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A NOTE ON SUBCLASSES OF STARLIKE AND CONVEX FUNCTIONS ASSOCIATED WITH BESSEL FUNCTIONS

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Abstract. The purpose of the present paper is to investigate some characterization for generalized Bessel functions of first kind to be in the new subclasses $\mathscr{P}_\lambda(\alpha)$ and $\mathscr{Q}_\lambda(\alpha)$ of analytic functions.

Keywords. Starlike functions; Convex functions; Uniformly starlike functions; Uniformly convex functions; Hadamard product.

1. Introduction

We recall here a generalized Bessel function $\omega(p,b,c) = \omega$ defined in [2] and given by

$$\omega(z) = \omega_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n\, c^n}{n!\,\Gamma(p+n+\frac{b+1}{2})}\left(\frac{z}{2}\right)^{2n+p}, \tag{1.1}$$

which is the particular solution of the second order linear homogeneous differential equation

$$z^2\omega''(z) + bz\omega'(z) + [cz^2 - p^2 + (1-b)p]\omega(z) = 0, \tag{1.2}$$

where $b,p,c \in \mathbb{C}$, which is natural generalization of Bessel's equation. The differential equation (1.2) permits the study of Bessel function, modified Bessel function, spherical Bessel function and modified spherical Bessel functions all together. Solutions of (1.2) are referred to as the generalized Bessel function of order p. The particular solution given by (1.1) is called the

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generalized Bessel function of the first kind of order p. Although the series defined above is convergent everywhere, the function $\omega_{p,b,c}$ is generally not univalent in $\mathbb{U}$. It is of interest to note that when $b=c=1$, we reobtain the Bessel function $\omega_{p,1,1}=J_p$, and for $c=-1, b=1$ the function $\omega_{p,1,-1}$ becomes the modified Bessel function $\mathscr{I}_p$. Now, consider the function $u_{p,b,c}$ defined by the transformation

$$u_{p,b,c}(z)=2^p\Gamma(p+\frac{b+1}{2})\, z^{\frac{-p}{2}}\ \omega_{p,b,c}\ (z^{\frac{1}{2}}).$$

By using well known Pochhammer symbol (or the shifted factorial) defined, in terms of the familiar Gamma function, by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & when \quad n=0, \\ \\ a(a+1)(a+2)\cdots(a+n-1) & when \quad n\in\mathbb{N}:=\{1,2,3,\cdots\}. \end{cases}$$

We represent $u_{p,b,c}$ as

$$u_{p,b,c}(z)=\sum_{n=0}^{\infty}\frac{(\frac{-c}{4})^n}{\left(p+\frac{b+1}{2}\right)_n}\left(\frac{z^n}{n!}\right),$$

where $\left(p+\frac{b+1}{2}\right)\neq 0,-1,-2,....$ This function is analytic on $\mathbb{C}$ and satisfies the second-order linear differential equation

$$4z^2u''(z)+2(2p+b+1)zu'(z)+cu(z)=0.$$

Let $\mathscr{A}$ be the class of analytic functions in the unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \quad z\in\mathbb{U}. \tag{1.3}$$

As usual, we denote by $\mathscr{S}$ the subclass of $\mathscr{A}$ consisting of functions which are normalized by $f(0)=0=f'(0)-1$ and also univalent in $\mathbb{U}$. Denote by $\mathscr{T}$ the subclass of $\mathscr{A}$ consisting of functions whose non-zero coefficients from second on, is given by

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n, \tag{1.4}$$

Also for functions $f\in\mathscr{A}$ given by (1.3) and $g\in\mathscr{A}$ given by $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$, we define the Hadamard product (or convolution) of f and g by

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n,\ \ z\in\mathbb{U}. \tag{1.5}$$

A function $f \in \mathscr{A}$ is said to be starlike of order α $(0 \leq \alpha < 1)$, if and only if $\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in \mathbb{U})$. This function class is denoted by $\mathscr{S}^*(\alpha)$. We also write $\mathscr{S}^*(0) =: \mathscr{S}^*$, where $\mathscr{S}^*$ denotes the class of functions $f \in \mathscr{A}$ that $f(\mathbb{U})$ is starlike with respect to the origin. A function $f \in \mathscr{A}$ is said to be convex of order α $(0 \leq \alpha < 1)$ if and only if $\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad (z \in \mathbb{U})$. This class is denoted by $\mathscr{K}(\alpha)$. Further, $\mathscr{K} = \mathscr{K}(0)$, the well-known standard class of convex functions. It is an established fact that $f \in \mathscr{K}(\alpha) \Longleftrightarrow zf' \in \mathscr{S}^*(\alpha)$.

A function $f \in \mathscr{A}$ is said to be in the class $\mathscr{R}^\tau(A,B)$, $(\tau \in \mathbb{C}\backslash\{0\},\ -1 \leq B < A \leq 1)$, if it satisfies the inequality

$$\left|\frac{f'(z)-1}{(A-B)\tau - B[f'(z)-1]}\right| < 1 \quad (z \in \mathbb{U}).$$

The class $\mathscr{R}^\tau(A,B)$ was introduced earlier by Dixit and Pal [9]. Two of the many interesting subclasses of the class $\mathscr{R}^\tau(A,B)$ are worthy of mention here. If we put

$$\tau = 1,\ A = \alpha \text{ and } B = -\alpha\ (0 < \alpha \leq 1),$$

we obtain the class of functions $f \in \mathscr{A}$ satisfying the inequality

$$\left|\frac{f'(z)-1}{f'(z)+1}\right| < \alpha \quad (z \in \mathbb{U}; 0 < \alpha \leq 1),$$

which was studied by (among others) Padmanabhan [13] and Caplinger and Causey [6].

Lemma 1.1. *A function $f(z)$ of the form (1.3) belongs to the class $\mathscr{R}^\tau(A,B)$ then*

$$|a_n| \leq \frac{(A-B)|\tau|}{n}, (n \in \mathbb{N} \setminus \{1\}). \tag{1.6}$$

The bounds given in (1.6) is sharp.

In this paper, motivated by the earlier work of Srivastava *et al.* [17] and Silverman [15] involving hypergeometric functions and recent work of Porwal and Dixit [14] involving Bessel function, we consider two subclasses of $\mathscr{S}$ namely $\mathscr{P}_\lambda(\alpha)$ and $\mathscr{Q}_\lambda(\alpha)$ due to Altintas *et al.* [1], to discuss some inclusion properties involving generalized Bessel functions; see, [2, 3, 4, 5, 11].

We recall the definition and sufficient conditions of the function class $\mathscr{P}_\lambda(\alpha)$ and $\mathscr{Q}_\lambda(\alpha)$ due to Altintas *et al.* [1].

For some α $(0 \leq \alpha < 1)$ and λ $(0 \leq \lambda \leq 1)$, we let $\mathscr{P}_\lambda(\alpha)$ be the subclass of $\mathscr{S}$ satisfying the analytic criteria

$$\mathscr{P}_\lambda(\alpha) := \left\{ f \in \mathscr{S} : \Re\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda z f'(z)} \right) > \alpha,\ z \in \mathbb{U} \right\}$$

and also let $\mathscr{Q}_\lambda(\alpha)$, be the subclass of $\mathscr{S}$ satisfying the analytic criteria

$$\mathscr{Q}_\lambda(\alpha) := \left\{ f \in \mathscr{S} : \Re\left(\frac{\lambda z^3 f'''(z) + (1+2\lambda) z^2 f''(z) + zf'(z)}{zf'(z) + \lambda z^2 f''(z)} \right) > \alpha, z \in \mathbb{U} \right\}.$$

Also denote $\mathscr{P}^*_\lambda(\alpha) = \mathscr{P}_\lambda(\alpha) \cap \mathscr{T}$ and $\mathscr{Q}^*_\lambda(\alpha) = \mathscr{Q}_\lambda(\alpha) \cap \mathscr{T}$, the subclasses of $\mathscr{T}$[16].

Remark 1.1. [16] *For some* $\alpha(0 \leq \alpha < 1)$*, and choosing* $\lambda = 0$ *and functions of the form (1.4), we note that* $\mathscr{P}^*_0(\alpha) \equiv \mathscr{S}^*(\alpha)$.

Remark 1.2. [16] *For choosing* $\lambda = 1$ *and functions of the form (1.4), we note that* $\mathscr{P}^*_1(\alpha) \equiv \mathscr{K}^*(\alpha)$. *We also note that* $\mathscr{P}^*_1(\alpha) \equiv \mathscr{K}^*(\alpha) \equiv \mathscr{Q}_0(\alpha)$.

Example 1.1. *For some* $\alpha(0 \leq \alpha < 1)$ *and choosing* $\lambda = 1$ *and functions of the form (1.4), we let* $\mathscr{Q}_1(\alpha) \equiv \mathscr{M}_(\alpha)$ *the subclass of* $\mathscr{S}$ *satisfying the analytic criteria*

$$\mathscr{M}(\alpha) := \left\{ f \in \mathscr{S} : \Re\left(\frac{z^3 f'''(z) + 3z^2 f''(z) + zf'(z)}{zf'(z) + z^2 f''(z)} \right) > \alpha, z \in \mathbb{U} \right\}.$$

Also denote that $\mathscr{M}^*_\lambda(\alpha) = \mathscr{M}_\lambda(\alpha) \cap \mathscr{T}$.

We recall the following necessary and sufficient conditions for functions $f \in \mathscr{P}^*_\lambda(\alpha)$ due to [1] and by using $zf'(z) \in \mathscr{P}^*_\lambda(\alpha) \Leftrightarrow f \in \mathscr{Q}^*_\lambda(\alpha)$ due to Le Branges [8], $f(z) = z + \sum_{n=2}^\infty n|a_n|z^n \in \mathscr{Q}^*_\lambda(\alpha)$ hence we state the necessary and sufficient conditions for $f \in \mathscr{Q}^*_\lambda(\alpha)$.

Lemma 1.2. *Let a function* $f(z)$ *of the form (1.3). Then*
(i) $f \in \mathscr{P}_\lambda(\alpha)$ [1] *if*

$$\sum_{n=2}^\infty (n\lambda - \lambda + 1)(n - \alpha)|a_n| \leq 1 - \alpha \tag{1.7}$$

and (ii)$f \in \mathscr{Q}_\lambda(\alpha)$ *if*

$$\sum_{n=2}^\infty n(n\lambda - \lambda + 1)(n - \alpha)|a_n| \leq 1 - \alpha. \tag{1.8}$$

Remark 1.3. *Let a function $f(z)$ of the form (1.4). Then*

(i) $f \in \mathscr{P}_\lambda^*(\alpha)$ [1] *if and only if*

$$\sum_{n=2}^{\infty}(n\lambda-\lambda+1)(n-\alpha)|a_n| \leq 1-\alpha \tag{1.9}$$

and (ii) $f \in \mathscr{Q}_\lambda^*(\alpha)$ *if and only if*

$$\sum_{n=2}^{\infty}n(n\lambda-\lambda+1)(n-\alpha)|a_n| \leq 1-\alpha. \tag{1.10}$$

By specializing the parameter $\lambda=0$ and $\lambda=1$ in Lemma 1.1, we get the results of Silverman [16].

Since $\mathscr{Q}_1^*(\alpha) \equiv \mathscr{M}^*(\alpha)$ from (1.10), we state the following lemma without proof.

Lemma 1.4. *A function $f(z)$ of the form (1.4) belongs to the class $\mathscr{M}^*(\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}n^2(n-\alpha)|a_n| \leq 1-\alpha. \tag{1.11}$$

The generalized Bessel function is a recent topic of study in geometric function theory; see, for example, [2, 3, 4, 5] and [11]. Motivated by results on connections between various subclasses of analytic univalent functions by using hypergeometric functions; see [7, 10, 12, 15] and by work of Baricz [2, 3, 4, 5], we obtain sufficient condition for function $z(2-u_p(z))$ belonging to the classes $\mathscr{P}_\lambda^*(\alpha)$ and $\mathscr{Q}_\lambda^*(\alpha)$ and connections between $\mathscr{R}^\tau(A,B)$.

2. Main Results

For convenience throughout in the sequel, we use the following notations

$$u_{p,b,c}(z)=u_p(z);\ m=p+\frac{b+1}{2}\neq 0,-1,-2,.....$$

and also let

$$zu_p(z)=z+\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n-1)!}z^n \tag{2.1}$$

and

$$z(2-u_p(z))=z-\sum_{n=2}^{\infty}\frac{(\frac{-c}{4})^{n-1}}{(m)_{n-1}\,(n-1)!}a_nz^n. \tag{2.2}$$

Further from (2.1), we get

$$z^2u_p'(z)+zu_p(z)=z+\sum_{n=2}^{\infty}n\frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n-1)!}z^n. \tag{2.3}$$

Differentiating (2.3), we obtain

$$z^3u_p''(z)+3z^2u_p'(z)+zu_p(z)=z+\sum_{n=2}^{\infty}n^2\frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n-1)!}z^n. \tag{2.4}$$

Differentiating (2.4) with respect to z, we obtain

$$z^4u_p'''(z)+7z^3u_p''(z)+6z^2u_p'(z)+zu_p(z)=z+\sum_{n=2}^{\infty}n^3\frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n-1)!}z^n. \tag{2.5}$$

Theorem 2.1. *If $c<0, m>0 (m\neq 0,-1,-2,...)$. Then $zu_p(z)\in\mathscr{P}_\lambda(\alpha)$ if*

$$\lambda u_p''(1)+[1-\lambda\alpha+2\lambda)]u_p'(1)+(1-\alpha)u_p(1)\leq 2(1-\alpha). \tag{2.6}$$

Proof. Since

$$zu_p(z)=z+\sum_{n=2}^{\infty}\frac{(\frac{-c}{4})^{n-1}}{(m)_{n-1}\,(n-1)!}a_nz^n,$$

by virtue of Lemma1.2 and (1.7) it suffices to show that

$$\sum_{n=2}^{\infty}(n\lambda-\lambda+1)(n-\alpha)\left(\frac{(\frac{-c}{4})^{n-1}}{(m)_{n-1}\,(n-1)!}\right)\leq 1-\alpha.$$

Letting

$$L(n,\lambda,\alpha)=\sum_{n=2}^{\infty}(n\lambda-\lambda+1)(n-\alpha)\left(\frac{(\frac{-c}{4})^{n-1}}{(m)_{n-1}\,(n-1)!}\right),$$

we can rewrite the above term as

$$L(n,\lambda,\alpha)=\lambda\sum_{n=2}^{\infty}n^2\left(\frac{(\frac{-c}{4})^{n-1}}{(m)_{n-1}\,(n-1)!}\right)+[1-\lambda(1+\alpha)]\sum_{n=2}^{\infty}n\left(\frac{(\frac{-c}{4})^{n-1}}{(m)_{n-1}\,(n-1)!}\right)$$
$$+\alpha(\lambda-1)\sum_{n=2}^{\infty}\left(\frac{(\frac{-c}{4})^{n-1}}{(m)_{n-1}\,(n-1)!}\right).$$

From (2.1), (2.3) and for $|z|=1$, we have

$$u_p(1)-1=\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n-1)!}$$

$$u_p'(1)+u_p(1)-1=\sum_{n=2}^{\infty}n\frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n-1)!}$$

$$u_p''(1)+3u_p'(1)+u_p(1)-1=\sum_{n=2}^{\infty} n^2 \frac{(-c/4)^{n-1}}{(m)_{n-1}\,(n-1)!}.$$

Hence we get

$$\begin{aligned} & L(n,\lambda,\alpha) \\ = \quad & \lambda(u_p''(1)+3u_p'(1)+u_p(1)-1)+[1-\lambda(1+\alpha)](u_p'(1)+u_p(1)-1) \\ + \quad & \alpha(\lambda-1)(u_p(1)-1) \\ = \quad & \lambda u_p''(1)+(1-\lambda\alpha+2\lambda)u_p'(1)+(1-\alpha)[u_p(1)-1]. \end{aligned}$$

But this expression is bounded above by $1-\alpha$ if (2.6) holds. Thus the proof is completed.

Remark 2.1. *The condition (2.6) is both necessary and sufficient if* $z(2-u_p(z)) \in \mathscr{P}_\lambda^*(\alpha)$.

Theorem 2.2. *If* $c<0, m>0 (m \neq 0,-1,-2,...$ *then* $zu_p(z) \in \mathscr{Q}_\lambda(\alpha)$ *if*

$$\lambda u_p'''(1)+[5\lambda-\lambda\alpha+1]u_p''(1)+[4\lambda-2\lambda\alpha+3-\alpha]u_p'(1)+(1-\alpha)u_p(1) \leq 2(1-\alpha). \tag{0.1}$$

Proof. Let f be of the form (1.3) belong to the class $\mathscr{S}$. By virtue of Lemma 1.2 and (1.8) it suffices to show that

$$\sum_{n=2}^{\infty} n(n\lambda-\lambda+1)(n-\alpha)\left(\frac{(\frac{-c}{4})^{n-1}}{(m)_{n-1}\,(n-1)!}\right) \leq 1-\alpha.$$

$$S(n,\lambda,\alpha)=\lambda \sum_{n=2}^{\infty} n^3\left(\frac{(\frac{-c}{4})^{n-1}}{(m)_{n-1}\,(n-1)!}\right)+[1-\lambda(1+\alpha)]\sum_{n=2}^{\infty} n^2\left(\frac{(\frac{-c}{4})^{n-1}}{(m)_{n-1}\,(n-1)!}\right) + \alpha(\lambda-1)\sum_{n=2}^{\infty} n\left(\frac{(\frac{-c}{4})^{n-1}}{(m)_{n-1}\,(n-1)!}\right).$$

From (2.3)-(2.5) and for $|z|=1$ and proceeding as in Theorem 2.1, we have

$$S(n,\lambda,\alpha)=\lambda[u_p'''(1)+7u_p''(1)+6u_p'(1)+u_p(1)-1] + (1-\lambda(1+\alpha))[u_p''(1)+3u_p'(1)+u_p(1)-1]+\alpha(\lambda-1)[u_p'(1)+u_p(1)-1]$$

and

$$S(n,\lambda,\alpha)=\lambda u_p'''(1)+(5\lambda-\lambda\alpha+1)u_p''(1)+(4\lambda-2\lambda\alpha+3-\alpha)u_p'(1)+(1-\alpha)(u_p(1)-1).$$

But this expression is bounded above by $1-\alpha$ if (2.7) holds. Thus the proof is completed.

Remark 2.2. *The condition (2.7) is both necessary and sufficient if* $z(2-u_p(z)) \in \mathcal{Q}^*_\lambda(\alpha)$.

Corollary 2.1. *If* $c<0, m>0 (m \neq 0,-1,-2,...)$ *then* $z(2-u_p(z))$*, is in* $\mathcal{M}^*(\alpha)$ *if and only if*

$$u_p'''(1)+(6-\alpha)u_p''(1)+(7-3\alpha)u_p'(1)+(1-\alpha)u_p(1) \leq 2(1-\alpha).$$

The proof follows by taking taking $\lambda = 1$ in above theorem.

3. Inclusion Properties

Now, we considered the linear operator

$$\mathcal{I}(c,m) : \mathcal{A} \to \mathcal{A}$$

defined by

$$\mathcal{I}(c,m)f(z) = zu_{p,b,c}(z) * f(z) = z + \sum_{n=2}^{\infty} \frac{(\frac{-c}{4})^{n-1}}{(m)_n \ (n-1)!} a_n z^n,$$

where $m = \left(p + \frac{b+1}{2}\right)$.

Making use of the Lemma 1.1, we will study the action of the Bessel function on the class $\mathcal{Q}_\lambda(\alpha)$.

Theorem 3.1. *Let* $c<0, m>0 (m \neq 0,-1,-2,...)$. *If* $f \in \mathcal{R}^\tau(A,B)$*, and if the inequality*

$$(A-B)|\tau|\{\lambda u_p''(1)+(1-\lambda\alpha+2\lambda)u_p'(1)+(1-\alpha)[u_p(1)-1]\} \leq 1-\alpha \qquad (3.1)$$

is satisfied, then $\mathcal{I}(c,m)(f) \in \mathcal{Q}_\lambda(\alpha)$.

Proof. Let f be of the form (1.3) belong to the class $\mathcal{R}^\tau(A,B)$. By virtue of Lemma 1.2 it suffices to show that

$$\sum_{n=2}^{\infty} n(n\lambda-\lambda+1)(n-\alpha)\left(\frac{(\frac{-c}{4})^{n-1}}{(m)_{n-1} \ (n-1)!}\right)|a_n| \leq 1-\alpha.$$

Since $f \in \mathcal{R}^\tau(A,B)$, we find from Lemma 1.1 that

$$|a_n| \leq (A-B)\frac{|\tau|}{n}.$$

Hence, we have

$$S(n,\lambda,\alpha) = \sum_{n=2}^{\infty} n(n\lambda-\lambda+1)(n-\alpha)\left(\frac{(\frac{-c}{4})^{n-1}}{(m)_{n-1} \ (n-1)!}\right)|a_n|$$

$$=(A-B)|\tau|\sum_{n=2}^{\infty}(n\lambda-\lambda+1)(n-\alpha)\left(\frac{(\frac{-c}{4})^{n-1}}{(m)_{n-1}\ (n-1)!}\right)$$

$$\begin{aligned} &= (A-B)|\tau|\lambda\sum_{n=2}^{\infty}n^2\left(\frac{(\frac{-c}{4})^{n-1}}{(m)_{n-1}\ (n-1)!}\right)+(A-B)|\tau|[1-\lambda(1+\alpha)]\sum_{n=2}^{\infty}n\left(\frac{(\frac{-c}{4})^{n-1}}{(m)_{n-1}\ (n-1)!}\right)\\ &+ (A-B)|\tau|\alpha(\lambda-1)\sum_{n=2}^{\infty}\left(\frac{(\frac{-c}{4})^{n-1}}{(m)_{n-1}\ (n-1)!}\right). \end{aligned}$$

Using (2.3), (2.4) and for $|z|=1$, we have

$$\begin{aligned} S(n,\lambda,\alpha) &\leq (A-B)|\tau|\{\lambda[u_p''(1)+3u_p'(1)+u_p(1)-1]+(1-\lambda(1+\alpha))[u_p'(1)+u_p(1)-1]\\ &\quad +\alpha(\lambda-1)[u_p(1)-1]\}\\ &= (A-B)|\tau|\{\lambda u_p''(1)+(1-\lambda\alpha+2\lambda)u_p'(1)+(1-\alpha)[u_p(1)-1]\}\leq 1-\alpha. \end{aligned}$$

But this expression is bounded above by $1-\alpha$ if and only if (3.1) holds. Thus the proof is completed.

Corollary 3.1. *Let $c<0, m>0 (m\neq 0,-1,-2,...)$. If $f\in\mathscr{R}^{\tau}(A,B)$, and if the inequality*

$$(A-B)|\tau|\{u_p''(1)+(3-\alpha)u_p'(1)+(1-\alpha)[u_p(1)-1]\}\leq 1-\alpha \tag{3.2}$$

is satisfied, then $\mathscr{I}(c,m)(f)\in\mathscr{M}(\alpha)$.

Remark 3.1. *The above conditions are also necessary for functions $z(2-u_p(z))$ of the form (2.2).*

Theorem 3.2. *Let $c<0, m>0 (m\neq 0,-1,-2,\dots)$, then*

$$\mathscr{L}(m,c,z)=\int_0^z(2-u_p(t))dt$$

is in $\mathscr{Q}_{\lambda}^{*}(\alpha)$ if and only if

$$\lambda u_p''(1)+[1-\lambda\alpha+2\lambda]u_p'(1)+(1-\alpha)u_p(1)\leq 2(1-\alpha). \tag{0.2}$$

Proof. Note that

$$\mathscr{L}(m,c,z)=z-\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}}\ \frac{z^n}{n!}.$$

Using Theorem 2.1, we only need to show that

$$\sum_{n=2}^{\infty} n(n\lambda-\lambda+1)(n-\alpha)\left(\frac{(\frac{-c}{4})^{n-1}}{(m)_{n-1}\; n!}\right) \leq 1-\alpha.$$

Note that

$$\sum_{n=2}^{\infty} n(n\lambda-\lambda+1)(n-\alpha)\left(\frac{(\frac{-c}{4})^{n-1}}{(m)_{n-1}\; n!}\right) \quad = \quad \sum_{n=2}^{\infty}(n\lambda-\lambda+1)(n-\alpha)\left(\frac{(\frac{-c}{4})^{n-1}}{(m)_{n-1}\;(n-1)!}\right).$$

Proceeding as in Theorem 2.1, we get

$$\sum_{n=2}^{\infty} n(n\lambda-\lambda+1)(n-\alpha)\left(\frac{(\frac{-c}{4})^{n-1}}{(m)_{n-1}\; n!}\right) = \lambda u_p''(1)+[1-\lambda\alpha+2\lambda]u_p'(1)+(1-\alpha)u_p(1),$$

which is bounded above by $1-\alpha$ if and only if (3.3) holds.

Corollary 3.2. *Let $c<0, m>0 (m \neq 0,-1,-2,\dots)$. Then*

$$\mathscr{L}(m,c,z) = \int_0^z (2-u_p(t))dt$$

is in $\mathscr{M}^*(\alpha)$ if and only if

$$u_p''(1)+(3-\alpha)u_p'(1)+(1-\alpha)u_p(1) \leq 2(1-\alpha).$$

Remark 3.2. *By taking $\lambda=0$ or $\lambda=1$ one can deduce above results obtained in* [5] *for various subclasses studied in* [16].

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