alpha,\beta,\gamma,A,B)$, $IF^{n}_{a,c,\lambda}(\alpha,\beta,\gamma,A,B)$, $IF^{n}(\lambda,\ell)(\alpha,\beta,\gamma,A,B)$, $TF_\lambda^{n}(\alpha,\beta,\gamma,A,B)$ and $TF^{n}(\alpha,\beta,\gamma,A,B)$, mentioned in the introduction.

**REFERENCES**


