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SOME PROPERTIES FOR A NEW CERTAIN SUBCLASS OF STARLIKE FUNCTIONS

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Abstract. In this paper, we obtained some properties for class of functions related to the class of starlike functions using a linear multiplier operator $D_{\lambda,\ell}^{n,q,s}f(z)$ $(n\in\mathbb{N}_0,\ \lambda\geq 0,\ \ell\geq 0)$, such as partial sums, integral means, square root and integral transform for these class are discussed.

Keywords. Analytic functions; Starlike functions; Convex functions; Partial sums; Integral means.

1. Introduction

Let $\mathscr{A}$ denote the class of functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k, \tag{1.1}$$

which are analytic in the open unit disc $U=\{z:|z|<1\}$. Further, by S we shall denote the class of all functions in $\mathscr{A}$ which are univalent and normalized by $f(0)=0=f'(0)-1$. We denote by $S^*(\alpha)$ and $K(\alpha)$ the subclasses of S consisting of all functions which are, respectively, starlike and convex of order α $(0\leq\alpha<1)$. Thus,

$$S^*(\alpha)=\left\{f\in S:\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha\ (0\leq\alpha<1;z\in U\right\} \tag{1.2}$$

and

$$K(\alpha)=\left\{f\in S:\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha\ (0\leq\alpha<1;\ z\in U)\right\}. \tag{1.3}$$

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The classes $S^*(\alpha)$ and $K(\alpha)$ were introduced by Robertson [21]. From (1.2) and (1.3) it follows that

$$f(z) \in K(\alpha) \Leftrightarrow zf'(z) \in S^*(\alpha). \tag{1.4}$$

We note that:

$$S^*(0) = S^*,\ K(0) = K.$$

Let $f \in \mathscr{A}$ be given by (1.1) and $g \in \mathscr{A}$ given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \qquad (b_k > 0). \tag{1.5}$$

We define the Hadamard product (or convolution) of f and g is defind by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.6}$$

Also denote by T the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \qquad (a_k \geq 0; z \in U). \tag{1.7}$$

For positive real values of $\alpha_1, ..., \alpha_q$ and $\beta_1, ..., \beta_s$ $(\beta_j \notin \mathbb{Z}_0^- = \{0, -1, -2, ...\};\ j = 1, 2, ..., s)$, we now define the generalized hypergeometric function ${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ by (see, for example, [32])

$$ {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k ... (\alpha_q)_k}{(\beta_1)_k ... (\beta_s)_k} \cdot \frac{z^k}{k!} \tag{1.8}$$

$$(q \leq s + 1; q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, ...\}; z \in U),$$

where $(a)_m$ is the Pochhammer symbol defined by

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = \begin{cases} 1 & (m = 0), \\ a(a+1)....(a+m-1) & (m \in \mathbb{N}). \end{cases} \tag{1.9}$$

Corresponding to the function $h(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ defined by

$$h(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = z\, {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z), \tag{1.10}$$

we consider a linear operator $H(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) : \mathscr{A} \longrightarrow \mathscr{A}$ which is defined by the following Hadamard product (or convolution):

$$H(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) f(z) = h(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) * f(z). \tag{1.11}$$

We observe that for function $f(z)$ of the form (1.1) we have

$$H(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) f(z) = z + \sum_{k=2}^{\infty} \Gamma_k(\alpha_1) a_k z^k, \tag{1.12}$$

where

$$\Gamma_k(\alpha_1) = \frac{(\alpha_1)_{k-1} ... (\alpha_q)_{k-1}}{(\beta_1)_{k-1} .... (\beta_s)_{k-1}} \cdot \frac{1}{(1)_{k-1}} \, (k \geq 2). \tag{1.13}$$

For convenience, we write

$$H_{q,s}[\alpha_1] = H(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s). \tag{1.14}$$

The linear operator $H_{q,s}[\alpha_1]$ was introduced and studied by Dziok and Srivastava [8]. This operator contains in turn many interesting operators such as the Hohlov operator [11], Carlson-Shaffer operator [6], Ruscheweyh derivative operator [22], and the Srivastava-Owa fractional derivative operator [19].

EL-Ashwah *et al.* [9] were defined the linear multiplier Dziok-Srivastava operator $D^{n,q,s}_{\lambda,\ell}f(z)$ as follows:

$$D^{n,q,s}_{\lambda,\ell}f(z) = z + \sum_{k=2}^{\infty} \Phi_{k,n}(\alpha_1,\lambda,\ell)a_k z^k \quad (n \in \mathbb{N}), \tag{1.15}$$

where

$$\Phi_{k,n}(\alpha_1,\lambda,\ell) = \left[\frac{\ell+1+\lambda(k-1)}{\ell+1}\Gamma_k(\alpha_1)\right]^n. \tag{1.16}$$

The operator $D^{n,q,s}_{\lambda,\ell}f(z)$, can be written in terms of convolution as

$$D^{n,q,s}_{\lambda,\ell}f(z) = [(h(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z) * g_{\lambda,\ell}(z)) * ...$$

$$\underbrace{*(h(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z) * g_{\lambda,\ell}(z))] *}_{n-\text{times}} f(z), \tag{1.17}$$

where

$$g_{\lambda,\ell}(z) = \frac{(\ell+1)z-(\ell+1-\lambda)z^2}{(\ell+1)(1-z)^2} = \frac{z-(1-\lambda/(\ell+1))z^2}{(1-z)^2}.$$

By specializing the parameters q, s, α_1, β_1, λ and ℓ, we obtain the following operators studied by various authors:

(i) For $q=2$, $s=1$, $\alpha_1=2$, $\alpha_2=1$, $\beta_1=2-\alpha$ $(\alpha \neq 2,3,...)$ and $\ell=0$, we have $D^{n,2,1}_{\lambda,0}f(z) = D^{n,\alpha}_{\lambda}f(z)$ (see Al-Oboudi and Al-Amoudi [2] and Aouf and Mostafa [3]);

(ii) For $q=2$, $s=1$, $\alpha_1=a$ $(a>0)$, $\alpha_2=1$, $\beta_1=c$ $(c>0)$ and $\ell=0$, we have $D^{n,2,1}_{\lambda,0}f(z) = I^n_{a,c,\lambda}f(z)$ (see Prajapat and Raina [20]);

(iii) For $q=2$, $s=1$ and $\alpha_1=\alpha_2=\beta_1=1$, we have $D^{n,2,1}_{\lambda,\ell}f(z) = I^n(\lambda,\ell)f(z)$ (see Catas [7]);

(iv) For $q=2$, $s=1$, $\alpha_1=\alpha_2=\beta_1=1$ and $\ell=0$, we have $D^{n,2,1}_{\lambda,0}f(z) = D^n_{\lambda}f(z)$ (see Al-Oboudi [1]) and $D^{n,2,1}_{0,0}f(z) = D^n f(z)$ (see Salagean [23]);

(v) $D^{1,q,s}_{0,0}f(z) = H_{q,s}[\alpha_1]$ (see Dziok and Sirvastava [8]).

For $0 \leq \delta \leq 1$, $0 < \beta \leq 1$, $-1 \leq A < B \leq 1$, $0 \leq \gamma \leq 1$, we let the class $TF^{n,q,s,\delta}_{\lambda,\ell}(\alpha,\beta,\gamma,A,B)$ denote the subclass of T consisting of functions of the form (1.7) and satisfing the condition:

$$\left|\frac{\frac{zF'_\delta(z)}{F_\delta(z)}-1}{(B-A)\gamma\left(\frac{zF'_\delta(z)}{F_\delta(z)}-\alpha\right)-B\left(\frac{zF'_\delta(z)}{F_\delta(z)}-1\right)}\right| < \beta \tag{1.18}$$

$$(0 \leq \alpha < 1; 0 < \beta \leq 1; z \in U),$$

where

$$\frac{zF_{\delta}'(z)}{F_{\delta}(z)} = \frac{z(D_{\lambda,\ell}^{n,q,s}f(z))' + \delta z^2(D_{\lambda,\ell}^{n,q,s}f(z))''}{(1-\delta)D_{\lambda,\ell}^{n,q,s}f(z) + \delta z(D_{\lambda,\ell}^{n,q,s}f(z))'} \quad (0 \le \delta \le 1). \tag{1.19}$$

We note that:

(i) For $n = 1$, $\lambda = 0$ and $\ell = 0$, we have $TF_{0,0}^{1,q,s,\delta}(\alpha,\beta,\gamma,A,B) = HF_{\gamma}^{\delta}(\alpha,\beta,A,B)$ (see Murugusundaramoothy et al. [17], also see Vijaya and Deepa [33]);

(ii) For $q = 2$, $s = 1$, $\alpha_1 = \alpha_2 = \beta_1 = \gamma = B = \beta = 1$, $\lambda = \ell = 0$ and $A = -1$, we have $TF_{0,0}^{n,2,1,0}(\alpha,\beta,1,-1,1) = Q(j,\delta,\alpha,n)$ (see Aouf and Silverman [4], with $j = 1$);

(iii) For $q = 2$, $s = 1$, $\alpha_1 = \alpha_2 = \beta_1 = \gamma = B = 1$, $\lambda = \delta = \ell = 0$ and $A = -1$, we have $TF_{0,0}^{n,2,1,0}(\alpha,\beta,1,-1,1) = S_n(\alpha,\beta)$ (see Salagean [25] and [26]);

(iv) For $q = 2$, $s = 1$, $\alpha_1 = \alpha_2 = \beta_1 = \gamma = B = 1$, $n = \lambda = \delta = 0$, $\ell = 0$ and $A = -1$, we have $TF_{0,0}^{0,2,1,0}(\alpha,\beta,1,-1,1) = S^*(\alpha,\beta)$ (see Gupta and Jain [10], also see Salagean [24]);

(v) For $q = 2$, $s = 1$, $\alpha_1 = \alpha_2 = \beta_1 = \beta = \gamma = B = 1$, $n = \lambda = \ell = 0$ and $A = -1$, we have $TF_{0,0,1}^{0,2,1,\delta}(\alpha,1,-1,1) = P(n,\delta,\alpha)$ (see Kim et al. [12], with $n = 1$);

(vi) For $q = 2$, $s = 1$, $\alpha_1 = \alpha_2 = \beta_1 = \beta = \gamma = B = 1$, $n = \lambda = \ell = 0$ and $A = -1$, we have $TF_{0,0}^{0,2,1,0}(\alpha,1,1,-1,1) = T^*(\alpha)$ and $TF_{0,0,1}^{1,q,s,1}(\alpha,1,-1,1) = \mathscr{C}(\alpha)$ (see Silverman [27]).

(vii) For $q = 2$, $s = 1$, $\alpha_1 = \alpha_2 = \beta_1 = \gamma = B = \beta = 1$, $\lambda = \delta = \ell = 0$ and $A = -1$, we have $TF_{0,0}^{n,2,1,0}(\alpha,\beta,-1,1,1) = S_n(\alpha,\beta)$ (see Salagean [23]);

Also, we note that:

(i) For $n = 0$, we have

$$\begin{aligned} &TF_{\lambda,\ell}^{0,q,s}(\alpha,\beta,\gamma,A,B) \\ &= TF_{\lambda,\ell}^{q,s}(\alpha,\beta,\gamma,A,B) = \left\{ f \in T : \left| \frac{\frac{zF_{\delta}'(z)}{F_{\delta}(z)} - 1}{(B-A)\gamma\left(\frac{zF_{\delta}'(z)}{F_{\delta}(z)} - \alpha\right) - B\left(\frac{zF_{\delta}'(z)}{F_{\delta}(z)} - 1\right)} \right| < \beta \ (z \in U) \right\}, \end{aligned}$$

where

$$\frac{zF_{\delta}'(z)}{F_{\delta}(z)} = \frac{zf'(z) + \delta z^2 f''(z)}{(1-\delta)f(z) + \delta z f'(z)} \quad (0 \le \delta \le 1);$$

(ii) For $q = 2$, $s = 1$, $\alpha_1 = 2$, $\alpha_2 = 1$, $\beta_1 = 2 - \alpha$ $(\alpha \ne 2,3,...)$ and $\ell = 0$, we have

$$\begin{aligned} &TF_{\lambda,0}^{n,2,1}(\alpha,\beta,\gamma,A,B) \\ &= TF_{\lambda}^{n,\alpha}(\alpha,\beta,\gamma,A,B) = \left\{ f \in T : \left| \frac{\frac{zF_{\delta}'(z)}{F_{\delta}(z)} - 1}{(B-A)\gamma\left(\frac{zF_{\delta}'(z)}{F_{\delta}(z)} - \alpha\right) - B\left(\frac{zF_{\delta}'(z)}{F_{\delta}(z)} - 1\right)} \right| < \beta \ (z \in U) \right\}, \end{aligned}$$

where

$$\frac{zF_{\delta}'(z)}{F_{\delta}(z)} = \frac{z(D_{\lambda}^{n,\alpha}f(z))' + \delta z^2(D_{\lambda}^{n,\alpha}f(z))''}{(1-\delta)D_{\lambda}^{n,\alpha}f(z) + \delta z(D_{\lambda}^{n,\alpha}f(z))'} \quad (0 \le \delta \le 1);$$

(iii) For $q=2$, $s=1$, $\alpha_1=a$ $(a>0)$, $\alpha_2=1$, $\beta_1=c$ $(c>0)$ and $\ell=0$, we have

$$TF_{\lambda,0}^{n,2,1}(\alpha,\beta,\gamma,A,B)$$
$$=IF_{a,c,\lambda}^{n}(\alpha,\beta,\gamma,A,B)=\left\{f\in T:\left|\frac{\frac{zF_{\delta}'(z)}{F_{\delta}(z)}-1}{(B-A)\gamma\left(\frac{zF_{\delta}'(z)}{F_{\delta}(z)}-\alpha\right)-B\left(\frac{zF_{\delta}'(z)}{F_{\delta}(z)}-1\right)}\right|<\beta\ (z\in U)\right\},$$

where

$$\frac{zF_{\delta}'(z)}{F_{\delta}(z)}=\frac{z(I_{a,c,\lambda}^{n}f(z))'+\delta z^{2}(I_{a,c,\lambda}^{n}f(z))''}{(1-\delta)I_{a,c,\lambda}^{n}f(z)+\delta z(I_{a,c,\lambda}^{n}f(z))'}\ \ (0\le\delta\le1);$$

(iv) For $q=2$, $s=1$ and $\alpha_1=\alpha_2=\beta_1=1$, we have

$$TF_{\lambda,\ell}^{n,2,1}(\alpha,\beta,\gamma,A,B)$$
$$=IF^{n}(\lambda,\ell)(\alpha,\beta,\gamma,A,B)=\left\{f\in T:\left|\frac{\frac{zF_{\delta}'(z)}{F_{\delta}(z)}-1}{(B-A)\gamma\left(\frac{zF_{\delta}'(z)}{F_{\delta}(z)}-\alpha\right)-B\left(\frac{zF_{\delta}'(z)}{F_{\delta}(z)}-1\right)}\right|<\beta\ (z\in U)\right\},$$

where

$$\frac{zF_{\delta}'(z)}{F_{\delta}(z)}=\frac{z(I^{n}(\lambda,\ell)f(z))'+\delta z^{2}(I^{n}(\lambda,\ell)f(z))''}{(1-\delta)I^{n}(\lambda,\ell)f(z)+\delta z(I^{n}(\lambda,\ell)f(z))'}\ \ (0\le\delta\le1);$$

(v) For $q=2$, $s=1$, $\alpha_1=\alpha_2=\beta_1=1$ and $\ell=0$, we have

$$TF_{\lambda,0}^{n,2,1}(\alpha,\beta,\gamma,A,B)$$
$$=TF_{\lambda}^{n}(\alpha,\beta,\gamma,A,B)=\left\{f\in T:\left|\frac{\frac{zF_{\delta}'(z)}{F_{\delta}(z)}-1}{(B-A)\gamma\left(\frac{zF_{\delta}'(z)}{F_{\delta}(z)}-\alpha\right)-B\left(\frac{zF_{\delta}'(z)}{F_{\delta}(z)}-1\right)}\right|<\beta\ (z\in U)\right\},$$

where

$$\frac{zF_{\delta}'(z)}{F_{\delta}(z)}=\frac{z(D_{\lambda}^{n}f(z))'+\delta z^{2}(D_{\lambda}^{n}f(z))''}{(1-\delta)D_{\lambda}^{n}f(z)f(z)+\delta z(D_{\lambda}^{n}f(z))'}\ \ (0\le\delta\le1);$$

(vi) For $q=2$, $s=1$, $\alpha_1=\alpha_2=\beta_1=1$, $\lambda=0$ and $\ell=0$, we have

$$TF_{0,0}^{n,2,1}(\alpha,\beta,\gamma,A,B)$$
$$=TF^{n}(\alpha,\beta,\gamma,A,B)=\left\{f\in T:\left|\frac{\frac{zF_{\delta}'(z)}{F_{\delta}(z)}-1}{(B-A)\gamma\left(\frac{zF_{\delta}'(z)}{F_{\delta}(z)}-\alpha\right)-B\left(\frac{zF_{\delta}'(z)}{F_{\delta}(z)}-1\right)}\right|<\beta\ (z\in U)\right\},$$

where

$$\frac{zF_{\delta}'(z)}{F_{\delta}(z)}=\frac{z(D^{n}f(z))'+\delta z^{2}(D^{n}f(z))''}{(1-\delta)D^{n}f(z)f(z)+\delta z(D^{n}f(z))'}\ \ (0\le\delta\le1).$$

Definition 1 [13]. (Subordination) For analytic functions f and g with $f(0)=g(0)$, f is said to be subordinate to g, denote by $f\prec g$, if there exists an analytic function w such that $w(0)=0$, $|w(z)|<1$ and $f(z)=g(w(z))$ $(z\in U)$, If $g(z)$ is univalent function, then $f\prec g$, if and only if

$$f(0)=g(0)\text{ and }f(U)\subset g(U).$$

In this paper, we obtain partial sums, square root, integral means and integral transform for the class $TF_{\lambda,\ell}^{n,q,s,\delta}(\alpha,\beta,\gamma,A,B)$.

2. Coefficient estimates

Unless otherwise mentioned, we assume in the reminder of this paper that $0 \leq \delta \leq 1$, $-1 \leq A < B \leq 1$, $0 < B \leq 1$, $q \leq s+1$, $q, s \in \mathbb{N}_0$, $\lambda \geq 0$, $\ell \geq 0$, $n \in \mathbb{N}_0$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 < \gamma \leq 1$ and $z \in U$.

Theorem 1. *Let the function $f(z)$ be defined by (1.7). Then $f(z)$ is in the class $TF_{\lambda,\ell}^{n,q,s,\delta}(\alpha,\beta,\gamma,A,B)$ if and only if*

$$\sum_{k=2}^{\infty} \psi_B^A(k,\delta,\alpha,\beta,\gamma)a_k \leq 1, \tag{2.1}$$

where

$$\psi_B^A(k,\delta,\alpha,\beta,\gamma) = \frac{(1+k\delta-\delta)[(1-\beta B)(k-1)+(B-A)\beta\gamma(k-\alpha)]\Phi_{k,n}(\alpha_1,\lambda,\ell)}{(B-A)\beta\gamma(1-\alpha)} \tag{2.2}$$

and $\Phi_{k,n}(\alpha_1,\lambda,\ell)$ is given by (1.16).

Proof. Assume that the inequality (2.1) holds true, we find from (1.7) and (1.22) that

$$\begin{aligned}
&\left|zF_\delta'(z) - F_\delta(z)\right| - \beta\left|(B-A)\gamma\left[zF_\delta'(z) - \alpha F_\delta(z)\right] - B\left[zF_\delta'(z) - F_\delta(z)\right]\right| \\
= &\left|\sum_{k=2}^{\infty}(1+k\delta-\delta)(k-1)\Phi_{k,n}(\alpha_1,\lambda,\ell)a_k z^k\right| - \beta\left|(B-A)\gamma(1-\alpha)z\right. \\
&\left.+\sum_{k=2}^{\infty}[(1+k\delta-\delta)(B-A)\gamma(k-\alpha) - B(k-1)]\Phi_{k,n}(\alpha_1,\lambda,\ell)a_k z^k\right| \\
&\leq \sum_{k=2}^{\infty}(1+k\delta-\delta)(k-1)\Phi_{k,n}(\alpha_1,\lambda,\ell)a_k r^k - (B-A)\beta\gamma(1-\alpha)r \\
&+\sum_{k=2}^{\infty}\beta[(1+k\delta-\delta)(B-A)\gamma(k-\alpha) - B(k-1)]\Phi_{k,n}(\alpha_1,\lambda,\ell)a_k r^k \\
&\leq \sum_{k=2}^{\infty}(1+k\delta-\delta)[(1-\beta B)(k-1)+(B-A)\beta\gamma(k-\alpha)]\Phi_{k,n}(\alpha_1,\lambda,\ell)a_k \\
&-(B-A)\beta\gamma(1-\alpha) \leq 0 \ (z \in U).
\end{aligned}$$

Hence, by the maximum modulus theorem, we have $f(z) \in TF_{\lambda,\ell}^{n,q,s,\delta}(\alpha,\beta,\gamma,A,B)$.

Conversely, Let

$$\left|\frac{\frac{zF_\delta'(z)}{F_\delta(z)} - 1}{(B-A)\gamma\left(\frac{zF_\delta'(z)}{F_\delta(z)} - \alpha\right) - B\left(\frac{zF_\delta'(z)}{F_\delta(z)} - 1\right)}\right|$$

$$= \left|\frac{\sum_{k=2}^{\infty}(1+k\delta-\delta)(k-1)\Phi_{k,n}(\alpha_1,\lambda,\ell)aa_k z^k}{(B-A)\gamma(1-\alpha)z - \sum_{k=2}^{\infty}(1+k\delta-\delta)[(B-A)\gamma(k-\alpha)+B(k-1)]\Phi_{k,n}(\alpha_1,\lambda,\ell)a_k z^k}\right| < \beta \ (z \in U).$$

Now since $\mathrm{Re}\{z\} \leq |z|$ for all z, we have

$$\mathrm{Re}\left\{\frac{\sum_{k=2}^{\infty}(1+k\delta-\delta)(k-1)\Phi_{k,n}(\alpha_1,\lambda,\ell)a_k z^{k-1}}{(B-A)\gamma(1-\alpha) - \sum_{k=2}^{\infty}(1+k\delta-\delta)[(B-A)\gamma(k-\alpha)-B(k-1)]\Phi_{k,n}(\alpha_1,\lambda,\ell)a_k z^{k-1}}\right\} < \beta. \tag{2.3}$$

Choos values of z on the real axis so that $f'(z)$ is real. Then upon clearing the denominator in (2.3) and letting $z \to 1^-$through real values, we have

$$\sum_{k=2}^{\infty}(1+k\delta-\delta)[(1-\beta B)(k-1)+(B-A)\beta\gamma(k-\alpha)]\Phi_{k,n}(\alpha_1,\lambda,\ell)a_k-(B-A)\beta\gamma(1-\alpha)\leq 0.$$

This completes the proof of Theorem 1.

3. Partial sums

Following the earlier works by Silverman [30] and Siliva [27] on partial sums of analytic functions, we consider in this section partial sums of functions in the class $TF_{\lambda,\ell}^{n,q,s,\delta}(\alpha,\beta,\gamma,A,B)$ and obtain sharp lower bounds for the ratios of real part of $f(z)$ to $f_m(z)$, $f_m(z)$ to $f(z)$, $f'(z)$ to $f'_m(z)$ and $f'_m(z)$ to $f'(z)$, respectively.

Theorem 2. *Let the function $f(z)$ of the form (1.7) be in the class $TF_{\lambda,\ell}^{n,q,s,\delta}(\alpha,\beta,\gamma,A,B)$. Define the partial sums $f_1(z)$ and $f_m(z)$, by*

$$f_1(z)=z \text{ and } f_m(z)=z-\sum_{k=2}^{m}a_kz^k \quad (m\in\mathbb{N}/\{1\}). \tag{3.1}$$

Then

$$\operatorname{Re}\left\{\frac{f(z)}{f_m(z)}\right\}>\frac{\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)-1}{\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)} \quad (z\in U;n\in\mathbb{N}), \tag{3.2}$$

and

$$\operatorname{Re}\left\{\frac{f_m(z)}{f(z)}\right\}>\frac{\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)}{1+\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)}. \tag{3.3}$$

The result is sharp for the extremal function is given by

$$f(z)=z+\frac{z^{m+1}}{\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)}. \tag{3.4}$$

Proof. For $\psi_B^A(k,\delta,\alpha,\beta,\gamma)$ given by (2.2) it is easily to show that $\psi_B^A(k+1,\delta,\alpha,\beta,\gamma)>\psi_B^A(k,\delta,\alpha,\beta,\gamma)>1$ $(k\geq 2)$. Therefore we have

$$\sum_{k=2}^{m}a_k+\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)\sum_{k=m+1}^{\infty}a_k\leq\sum_{k=2}^{\infty}\psi_B^A(k,\delta,\alpha,\beta,\gamma)a_k\leq 1. \tag{3.5}$$

By setting

$$g_1(z)=\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)\left\{\frac{f(z)}{f_m(z)}-\left(\frac{\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)-1}{\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)}\right)\right\}$$

$$=1-\frac{\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)\sum_{k=m+1}^{\infty}a_kz^{k-1}}{1-\sum_{k=2}^{m}a_kz^{k-1}}, \tag{3.6}$$

we see from (3.5) that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq \frac{\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)\sum_{k=m+1}^{\infty} a_k}{2-2\sum_{k=2}^{m} a_k - \psi_B^A(m+1,\delta,\alpha,\beta,\gamma)\sum_{k=m+1}^{\infty} a_k} \leq 1 \ (z \in U), \tag{3.7}$$

which yields the assertion (3.2) of Theorem 2. For $z = re^{\frac{i\pi}{m}}$ that

$$\frac{f(z)}{f_m(z)} = 1 + \frac{z^m}{\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)} \to 1 - \frac{1}{\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)},$$

as $r \to 1^-$. Similarly, if we take

$$g_2(z) = (1+\psi_B^A(m+1,\delta,\alpha,\beta,\gamma))\left\{\frac{f_m(z)}{f(z)} - \frac{\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)}{1+\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)}\right\}$$

$$= 1 + \frac{(1+\psi_B^A(m+1,\delta,\alpha,\beta,\gamma))\sum_{k=m+1}^{\infty} a_k z^{k-1}}{1-\sum_{k=2}^{\infty} a_k z^{k-1}}, \tag{3.8}$$

and using (3.5), we have

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \leq \frac{(1+\psi_B^A(m+1,\delta,\alpha,\beta,\gamma))\sum_{k=m+1}^{\infty} a_k}{2-2\sum_{k=2}^{\infty} a_k - (1+\psi_B^A(m+1,\delta,\alpha,\beta,\gamma))\sum_{k=m+1}^{\infty} a_k}, \tag{3.9}$$

which leads us immediately to the assertion (3.3). This completes the proof of Theorem 2.

Theorem 3. *Let the function $f(z)$ defined by (1.1) be in the class $TF_{\lambda,\ell}^{n,q,s,\delta}(\alpha,\beta,\gamma,A,B)$ satisfies the condition (2.1), then*

$$\operatorname{Re}\left\{\frac{f'(z)}{f'_m(z)}\right\} \geq \frac{\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)-(m+1)}{\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)}, \tag{3.10}$$

and

$$\operatorname{Re}\left\{\frac{f'_m(z)}{f'(z)}\right\} \geq 1 - \frac{\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)}{m+1+\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)}. \tag{3.11}$$

The result is sharp for the extremal function is given by (3.4).

Proof. Let

$$\begin{aligned} g_3(z) &= \psi_B^A(m+1,\delta,\alpha,\beta,\gamma)\left\{\frac{f'(z)}{f'_m(z)} - \left(\frac{\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)-(m+1)}{\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)}\right)\right\} \\ &= 1 - \frac{\left(\frac{\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)}{m+1}\right)\sum_{k=m+1}^{\infty} ka_k z^{k-1}}{1-\sum_{k=2}^{m} ka_k z^{k-1}}. \end{aligned}$$

Using (3.5), we have

$$\left|\frac{g_3(z)-1}{g_3(z)+1}\right| \leq \frac{\left(\frac{\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)}{m+1}\right)\sum_{k=m+1}^{\infty} ka_k}{2-2\sum_{k=2}^{\infty} ka_k - \left(\frac{\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)}{m+1}\right)\sum_{k=m+1}^{\infty} ka_k} \leq 1, \tag{3.12}$$

if

$$\sum_{k=2}^{m} ka_k + \left(\frac{\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)}{m+1}\right)\sum_{k=m+1}^{\infty} ka_k \leq 1. \tag{3.13}$$

Since the left hand of (3.13) is bounded above by $\sum_{k=2}^{m}\psi_B^A(k,\delta,\alpha,\beta,\gamma)a_k$ if

$$\sum_{k=2}^{m}(\psi_B^A(k,\delta,\alpha,\beta,\gamma)-k)a_k+\sum_{k=m+1}^{\infty}\psi_B^A(k,\delta,\alpha,\beta,\gamma)-\frac{k\psi_B^A(m+1,\delta,\alpha,\beta,\gamma)}{m+1}a_k\geq 0, \tag{3.14}$$

which proves the assertion (3.10) of Theorem 3. The proof of the assertion (3.11) is similar, thus, we omit it.

4. Integral means

In this section integral means for functions belonging to the class $TF_{\lambda,\ell}^{n,q,s,\delta}(\alpha,\beta,\gamma,A,B)$ are obtained. In [28], Silverman found that the function $f_2(z)=z-\frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality, conjectured in [29] and settled in [31], that

$$\int_0^{2\pi}\left|g(re^{i\phi})\right|^{\eta}d\phi\leq\int_0^{2\pi}\left|f(re^{i\phi})\right|^{\eta}d\phi,$$

for all $f\in T$, $\eta>0$ and $0<r<1$. In [31], he also proved his conjecture for the subclasses $T^*(\alpha)$ and $C(\alpha)$ of T. In 1925, Littlewood [14] proved the following subordination theorem.

Lemma 1. *If the functions f and g are analytic in U with $g\prec f$, then for $\eta>0$, and $0<r<1$,*

$$\int_0^{2\pi}\left|g(re^{i\phi})\right|^{\eta}d\phi\leq\int_0^{2\pi}\left|f(re^{i\phi})\right|^{\eta}d\phi. \tag{4.1}$$

Using Theorem 1 and Lemma 1, we prove the following result.

Theorem 4. *Let $f(z)\in TF_{\lambda,\ell}^{n,q,s,\delta}(\alpha,\beta,\gamma,A,B)$, $\eta>0$, $0\leq\alpha<1$, $0\leq\gamma<1$, $n\geq 0$ and $f_2(z)$ is given by*

$$f_2(z)=z-\psi_B^A(2,\delta,\alpha,\beta,\gamma)z^2,$$

where

$$\psi_B^A(2,\delta,\alpha,\beta,\gamma)=\frac{(1+\delta)[(1-\beta B)+(B-A)\beta\gamma(2-\alpha)]\Phi_{2,n}(\alpha_1,\lambda,\ell)}{(B-A)\beta\gamma(1-\alpha)}. \tag{4.2}$$

Then for $z=re^{i\phi}$, $0<r<1$, we have

$$\int_0^{2\pi}|f(z)|^{\eta}d\phi\leq\int_0^{2\pi}|f_2(z)|^{\eta}d\phi. \tag{4.3}$$

Proof. For $f(z)$ is given by (1.7), (4.3) is equivalent to prove that

$$\int_0^{2\pi}\left|1-\sum_{k=2}^{\infty}|a_k|z^{k-1}\right|^{\eta}d\phi\leq\int_0^{2\pi}\left|1-\psi_B^A(2,\delta,\alpha,\beta,\gamma)\right|^{\eta}d\phi.$$

By Lemma 2, it suffices to show that

$$1-\sum_{k=2}^{\infty}|a_k|z^{k-1}\prec 1-\psi_B^A(2,\delta,\alpha,\beta,\gamma)z.$$

Setting

$$1-\sum_{k=2}^{\infty}a_k z^{k-1}=1-\psi_B^A(2,\delta,\alpha,\beta,\gamma)w(z), \tag{4.4}$$

we see from (4.4) and (2.1) that

$$\begin{aligned}|w(z)| &= \left|\sum_{k=2}^{\infty}\psi_B^A(2,\delta,\alpha,\beta,\gamma)a_k z^{k-1}\right| \\ &\leq |z|\sum_{k=2}^{\infty}\psi_B^A(k,\delta,\alpha,\beta,\gamma)a_k\leq|z|.\end{aligned}$$

This completes the proof of Theorem 4.

5. Square root transformation

Defintion 2. Let $f(z)\in S$ and $h(z)=\sqrt{f(z^2)}$, then $h(z)\in S$ and $h(z)=z+\sum_{k=2}^{\infty}c_{2k-1}z^{2k-1}$ for $|z|<1$, the function h is called a square root transformation of $f(z)$.

Theorem 5. *Let the function $f(z)$ defined by (1.7) be in the class $TF_{\lambda,\ell}^{n,q,s,\delta}(\alpha,\beta,\gamma,A,B)$ and $h(z)$ be the square root transformation of $f(z)$, then*

$$r\sqrt{1-\psi_B^A(2,\delta,\alpha,\beta,\gamma)r^2}\leq|h(z)|\leq r\sqrt{1+\psi_B^A(2,\delta,\alpha,\beta,\gamma)r^2}, \tag{5.1}$$

where

$$f(z)=z-\psi_B^A(2,\delta,\alpha,\beta,\gamma)z^2\ (|z|=\pm r) \tag{5.2}$$

and $\psi_B^A(2,\delta,\alpha,\beta,\gamma)$ is given by (4.2).

Proof. We have

$$r^2-\psi_B^A(2,\delta,\alpha,\beta,\gamma)r^4\leq\left|f(z^2)\right|\leq r^2+\psi_B^A(2,\delta,\alpha,\beta,\gamma)r^4. \tag{5.3}$$

Using (5.3) in the defintion 2 we find

$$\begin{aligned}|h(z)| &= \sqrt{|f(z^2)|} \\ &\leq \sqrt{r^2+\psi_B^A(2,\delta,\alpha,\beta,\gamma)r^4}\end{aligned}$$

$$=r\sqrt{1+\psi_B^A(2,\delta,\alpha,\beta,\gamma)r^2}. \tag{5.4}$$

Since, $1 \leq \psi_B^A(2,\delta,\alpha,\beta,\gamma)$ and $r = |z| < 1$, we have

$$1+\psi_B^A(2,\delta,\alpha,\beta,\gamma)r^2 \geq 1-\psi_B^A(2,\delta,\alpha,\beta,\gamma)r^2, \tag{5.5}$$

and hence,

$$\begin{aligned} |h(z)| &= \sqrt{|f(z^2)|} \\ &\geq \sqrt{r^2-\psi_B^A(2,\delta,\alpha,\beta,\gamma)r^4} \\ &= r\sqrt{1-\psi_B^A(2,\delta,\alpha,\beta,\gamma)r^2}. \end{aligned} \tag{5.6}$$

This completes the proof of Theorem 5.

5. Integral transform

For $f \in S$ we define the integral transform

$$V_\mu(f(z)) = \int_0^1 \mu(t)\frac{f(tz)}{t}dt, \tag{6.1}$$

where $\mu(t)$ is a real valued, non-negative weight function normalized so that $\int_0^1 \mu(t)dt = 1$. Since special cases of $\mu(t)$ are particularly interesting such as $\mu(t) = (1+c)t^c$, $c > -1$, for which V_μ is known as the Bernardi operator [5], and

$$\mu(t) = \frac{(c+1)^\eta}{\Gamma(\eta)}t^c\left(\log\frac{1}{t}\right)^{\eta-1} \quad (c > -1; \eta \geq 0), \tag{6.2}$$

which gives the Komatu operator [13], see also [18].

Now we show that the class $TF_{\lambda,\ell}^{n,q,s,\delta}(\alpha,\beta,\gamma,A,B)$ is closed under $V_\mu(f)$.

Theorem 6. *Let the function $f(z)$ defined by (1.7) be in the class $TF_{\lambda,\ell}^{n,q,s,\delta}(\alpha,\beta,\gamma A,B)$, then*

$$V_\mu(f(z)) \in TF_{\lambda,\ell}^{n,q,s,\delta}(\alpha,\beta,\gamma,A,B).$$

Proof. From (6.1), we have

$$\begin{aligned} V_\mu(f(z)) &= \frac{(c+1)^\eta}{\Gamma(\eta)}\int_0^1(-1)^{\eta-1}t^c(\log t)^{\eta-1}\left(z-\sum_{k=2}^{\infty}a_kz^kt^{k-1}\right)dt \\ &= \frac{(-1)^{\eta-1}(c+1)^\eta}{\Gamma(\eta)}\lim_{r\to 0^+}\left\{\int_r^1 t^c(\log t)^{\eta-1}\left(z-\sum_{k=2}^{\infty}a_kz^kt^{k-1}\right)dt\right\} \\ &= z-\sum_{k=2}^{\infty}\left(\frac{c+1}{c+k}\right)^\eta a_kz^k. \end{aligned}$$

We need to prove that

$$\sum_{k=2}^{\infty}\frac{(1+k\delta-\delta)[(1-\beta B)(k-1)+(B-A)\beta\gamma(1-\alpha)]\Phi_{k,n}(\alpha_1,\lambda,\ell)}{(B-A)\beta\gamma(1-\alpha)}\left(\frac{c+1}{c+k}\right)^{\eta}a_k\leq 1. \tag{6.3}$$

On the other hand by Theorem 1, $f(z)\in TF_{\lambda,\ell}^{n,q,s,\delta}(\alpha,\beta,\gamma,A,B)$ if and only if

$$\sum_{k=2}^{\infty}\psi_B^A(k,\delta,\alpha,\beta,\gamma)a_k\leq 1.$$

where $\psi_B^A(2,\delta,\alpha,\beta,\gamma)$ is given by (2.2). Since $\dfrac{c+1}{c+k}<1$ $(k\geq 2)$, therefore (6.3) holds and the proof of Theorem 6 is completed.

Theorem 7. *Let the function $f(z)$ defined by (1.7) be in the class $TF_{\lambda,\ell}^{n,q,s,\delta}(\alpha,\beta,\gamma,A,B)$. Then $V_\mu(f(z))$ is starlike of order ξ $(0\leq\xi<1)$ in the disc $|z|<r_1$, where*

$$r_1=\inf_{k\geq 2}\left[\left(\frac{c+k}{c+1}\right)^{\eta}\frac{(1-\xi)\psi_B^A(k,\delta,\alpha,\beta,\gamma)}{(k-\xi)}\right]^{\frac{1}{k-1}}, \tag{6.4}$$

where $\psi_B^A(2,\delta,\alpha,\beta,\gamma)$ is given by (2.2).

Proof. It is sufficient to show that

$$\left|\frac{z[V_\mu(f(z))]'}{V_\mu(f(z))}-1\right|\leq 1-\xi \text{ for } |z|<r_1, \tag{6.5}$$

where r_1 is given by (6.4). We have

$$\left|\frac{z[V_\mu(f(z))]'}{V_\mu(f(z))}-1\right|\leq\frac{\sum\limits_{k=2}^{\infty}(k-1)\left(\frac{c+1}{c+k}\right)^{\eta}a_k|z|^{k-1}}{1-\sum\limits_{k=2}^{\infty}\left(\frac{c+1}{c+k}\right)^{\eta}a_k|z|^{k-1}}.$$

Thus

$$\left|\frac{z[V_\mu(f(z))]'}{V_\mu(f(z))}-1\right|\leq 1-\xi,$$

if

$$\sum_{k=2}^{\infty}\left(\frac{c+1}{c+k}\right)^{\eta}\left(\frac{k-\xi}{1-\xi}\right)a_k|z|^{k-1}\leq 1. \tag{6.6}$$

But, by Theorem 1, (6.6) will be true if

$$\left(\frac{c+1}{c+k}\right)^{\eta}\left(\frac{k-\xi}{1-\xi}\right)|z|^{k-1}\leq\psi_B^A(k,\delta,\alpha,\beta,\gamma),$$

that is, if

$$r_1=|z|\leq\left[\left(\frac{c+k}{c+1}\right)^{\eta}\frac{(1-\xi)\psi_B^A(k,\delta,\alpha,\beta,\gamma)}{(k-\xi)}\right]^{\frac{1}{k-1}}. \tag{6.7}$$

Theorem 7 follwos easily from (6.7).

Using arguments similar to the proof of Theorem 6, we obtain the following theorems.

Theorem 8. *(i) Let the function $f(z)$ is starlike of order ρ, then $V_\mu(f(z)$ is also starlike of order α.*

(ii) Let the function $f(z)$ is convex of order ρ, then $V_\mu(f(z)$ is also convex of order α.

Theorem 9. *Let the function $f(z)$ defined by (1.7) be in the class $TF_{\lambda,\ell}^{n,q,s,\delta}(\alpha,\beta,\gamma,A,B)$. Then $V_\mu(f(z))$ is convex of order ξ $(0 \le \xi < 1)$ in the disc $|z| < r_2$, where*

$$r_2 = \inf_{k\ge 2}\left[\left(\frac{c+k}{c+1}\right)^{\eta}\frac{(1-\xi)\psi_B^A(k,\delta,\alpha,\beta,\gamma)}{k(k-\xi)}\right]^{\frac{1}{k-1}}, \tag{6.8}$$

where $\psi_B^A(k,\delta,\alpha,\beta,\gamma)$ is given by (2.2).

Remarks. (i) Putting $n=\delta=0$ and $(B-A)=2(B-A)$ $(-1 \le A < B \le 1)$ in the above results, we obtain the corresponding results obtained by Magesh et al. [15, with $m=0$ and $\alpha=\frac{1}{2}$];
(ii) Putting $n=1$, $\lambda=0$ and $\ell=0$, in the above results, we obtain the corresponding results obtained by Vijaya and Deepa [33]);
(iii) Putting $n=1$, $\lambda=0$ and $\ell=0$, in Theorems 4, 5, 6 and 7, respectively we obtain the corresponding results obtained by Murugusundaramoothy et al. [17, Theorms 4.1, 4.2, 4.3 and 4.4] respectively;
(iv) Specializing the parameters n, q, s, α_1, β_1, λ and ℓ, we obtain results corresponding to the classes $TF_{\lambda,\ell}^{q,s}(\alpha,\beta,\gamma,A,B)$, $TF_{\lambda}^{n,\alpha}(\alpha,\beta,\gamma,A,B)$, $IF_{a,c,\lambda}^{n}(\alpha,\beta,\gamma,A,B)$, $IF^{n}(\lambda,\ell)(\alpha,\beta,\gamma,A,B)$, $TF_{\lambda}^{n}(\alpha,\beta,\gamma,A,B)$ and $TF^{n}(\alpha,\beta,\gamma,A,B)$, mentioned in the introduction.

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