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SOME EXTENSION RESULTS OF A COUPLED SYSTEM OF 3D AXISYMMETRIC INVISCID STAGNATION FLOWS

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Abstract. A system of two integral equations is presented to describe a coupled three-dimensional differential system related to Navier-Stokes equations. By the study of positive solutions of this integral system, we prove the existence of solutions analytically and obtain upper and lower bounds of the shear stress functions for the system. Some recent results are extended.

Keywords. Three-dimensional flows; Systems; Existence of solutions; Upper and lower bounds.

1. Introduction

In this paper, we investigate analytically the existence and upper and lower bounds of second derivative of solutions for the following coupled system of two third order differential equations when $\lambda \geq 1$

$$f'''(\eta)+(f(\eta)+\lambda g(\eta))f''(\eta)+(1-f'^2(\eta))=0 \quad on \quad \mathbb{R}^+, \tag{1.1}$$

$$g'''(\eta)+(f(\eta)+\lambda g(\eta))g''(\eta)+\lambda(1-g'^2(\eta))=0 \quad on \quad \mathbb{R}^+ \tag{1.2}$$

with boundary conditions

$$f(0)=0, f'(0)=0, f'(\infty)=1, g(0)=0, g'(0)=0, g'(\infty)=1, \tag{1.3}$$

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where $\mathbb{R}^+ = [0,\infty)$.

The system (1.1)-(1.3) has been used to describe the three-dimensional (3D) axisymmetric inviscid stagnation flows related to Navier-Stokes equations; see [1] (Eq. (8)-(11)) or [2] (Eq. (12)-(14)) or [3] (Eq. (3.7)-(3.9)). λ is a parameter related to the external flow components. A solution of (1.1)-(1.3) is called a similarity solution, $f(\eta)$ and $g(\eta)$ are the similarity stream functions, $f'(\eta)$ and $g'(\eta)$ are the velocity functions and $f''(\eta)$ and $g''(\eta)$ are the shear stress (or the skin friction) functions.

Regarding the study of (1.1)-(1.3), Howarth [2] presented numerical study for the case $0 < \lambda < 1$ which can be applied to the stagnation region of an ellipsoid. Davey [3] investigated numerically the stagnation region near a saddle point $(-1 < \lambda < 0)$. The two-dimensional cases, $\lambda = g = 0$ or $\lambda = 1$ and $g = f$, the special cases of the Falkner-Skan equations, were solved by Hiemenz and Homann [1], respectively. Some recent results of the Falkner-Skan equations and other boundary layer problems can be found in [4]-[11].

Very recently, the authors [12] studied analytically the existence of solutions of the system (1.1)-(1.3) when $\lambda < 1$ (but not all) via introducing an integral system and studying the existence of positive solutions of the integral system.

A natural question arises: is there an analytic solution of (1.1)-(1.3) when $\lambda \geq 1$?

It is well-known that numerical and analytical study of the shear stress (or the skin friction) functions is of great importance, one may refer to, for example, [13]-[16] and references therein.

In this paper, we study the existence of solutions of (1.1)-(1.3) when $\lambda \geq 1$ and present the analytical results for the shear stress functions.

This paper is organized as follows. In Section 2, we introduce an integral system (see (2.8) and (2.9)) to describe the system (1.1)-(1.3) under suitable conditions. In Sections 3, we prove the existence of solutions of (1.1)-(1.3) when $\lambda \geq 1$. In Sections 4, we present the analytical results for the shear stress functions.

2. An integral system describing (1.1)-(1.3)

Our work is restricted to $\lambda \geq 1$. We first prove the following proposition and introduce a system of two integral equations to describe (1.1)-(1.3) under suitable conditions.

Proposition 2.1. *Let $(f,g)\in C^3(\mathbb{R}^+)\times C^3(\mathbb{R}^+)$ be a solution of (1.1)-(1.3) satisfying $f''(\eta)>0$ and $g'(\eta)\geq 0$ on $\mathbb{R}^+$. Then the following assertions hold:*

(P_1) *$g''(\eta)\geq 0$ and g'' is decreasing on $\mathbb{R}^+$.*

(P_2) *$0\leq f'(\eta)\leq g'(\eta)\leq 1$ on $\mathbb{R}^+$.*

(P_3) *$f''(\eta)$ is decreasing on $\mathbb{R}^+$ and $\lim_{\eta\to\infty} f''(\eta)=0$.*

Proof. (P_1) We first prove $g'(\eta)\leq 1$ and $g''(\eta)\geq 0$ on $\mathbb{R}^+$. In fact, if there exists $\eta_0\in\mathbb{R}^+$ such that $g'(\eta_0)>1$, by $g'(0)=0$ and $g'(\infty)=1$, we know that there is $\eta_1\in\mathbb{R}^+$ to satisfy $\eta_1>\eta_0$ and $g'(\eta_1)<g'(\eta_0)$. Let $\xi\in(0,\eta_1)$ such that

$$g'(\xi)=\max\{g''(\eta):\eta\in[0,\eta_1]\}>1.$$

Hence, $g''(\xi)=0$. The test of second derivative shows $g'''(\xi)\leq 0$. However, by (1.2), we have $g'''(\xi)=-\lambda(1-g'^2(\xi))>0$, which is a contradiction. If there exists $\eta_0\in\mathbb{R}^+$ such that $g''(\eta_0)<0$, by $g'(\eta_0)\leq 1$, we know that there exists $\eta_1\in\mathbb{R}^+$ with $\eta_1>\eta_0$ such that $g'(\eta_1)<g'(\eta_0)\leq 1$. Since $g'(\infty)=1$, we choose $\eta_2\in\mathbb{R}^+$ with $\eta_2>\eta_1$ to satisfy $g'(\eta_2)>g'(\eta_1)$. Let $\xi\in(\eta_0,\eta_2)$ such that

$$0\leq g'(\xi)=\min\{g'(\eta):\eta\in[\eta_0,\eta_2]\}<1.$$

Hence, $g''(\xi)=0$ and $g'''(\xi)\geq 0$. However, by (1.2), we have $g'''(\xi)=-\lambda(1-g'^2(\xi))<0$, which is a contradiction. Hence, $g''(\eta)\geq 0$ on $\mathbb{R}^+$.

Since $0\leq g'(\eta)\leq 1$, $g''(\eta)\geq 0$ and $f''(\eta)>0$, by (1.3), we see easily $g(\eta)\geq 0$, $0\leq f'(\eta)<1$ and $f(\eta)\geq 0$ on $\mathbb{R}^+$. By (1.2), we obtain

$$g'''(\eta)=-(f(\eta)+\lambda g(\eta))g''(\eta)-\lambda(1-g'^2(\eta))\leq 0 \text{ on } \mathbb{R}^+$$

and g'' is decreasing on $\mathbb{R}^+$.

(P_2) It follows from $g''(\eta)\geq 0$ that $g'(\eta)$ is increasing on $\mathbb{R}^+$. This, together with (1.3), implies $0\leq g'(\eta)\leq 1$ on $\mathbb{R}^+$.

If there exists $\eta_0\in\mathbb{R}^+$ such that $f'(\eta_0)>g'(\eta_0)$. Let $\tau(\eta)=f'(\eta)-g'(\eta)$. By $f'(0)=0=g'(0)$ and $f'(\infty)=1=g'(\infty)$, then $\tau(0)=0=\tau(\infty)$ and $\tau(\eta_0)>0$. We choose $\eta_1>\eta_0$ to satisfy $\tau(\eta_1)<\tau(\eta_0)$ and $\xi\in(0,\eta_1)$ such that

$$\tau(\xi)=\max\{\tau(\eta):\eta\in[0,\eta_1]\}>0.$$

Hence $\tau'(\xi)=0$ and $\tau''(\xi)\le 0$. This shows that $f''(\xi)=g''(\xi)$ and $f'(\xi)>g'(\xi)$.

On the other hand, by (1.1) and (1.2), we have

$$\begin{aligned}\tau''(\xi)=f'''(\xi)-g'''(\xi) &= -(1-f'^2(\xi))+\lambda(1-g'^2(\xi))\\ &\ge -(1-f'^2(\xi))+(1-g'^2(\xi))=f'^2(\xi)-g'^2(\xi)>0,\end{aligned}$$

which is a contradiction.

(P_3) It follows from (1.1) that

$$f'''(\eta)=-(f(\eta)+\lambda g(\eta))f''(\eta)-(1-f'^2(\eta))<0 \text{ on } \mathbb{R}^+.$$

This implies that $f''(\eta)$ is decreasing on $\mathbb{R}^+$ and $\lim_{\eta\to\infty} f''(\eta)$ exists.

Since $f'(+\infty)=1$ implies $\liminf_{\eta\to\infty} f''(\eta)=0$, we conclude therefore that $\lim_{\eta\to\infty} f''(\eta)=\liminf_{\eta\to\infty} f''(\eta)=0$. This completes the proof.

For the sake of convenience, we define a few functions

$$\alpha(\lambda,x,t)=1+2t+\lambda x,$$

$$\beta(\lambda,x,t)=t+\lambda x$$

and

$$h(\lambda,x,y,t)=(t^2-1)x+\lambda(1-y^2),$$

where $t\in[0,1]$ and $x,y\in\mathbb{R}=(-\infty,\infty)$.

Let

$$Q^*=Q_1\times Q_2,$$

where

$$Q_1=\{z\in C[0,1]: z(t)>0, t\in[0,1)\},$$

$$Q_2=\{w\in C[0,1]\cap C^1[0,1): 1\ge w(t)\ge t, t\in[0,1)\}$$

and

$$\Gamma=\{(f,g)\in C^3(\mathbb{R}^+)\times C^3(\mathbb{R}^+): f''(\eta)>0, g'(\eta)\ge f'(\eta), \eta\in\mathbb{R}^+\}.$$

Inspired by Theorems 2.1 and 4.1 [12], we prove the following Theorem 2.2.

Theorem 2.2. *(1) If $(f,g)\in\Gamma$ is a solution of (1.1)-(1.3), then*

$$z'(t) = -\int_0^t \frac{\beta(\lambda,w(s),s)}{z(s)}ds - \frac{1-t^2}{z(t)} \text{ for } t\in[0,1) \text{ and } z(1)=0 \tag{2.1}$$

and

$$w''(t) = -\frac{h(\lambda,w'(t),w(t),t)}{z^2(t)} \text{ for } t\in[0,1), w(0)=0 \text{ and } w(1)=1 \tag{2.2}$$

has a solution $(z,w)\in Q^*$, where $z(t)=f''(\eta)$, $w(t)=g'(\eta)$, $t=f'(\eta)$ and $w'(t)z(t)=g''(\eta)$.

(2) If $(z,w)\in Q^*$ is a solution of (2.1)-(2.2), then (1.1)-(1.3) has a solution $(f,g)\in\Gamma$.

Proof. (1) Assume that $(f,g)\in\Gamma$. Let $\eta := \eta(t) = (f')^{-1}(t)$ for $t\in[0,1)$ be the inverse function to $t=f'(\eta):\mathbb{R}^+\to[0,1)$. It follows that f' is strictly increasing on $\mathbb{R}^+$ and $\eta(t) = (f')^{-1}(t):[0,1)\to\mathbb{R}^+$ with $(f')^{-1}(0)=0$, $\lim_{t\to1^-}(f')^{-1}(t)=\infty$. Let $z(t)=f''(\eta)>0$ for $t\in[0,1)$, by Proposition 2.1 (P_3), then $z(t)>0$ for $t\in[0,1)$ and z is continuous on $[0,1)$. By Proposition 2.1 (P_3), $z(1)=\lim_{\eta\to\infty}f''(\eta)=0$. Hence, we have $z(t)\in C[0,1]$ and $z(1)=0$.

Using the Chain Rule to $z(t)=f''(\eta)$, we obtain $f'''(\eta)\frac{d\eta}{dt}=z'(t)$ and by the Inverse Function Theorem, we have

$$\frac{d\eta}{dt} = \frac{1}{f''(\eta)} = \frac{1}{z(t)} \text{ for } t\in[0,1).$$

This, together with $f'(\eta)=t$, implies

$$f'''(\eta)=z'(t)z(t), \quad \eta=\int_0^t \tfrac{1}{z(s)}ds \text{ and } f'(\eta)\tfrac{d\eta}{dt}=\tfrac{t}{z(t)} \text{ for } t\in[0,1).$$

Integrating the last equality from 0 to t implies

$$f(\eta(t)) = \int_0^t \tfrac{s}{z(s)}ds \text{ for } t\in[0,1).$$

Let

$$w(t)=g'(\eta)=g'(\int_0^t \tfrac{1}{z(s)}ds) \text{ for } t\in[0,1).$$

Then $w(t)\in C[0,1)$ and $w(0)=0$. By $g'(\infty)=1$, we know that w is continuous from the left at 1. Hence, we have $w\in C[0,1]$ and $w(1)=1$.

Notice that $g'(\eta)\frac{d\eta}{dt}=\frac{w(t)}{z(t)}, t\in[0,1)$, we have $g(\eta)=\int_0^t\frac{w(s)}{z(s)}ds$. Differentiating $w(t)$ with respect to t, we have

$$w'(t)=g''(\eta)\tfrac{d\eta}{dt}=\tfrac{g''(\eta)}{z(t)} \text{ for } t\in[0,1).$$

From this, we have $g''(\eta) = w'(t)z(t)$ for $\eta \in \mathbb{R}^+$. Differentiating $g''(\eta)$ with respect to t and utilizing $\frac{d\eta}{dt} = \frac{1}{z(t)}$, we have

$$\frac{g'''(\eta)}{z(t)} = w''(t)z(t) + w'(t)z'(t).$$

Hence

$$g'''(\eta) = w''(t)z^2(t) + w'(t)z(t)z'(t).$$

Substituting f, f', f'', f''' and g into (1.1) implies

$$z'(t) = -\int_0^t \frac{\beta(\lambda, w(s), s)}{z(s)} ds - \frac{1-t^2}{z(t)} \quad for \quad t \in [0,1).$$

Hence, (z,w) satisfies (2.1). Substituting g, g', g'', g''' and f into (1.2) implies

$$w''(t)z^2(t) + w'(t)z(t)z'(t) + w'(t)z(t)\int_0^t \frac{\beta(\lambda, w(s), s)}{z(s)} ds + \lambda(1 - w^2(t)) = 0.$$

By $\int_0^t \frac{\beta(\lambda, w(s), s)}{z(s)} ds = \frac{t^2-1}{z(t)} - z'(t)$, we have

$$w''(t) + \frac{h(\lambda, w'(t), w(t), t)}{z^2(t)} = 0$$

and (z,w) satisfies (2.2). Proposition 2.1 (P_2) implies $w(t) \geq t$ for $t \in [0,1]$. Hence, $(z,w) \in Q^*$.

(2) Before utilizing a solution of (2.1)-(2.2) in Q^* to construct a solution of (1.1)-(1.3), we need to prove $\int_0^1 \frac{1}{z(s)} ds = \infty$.

In fact, if $\int_0^1 \frac{1}{z(s)} ds < \infty$. Let $\sigma = \int_0^1 \frac{1}{z(s)} ds$, then $0 < \sigma < \infty$. Integrating (2.1) from t to 1 and utilizing $z(1) = 0$, we have

$$\begin{aligned} z(t) &= \int_t^1 \int_0^s \frac{\beta(\lambda, w(\tau), \tau)}{z(\tau)} d\tau ds + \int_t^1 \frac{1-s^2}{z(s)} ds \\ &\leq \int_t^1 \int_0^s \frac{1+\lambda}{z(\tau)} d\tau ds + (1-t)\int_t^1 \frac{1+s}{z(s)} ds \\ &\leq \int_t^1 \int_0^1 \frac{1+\lambda}{z(\tau)} d\tau ds + (1-t)\int_0^1 \frac{2}{z(s)} ds \\ &= \sigma(3+\lambda)(1-t) \quad for \quad t \in [0,1). \end{aligned}$$

And then $\int_0^1 \frac{1}{z(s)} ds \geq \int_0^1 \frac{ds}{\sigma(3+\lambda)(1-s)} = \infty$, which is a contradiction.

Let

$$\eta := \eta(t) = \int_0^t \frac{1}{z(s)} ds, \quad 0 \leq t < 1. \tag{2.3}$$

Then $\eta(t)$ is strictly increasing on $[0,1)$ and

$$\eta(0)=0, \quad \eta(1-0)=\int_0^1 \frac{1}{z(s)}ds=+\infty.$$

Let $t=h(\eta)$ be the inverse function to $\eta=\eta(t)$. We define the function

$$f(\eta)=\int_0^\eta h(s)ds, \quad g(\eta)=\int_0^\eta w(h(s))ds, \quad \eta\in\mathbb{R}^+.$$

Then

$$f'(\eta)=h(\eta), f(0)=0, f'(0)=0, f'(\infty)=1,$$

$$g'(\eta)=w(h(\eta)), g(0)=0, g'(0)=0, g'(\infty)=1,$$

$g'(\eta)=w(h(\eta))=w(t)\geq t=f'(\eta)$ and (f,g) satisfies (1.3). From (2.3), we have

$$\eta=\eta(f'(\eta))=\int_0^{f'(\eta)} \frac{1}{z(s)}ds, \eta\in\mathbb{R}^+. \tag{2.4}$$

Differentiating (2.4) with respect to η, we have

$$f''(\eta)=z(f'(\eta))=z(t), \eta\in\mathbb{R}^+. \tag{2.5}$$

Then $f''(\eta)>0$ for $\eta\in\mathbb{R}^+$. Differentiating (2.5) with respect to η, we have

$$f'''(\eta)=z'(f'(\eta))f''(\eta)=z'(f'(\eta))z(f'(\eta))=z'(t)z(t), \quad 0\leq t<1. \tag{2.6}$$

By setting $s=f'(\sigma)$ and utilizing $t=f'(\eta)$ and (2.5), we have

$$\begin{aligned}\int_0^t \frac{\beta(\lambda,w(s),s)}{z(s)}ds &= \int_0^{f'(\eta)} \frac{s+\lambda w(s)}{z(s)}ds \\ &= \int_0^\eta (f'(\sigma)+\lambda g'(\sigma))d\sigma = f(\eta)+\lambda g(\eta).\end{aligned} \tag{2.7}$$

It follows from (2.1), (2.5), (2.6) and (2.7) that

$$f'''=-(f+\lambda g)f''+f'^2-1.$$

Hence, (f,g) satisfies (1.1). By (2.3), we have $\frac{dt}{d\eta}=z(t)$. Differentiating $g'(\eta)$ with respect to η, we have

$$g''(\eta)=w'(t)\frac{dt}{d\eta}=w'(t)z(t), \quad g'''(\eta)=w''(t)z^2(t)+w'(t)z'(t)z(t).$$

This, together with (2.1), (2.2) and (2.7) implies

$$\begin{aligned}
& g''' + (f+\lambda g)g'' + \lambda(1-g'^2) \\
= \;& w''(t)z^2(t) + w'(t)z'(t)z(t) + w'(t)z(t)\int_0^t \frac{\beta(\lambda,w(s),s)}{z(s)}ds + \lambda(1-w^2(t)) \\
= \;& w''(t)z^2(t) + w'(t)z(t)[z'(t) + \int_0^t \frac{\beta(\lambda,w(s),s)}{z(s)}ds] + \lambda(1-w^2(t)) \\
= \;& w''(t)z^2(t) + (t^2-1)w'(t) + \lambda(1-w^2(t)) \\
= \;& z^2(t)[w''(t) + \frac{h(\lambda,w'(t),w(t),t)}{z^2(t)}] = 0.
\end{aligned}$$

Hence, (f,g) satisfies (1.2). Obviously, $(f,g)\in\Gamma$ and the proof is completed.

The following theorem shows an relation between the system (2.1)-(2.2) and a system of two integral equations that is used to prove Theorems 3.6 and 4.1.

Theorem 2.3. *Let $(z,w)\in Q^*$. Then (z,w) is a solution of (2.1)-(2.2) if and only if (z,w) satisfies*

$$z(t) = \int_t^1 \frac{\alpha(\lambda,w(s),s)(1-s)}{z(s)}ds + (1-t)\int_0^t \frac{\beta(\lambda,w(s),s)}{z(s)}ds, \tag{2.8}$$

$$w(t) = \int_0^b G_{0,b}(t,s)\frac{h(\lambda,w'(s),w(s),s)}{z^2(s)}ds + \frac{tw(b)}{b} \text{ for any } b\in(0,1), \tag{2.9}$$

where $G_{0,b}(t,s)$ denotes Green function for $u''(t)=0$ with $u(0)=0$ and $u(b)=0$ defined by

$$G_{0,b}(t,s) = \begin{cases} \frac{t(b-s)}{b}, & 0\le t\le s\le b, \\ \frac{s(b-t)}{b}, & 0\le s\le t\le b. \end{cases} \tag{2.10}$$

Proof. Let $(z,w)\in Q^*$ be a solution of (2.1)-(2.2). Since two terms in the right hand of (2.1) have same sign, integrating (2.1) from t to 1 and utilizing

$$\int_t^1\int_0^\sigma \frac{\beta(\lambda,w(s),s)}{z(s)}dsd\sigma = \int_0^t\int_t^1 \frac{\beta(\lambda,w(s),s)}{z(s)}d\sigma ds + \int_t^1\int_s^1 \frac{\beta(\lambda,w(s),s)}{z(s)}d\sigma ds,$$

we have

$$\begin{aligned}
z(1)-z(t) &= -\int_t^1\int_0^\sigma \frac{\beta(\lambda,w(s),s)}{z(s)}dsd\sigma - \int_t^1 \frac{1-s^2}{z(s)}ds \\
&= -\int_t^1 \frac{\alpha(\lambda,w(s),s)(1-s)}{z(s)}ds - (1-t)\int_0^t \frac{\beta(\lambda,w(s),s)}{z(s)}ds.
\end{aligned}$$

By $z(1)=0$, then

$$z(t)=\int_t^1 \frac{\alpha(\lambda,w(s),s)(1-s)}{z(s)}ds+(1-t)\int_0^t \frac{\beta(\lambda,w(s),s)}{z(s)}ds$$

and (z,w) satisfies (2.8). Utilizing (2.10) and $w(0)=0$, we know that (z,w) satisfies (2.9). Conversely, we obtain easily $z(1)=0$. In fact, if $z(1)\neq 0$, then $z(t)>0$ on $[0,1]$ since $z(t)>0$ on $[0,1)$. This implies that two integrands in the right side of (2.8) are continuous, we have $\lim_{t\to 1^-} z(t)=0$, a contradiction. Differentiating (2.9) with t, we know that (z,w) satisfies (2.2).

Since $t\leq w(t)\leq 1$ for $[0,1)$, we have $w(1)=1$. Putting $t=0$ in (2.8), we obtain $w(0)=0$. For $t\in[0,1)$, taking $b\in(t,1)$ and differentiating (2.9) with t twice, we know that (z,w) satisfies (2.2).

Combining Theorems 2.2 and 2.3, we obtain the following equivalent result between (1.1)-(1.3) and (2.8)-(2.9) as follows.

Theorem 2.4. *The system of (1.1)-(1.3) has a solution $(f,g)\in\Gamma$ if and only if the system of (2.8)-(2.9) has a solution $(z,w)\in Q^*$.*

3. Existence of solutions of (1.1)-(1.3)

In this section, we study the existence of solutions of (1.1)-(1.3).

We denote the norms of the Banach spaces $C[0,1]$ and $C[0,1]\times C^1[0,1]$ by

$$\|z\|=\max\{|z(t)|:t\in[0,1]\}, \|w\|=\|w\|+\|w'\|,$$

respectively.

In order to study the existence of solutions of (2.8)-(2.9), we define a map F from $C[0,1]\times C^1[0,1]$ into $C[0,1]\times C^1[0,1]$ by

$$F(z,w)=(A(z,w),B(z,w)),$$

where

$$A(z,w)(t)=S(z,w)(t)+(1-t)T(z,w)(t)+\frac{1}{n},$$

$$B(z,w)(t)=\int_0^1 G_{0,1}(t,s)\frac{h(\lambda,w'(s),\theta w(s),s)}{\varphi^2(z(s),n)}ds+t,$$

$$S(z,w)(t)=\int_t^1 \frac{\alpha(\lambda,w(s),s)(1-s)}{\varphi(z(s),n)}ds,\quad T(z,w)(t)=\int_0^t \frac{\beta(\lambda,w(s),s)}{\varphi(z(s),n)}ds,$$

$$\varphi(z(t),n)=\max\{z(t),c(t),\frac{1}{n}\},\quad \theta w(t)=\max\{w(t),0\},\quad c(t)=\frac{1}{2}(1-t),t\in[0,1]$$

and $n>1$ is a natural number.

We first prove the following preliminary results.

Lemma 3.1. *Let* $(\lambda,z,w)\in[1,\infty)\times C[0,1]\times C^1[0,1]$ *and* $0<\mu\le 1$ *such that* $(z,w)=\mu F(z,w)$*, that is,*

$$z(t)=\mu A(z,w)(t),\quad w(t)=\mu B(z,w)(t). \tag{3.1}$$

Then the following assertions hold:

(i) $0\le w(t)\le 1$ *for* $t\in[0,1]$.

(ii) $0\le z(t)\le 9+4\lambda$ *for* $t\in[0,1]$.

(iii) *Let* $\xi\in(0,1]$ *such that* $w'(\xi)\ge 0$. *Then* $w'(t)\ge 0$ *on* $[0,\xi]$. *Hence, if* $\nu\in(0,1)$ *such that* $w'(\nu)<0$, *then* $w'(t)<0$ *on* $[\nu,1]$.

(vi) *If* $\mu=1$, *then* $t\le w(t)\le 1$ *and* $w'(t)\ge 0$ *on* $[0,1]$.

Proof. (i) If there exists $t_0\in(0,1)$ such that $w(t_0)>1$, it follows from $w(0)=0<\mu=w(1)\le 1$ that there exists $t_*\in(0,1)$ such that

$$w(t_*)=\max\{w(t):t\in[0,1]\}>1.$$

Differentiating $w(t)$ with t twice, we have

$$w''(t)=-\mu\frac{h(\lambda,w'(t),\theta w(t),t)}{\varphi^2(z(t),n)}=-\mu\frac{(t^2-1)w'(t)+\lambda(1-(\theta w)^2(t))}{\varphi^2(z(t),n)}. \tag{3.2}$$

It follows from $w'(t_*)=0$ and (3.2) that

$$w''(t_*)=-\frac{\mu\lambda(1-w^2(t_*))}{\varphi^2(z(t_*),n)}>0,$$

which contradicts $w''(t_*)\le 0$ by the test of second derivative .

If there exists $t_0\in(0,1)$ such that $w(t_0)<0$, by $w(0)=0$ and $w(1)=\mu>0$, we know that there exists $t_*\in(0,1)$ such that

$$w(t_*)=\min\{w(t):t\in[0,1]\}<0.$$

By $w'(t_*) = 0$, $0 \le \theta w(t_*) < 1$ and (3.2), we know

$$w''(t_*) = -\frac{\mu\lambda(1-(\theta w)^2(t_*))}{\varphi^2(z(t_*),n)} < 0,$$

which contradicts $w''(t_*) \ge 0$. Hence (i) holds.

(ii) From (i), we obtain $0 \le \alpha(\lambda, w(s), s) \le 3+\lambda$ and $0 \le \beta(\lambda, w(s), s) \le 1+\lambda$. This, together with $\varphi(z(t),n) \ge c(t)$ and $1-t \le 1-s$ when $t \ge s$, implies

$$0 \le S(z,w)(t) \le \int_t^1 \frac{\alpha(\lambda,w(s),s)(1-s)}{c(s)}ds \le \int_0^1 \frac{(3+\lambda)(1-s)}{c(s)}ds = 6+2\lambda,$$

$$T(z,w)(t) \le \int_0^t \frac{\beta(\lambda,w(s),s)(1-s)}{c(s)}ds \le \int_0^1 \frac{(1+\lambda)(1-s)}{c(s)}ds = 2+2\lambda.$$

Hence

$$0 \le z(t) \le A(z,w)(t) \le S(z,w)(t) + (1-t)T(z,w)(t) + 1 \le 9+4\lambda$$

and (ii) holds

(iii) It follows from $w(0) = 0$ and (i) that $w'(0) \ge 0$. If there exists $t \in (0,\xi)$ such that $w'(t) < 0$, then there must be $t_* \in (0,\xi)$ such that

$$w'(t_*) = \min\{w'(t) : t \in [0,\xi] < 0$$

and $w''(t_*) = 0$. However, by (3.2) and (i), we obtain

$$w'(t_*) = \frac{\lambda(1-(\theta w)^2(t_*))}{1-t_*^2} \ge 0,$$

which is a contradiction. Hence (iii) holds.

(vi) Assume that $\mu = 1$ and there exists $t_0 \in (0,1)$ such that $w(t_0) < t_0$. Let $\psi(t) = t - w(t)$. By $\psi(0) = 0 = \psi(1)$ and $\psi(t_0) > 0$, then there exists $t_* \in (0,1)$ such that

$$\psi(t_*) = \max\{\psi(t) : t \in [0,1]\} > 0.$$

It follows from $\psi'(t_*) = 0$, $w'(t_*) = 1$, $0 \le \theta w(t_*) < t_*$ and $\lambda \ge 1$ that

$$\begin{aligned}\psi''(t_*) = -w''(t_*) &= \mu\frac{(t_*^2-1)+\lambda(1-(\theta w)^2(t_*))}{\varphi^2(z(t_*),n)} \\ &> \mu\frac{(t_*^2-1)+\lambda(1-t_*^2)}{\varphi^2(z(t_*),n)} = \frac{\mu(\lambda-1)(1-t_*^2)}{\varphi^2(z(t_*),n)} \ge 0,\end{aligned}$$

which contradicts $\psi''(t_*) \le 0$.

Note that $w(1)-w(t)=1-w(t)\geq 0$ for $t\in[0,1)$, we know $w'(1)\geq 0$. By (iii), $w'(t)\geq 0$ for $t\in[0,1]$. Hence (vi) holds.

The following Lemmas will be used to investigate the existence of solutions of (2.8)-(2.9), which can be found in [17], [18] or [9, Lemma 2.4].

Lemma 3.2. [17] *Let E be a Banach space, D be a bounded open set of E and $\theta\in D$. $F:\overline{D}\to E$ is compact. If $x\neq \mu Fx$ for any $0<\mu<1$ and $x\in\partial D$, then F has a fixed point in $\overline{D}$.*

Lemma 3.3. [9, 18] (Helly selection principle) *Let $\{u_n(t)\}\subset BV[a,b]$ be an infinite sequence. Assume that $\{V_{u_n}\}$ is bounded and there exists $K>0$ such that $|u_n(t)|\leq K$ for $t\in[a,b]$ and $n\in\mathbb{N}$. Then there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u\in BV[a,b]$ such that $u_{n_k}(t)\to u(t)$ for each $t\in[a,b]$.*

Lemma 3.4. *For any $n>1$, there exists $(z_n,w_n)\in C[0,1]\times C^1[0,1]$ such that*

$$z_n(t)=\int_t^1\frac{\alpha(\lambda,w_n(s),s)(1-s)}{\varphi(z_n(s),n)}ds+(1-t)\int_0^t\frac{\beta(\lambda,w_n(s),s)}{\varphi(z_n(s),n)}ds+\frac{1}{n},\tag{3.3}$$

$$w_n(t)=\int_0^1 G_{0,1}(t,s)\frac{h(\lambda,w_n'(s),w_n(s),s)}{\varphi^2(z_n(s),n)}ds+t.\tag{3.4}$$

Proof. Let

$$\Omega=\{(z,w):(z,w)\in C[0,1]\times C^1[0,1],||(z,w)||<R\},$$

where $R=15(1+\lambda)n^2$. We prove $(z,w)\neq\mu F(z,w)$ for $0<\mu<1$ and $||(z,w)||=R$.

In fact, if there exist (z,w) with $||(z,w)||=R$ and $0<\mu<1$ such that $(z,w)=\mu F(z,w)$. By Lemma 3.1 (i) and (ii), we have $0\leq w(t)\leq 1$ for $t\in[0,1]$ and then $\theta w(t)=w(t)$, $\|w\|\leq 1$ and $\|z\|\leq 9+4\lambda$. By (3.1), we have

$$w'(t)=\mu[-\int_0^t s\frac{h(\lambda,w'(s),w(s),s)}{\varphi^2(z(s),n)}ds+\int_t^1(1-s)\frac{h(\lambda,w'(s),w(s),s)}{\varphi^2(z(s),n)}ds+1].\tag{3.5}$$

Noticing that $|h(\lambda,w'(s),w(s),s)|\leq|w'(s)|+\lambda$ and $\varphi(z(s),n)\geq\dfrac{1}{n}$ for $s\in[0,1]$, we obtain

$$\frac{|h(\lambda,w_n'(s),w_n(s),s)|}{\varphi^2(z(s),n)}\leq(|w'(s)|+\lambda)n^2\text{ for }s\in[0,1].$$

By Lemma 3.1 (iii), if $w'(1)\geq 0$ on $[0,1]$, then $\int_0^1|w'(s)|ds=\int_0^1 w'(s)ds=w(1)-w(0)=1$. If $w'(1)<0$, it follows from Lemma 3.1 (i) and $w(1)=\mu>0$ that $w'(0)\geq 0$, and we conclude therefore that there must be $\nu \in (0,1)$ such that $w'(\nu) = 0$ and $w'(\nu) < 0$ on $(\nu, 1]$. From this, we get

$$\begin{aligned}\int_0^1 |w'(s)|ds &= \int_0^\nu |w'(s)|ds + \int_\nu^1 |w'(s)|ds \\ &= \int_0^\nu w'(s)ds - \int_\nu^1 w'(s)ds = 2w(\nu) - \mu \leq 2\end{aligned}$$

and

$$\begin{aligned}|w'(t)| &\leq \int_0^1 \frac{|h(\lambda, w'(s), w(s), s)|}{\varphi^2(z(s), n)} ds + \int_0^1 \frac{|h(\lambda, w'(s), w(s), s)|}{\varphi^2(z(s), n)} ds + 1 \\ &\leq 2(\int_0^1 |w'(s)|ds + \lambda)n^2 + 1.\end{aligned}$$

Hence, we obtain $\|w'\| \leq 2(2+\lambda)n^2 + 1$ and

$$\begin{aligned}\|(z,w)\| &= \|z\| + \|w\| + \|w'\| \leq (9+4\lambda) + 1 + 2(2+\lambda)n^2 + 1 \\ &< 15(1+\lambda)n^2 = R,\end{aligned}$$

a contradiction. Since $\varphi(z(t), n) \geq \frac{1}{n}$ on $[0,1]$, we know easily that F is compact from $C[0,1] \times C^1[0,1]$ into $C[0,1] \times C^1[0,1]$. It follows from Lemma 3.2 that F has a fixed point (z_n, w_n) in $C[0,1] \times C^1[0,1]$, which satisfies (3.3)-(3.4).

Lemma 3.5. *Let (z_n, w_n) be obtained in Lemma 3.4. Then*

(i) $t \leq w_n(t) \leq 1$ and $w_n(t)$ is increasing in $[0,1]$.

(ii) $\{w_n'(t)\}$ and $\{w_n''(t)\}$ are bounded on $[0,b]$ for any $b \in (\frac{1}{2}, 1)$.

Proof. (i) By Lemma 3.1 (vi), (i) holds and $\theta w_n(t) = w_n(t)$.

(ii) Differentiating $w_n(t)$ with t twice, we have

$$w_n''(t) = -\frac{h(\lambda, w_n'(t), w_n(t), t)}{\varphi^2(z_n(t), n)}. \tag{3.6}$$

Utilizing the Green function defined in (2.10), we obtain

$$w_n(t) = \int_0^b G_{0,b}(t,s) \frac{h(\lambda, w_n'(s), w_n(s), s)}{\varphi^2(z_n(s), n)} ds + \frac{w_n(b)}{b} t, \quad t \in [0,b]. \tag{3.7}$$

Letting $h(\lambda, w_n'(s), w_n(s), s) = h_n(\lambda, s)$ and differentiating $w_n(t)$ in (3.7) with t, we have

$$w_n'(t) = -\int_0^t \frac{s}{b} \frac{h_n(\lambda, s)}{\varphi^2(z_n(s), n)} ds + \int_t^b \frac{b-s}{b} \frac{h_n(\lambda, s)}{\varphi^2(z_n(s), n)} ds + \frac{w_n(b)}{b}$$

and

$$\begin{aligned}|w_n'(t)| &\le \int_0^t \frac{s}{b}\frac{|h_n(\lambda,s)|}{\varphi^2(z_n(s),n)}ds+\int_t^b \frac{b-s}{b}\frac{|h_n(\lambda,s)|}{\varphi^2(z_n(s),n)}ds+\frac{w_n(b)}{b}\\ &\le \int_0^b \frac{|h_n(\lambda,s)|}{\varphi^2(z_n(s),n)}ds+\int_0^b \frac{|h_n(\lambda,s)|}{\varphi^2(z_n(s),n)}ds+\frac{1}{b}\\ &\le 2\int_0^b \frac{|h_n(\lambda,s)|}{\varphi^2(z_n(s),n)}ds+\frac{1}{b}.\end{aligned}$$

Notice that

$$\varphi(z_n(s),n)\ge \frac{1}{2}(1-b), |h_n(\lambda,s)|\le w_n'(s)+\lambda \text{ on } [0,b], \tag{3.8}$$

we get

$$\frac{|h_n(\lambda,s)|}{\varphi^2(z_n(s),n)}\le \frac{4(w_n'(s)+\lambda)}{(1-b)^2}$$

and

$$\int_0^b \frac{w_n'(s)+\lambda}{(1-b)^2}ds\le \int_0^1 \frac{w_n'(s)+\lambda}{(1-b)^2}ds=\frac{1+\lambda}{(1-b)^2}.$$

Hence $|w_n'(t)|\le \frac{8(1+\lambda)}{(1-b)^2}+\frac{1}{b}$ on $[0,b]$. This, together with (3.6) and (3.8), implies that $\{w_n''(t)\}$ is bounded on $[0,b]$. Hence, (ii) holds.

Now, we prove one of main results of this paper.

Theorem 3.6. *For $\lambda\ge 1$, the system of (1.1)-(1.3) has a solution (f,g) satisfying $f''(\eta)>0$ and $g'(\eta)\ge f'(\eta)$ on $\mathbb{R}^+$.*

Proof. The proof is divided into two steps.

Step 1. (2.8)-(2.9) has one solution $(z,w)\in Q^*$.

Let (z_n,w_n) be obtained in Lemma 3.4. By Lemma 3.5 (i), we know that $t\le w_n(t)\le 1$ for $t\in[0,1]$, $w_n(t)$ is increasing and $w_n'(t)\ge 0$ on $(0,1)$. It follows from (3.3) that

$$z_n'(t)=-\frac{1-t^2}{\varphi(z_n(t),n)}-\int_0^t \frac{\beta(\lambda,w_n(s),s)}{\varphi(z_n(s),n)}ds, t\in[0,1].$$

Since $w_n(t)\ge 0$ for $t\in[0,1]$, we know $z_n'(t)<0$ for $t\in[0,1]$, that is, $z_n(t)$ is decreasing in $[0,1]$ and $\varphi(z_n(t),n)\ge\varphi(z_n(s),n)$ for $s\ge t$. This and (3.3) imply

$$\begin{aligned}\varphi(z_n(t),n)\ge z_n(t) &\ge \frac{1}{\varphi(z_n(t),n)}\int_t^1 \alpha(\lambda,w_n(s),s)(1-s)ds\\ &\ge \frac{1}{\varphi(z_n(t),n)}\int_t^1 (1-s)ds\end{aligned}$$

and $\varphi(z_n(t),n) \geq \frac{\sqrt{2}}{2}(1-t)$ on $[0,1]$. It follows from Lemma 3.1 (ii) that $|z_n(t)| \leq 9+4\lambda$ for $t \in [0,1]$. By Lemma 3.3, there exist $z,w \in BV[0,1]$ and two subsequences $\{z_{n_k}(t)\}$ of $\{z_n(t)\}$ and $\{w_{n_i}(t)\}$ of $\{w_n(t)\}$ such that $z_{n_k}(t) \to z(t)$, $w_{n_i}(t) \to w(t)$ for each $t \in [0,1]$, respectively, and $z(t) > 0, t \leq w(t) \leq 1$ for $t \in [0,1)$, $z(1) = 0$ and $w(1) = 1$. Without loss of generality, we assume that $\{z_{n_k}(t)\}$ is $\{z_n(t)\}$ and $\{w_{n_i}(t)\}$ is $\{w_n(t)\}$. By Lemma 3.5, we know that $\{w'_n(t)\}$ is equicontinuous and is bounded on $[0,b]$. Letting $b = 1-\frac{1}{k}(k>1)$, utilizing the diagonal principle and the Arzela-Ascoli theorem, we know that there exist the subsequence $\{w'_{n_k}(t)\}$ of $\{w'_n(t)\}$ and $w_0 \in C[0,1)$ such that $w'_{n_k}(t) \to w_0(t)$ for each $t \in [0,1)$. For the sake of convenience, let $\{w_{n_k}(t)\}$ be itself of $\{w_n(t)\}$. By $w_n(t) = \int_0^t w'_n(s)ds + w_n(0)$, we obtain $w(t) = \int_0^t w_0(s)ds + w(0)$ and then $w_0(t) = w'(t)$ for $t \in [0,1)$. Since $\lim_{n\to\infty} \varphi(z_n(t),n) = \max\{z(t),c(t)\} \geq \frac{\sqrt{2}}{2}(1-t)$ on $[0,1]$ and $c(t) < \frac{\sqrt{2}}{2}(1-t)$ on $[0,1)$, we obtain $\lim_{n\to\infty} \varphi(z_n(t),n) = z(t)$ on $[0,1]$. Letting $n \to \infty$ in (3.7), we obtain easily that (z,w) satisfies (2.9) and $w \in C[0,1)$. Meanwhile, $t \leq w(t) \leq 1$ for $t \in [0,1)$. This implies $\lim_{t\to 1^-} w(t) = 1 = w(1)$ and $w \in C[0,1] \cap C^1[0,1)$. For $t \in [0,1)$, we may choose $b \in (0,1)$ to satisfy $t \leq b$, letting $n \to \infty$ in (3.3) and noticing $\varphi(z_n(t),n) \geq \frac{\sqrt{2}}{2}(1-b)$ on $[0,b]$, we obtain

$$z(t) = \lim_{n\to\infty} \gamma_n^b + \int_t^b \frac{\alpha(\lambda,w(s),s)(1-s)}{z(s)}ds + (1-t)\int_0^t \frac{\beta(\lambda,w(s),s)}{z(s)}ds, \tag{3.9}$$

where $\gamma_n^b = \int_b^1 \frac{\alpha(\lambda,w_n(s),n)(1-s)}{\varphi(z_n(s),n)}ds$. Since

$$0 \leq \alpha(\lambda,w_n(s),s) \leq 3+\lambda \text{ and } \varphi(z_n(t),n) \geq \frac{\sqrt{2}}{2}(1-t) \text{ on } [0,1],$$

we have

$$0 \leq \frac{\alpha(\lambda,w_n(s),s)(1-s)}{\varphi(z_n(s),n)} \leq \sqrt{2}(3+\lambda) \text{ and } 0 \leq \gamma_n^b \leq \sqrt{2}(3+\lambda)(1-b).$$

This implies $\lim_{b\to 1^-}\lim_{n\to\infty} \gamma_n^b = 0$. Letting $b \to 1^-$ in (3.9), we know easily that (z,w) satisfies (2.8) and $z \in C[0,1)$. We show $\lim_{t\to 1^-} z(t) = 0 = z(1)$. In fact, if $\lim_{t\to 1^-} z(t) := \sigma > 0$, since $z(t)$ is decreasing on $[0,1]$, then $z(t) \geq \sigma$ on $[0,1)$ and two integrands in the right side of (2.8) are bounded on $[0,1)$. This implies $\lim_{t\to 1^-} z(t) = 0$, which is a contradiction. Hence, $z \in C[0,1]$.

Step 2. Since (2.8)-(2.9) has one solution $(z,w) \in Q^*$, it follows from Theorem 2.4 that the system of (1.1)-(1.3) has a solution $(f,g) \in \Gamma$.

Remark 3.7. In [12], the authors studied the existence of solutions of (1.1)-(1.3) for $\lambda < 1$. Theorem 2.4 extends the study [12].

4. Upper and lower bounds of the shear stress functions

In this section, we utilize the equivalent Theorem 2.4 to prove another of main results of this paper.

Theorem 4.1. *Let $\lambda \geq 1$ and (f,g) be a solution of (1.1)-(1.3) satisfying $f''(\eta) > 0$ and $g'(\eta) \geq 0$ on $\mathbb{R}^+$. Then*

(i) $1 \leq \|f''\| \leq 2(\sqrt{2}-1)\lambda + \dfrac{2(\sqrt{2}+1)}{3}$.

(ii) $1 \leq \|g''\| \leq [2(\sqrt{2}-1)\lambda + \dfrac{2(\sqrt{2}+1)}{3}](\lambda+1)$,

where $\|f''\| = \sup\{f''(\eta) : \eta \in \mathbb{R}^+\}$.

Proof. (i) Let $(f,g) \in \Gamma$ be a solution of (1.1)-(1.3). Proposition 2.1 (P_1) and (P_3) show that f'' and g'' are decreasing $[0,\infty)$ and $\|f''\| = f''(0)$ and $\|g''\| = g''(0)$.

It follows from Theorems 2.2 and 2.3 that (2.8)-(2.9) has a solution $(z,w) \in Q^*$ satisfying $f''(\eta) = z(t)$ and $g''(\eta) = w'(t)z(t)$. By (2.1) and $t \leq w(t) \leq 1$ on $[0,1]$, we know $z'(t) < 0$ on $[0,1)$, that is, $z(t)$ is decreasing on $[0,1]$. From (2.8), we obtain

$$z(t) \geq \int_t^1 \frac{\alpha(\lambda, w(s), s)(1-s)}{z(s)} ds \geq \frac{1}{z(t)} \int_t^1 \alpha(\lambda, w(s), s)(1-s) ds$$

and by $\alpha(\lambda, w(s), s) \geq 1 + 3s$

$$z(t) \geq \frac{1}{z(t)} \int_t^1 (1+3s)(1-s) ds = \frac{1}{z(t)} (1+t)(1-t)^2, \quad t \in [0,1]$$

and

$$z(t) \geq (1-t)\sqrt{1+t} \text{ on } [0,1]. \tag{4.1}$$

This implies $f''(0) = z(0) \geq 1$ and the left side of (i) holds. It follows from (2.8) and (4,1) that

$$\begin{aligned} z(0) &= \int_0^1 \frac{\alpha(\lambda, w(s), s)(1-s)}{z(s)} ds \\ &\leq \int_0^1 \frac{(1+2s+\lambda)(1-s)}{\sqrt{1+s}(1-s)} ds = 2(\sqrt{2}-1)\lambda + \frac{2(\sqrt{2}+1)}{3}. \end{aligned}$$

Hence, the right side of (i) holds.

(ii) By $w(t) \geq t$ and $w(0) = 0$, we obtain $w'(0) \geq 1$ and the left side of (ii) holds by (i).

Differentiating (2.9) with t, we have for $t \leq b < 1$

$$\begin{aligned} w'(t) &= -\int_0^t \frac{s}{b} \frac{h(\lambda, w'(s), w(s), s)}{z^2(s)} ds \\ &+ \int_t^b \frac{(b-s)}{b} \frac{h(\lambda, w'(s), w(s), s)}{z^2(s)} ds + \frac{w(b)}{b} \end{aligned}$$

and by (4.1) and $w'(s) \geq 0$ on $[0, 1)$

$$\begin{aligned} w'(0) &= \int_0^b \frac{(b-s)}{b} \frac{h(\lambda, w'(s), w(s), s)}{z^2(s)} ds + \frac{w(b)}{b} \\ &\leq \frac{\lambda}{b} \int_0^b \frac{(b-s)(1-w^2(s))}{z^2(s)} ds + \frac{w(b)}{b} \end{aligned}$$

and by $w(s) \geq s$ and (4.1)

$$\begin{aligned} w'(0) &\leq \frac{\lambda}{b} \int_0^b \frac{(1-s)(1-s^2)}{z^2(s)} ds + \frac{w(b)}{b} \\ &\leq \frac{\lambda}{b} \int_0^b ds + \frac{w(b)}{b} = \lambda + \frac{w(b)}{b}. \end{aligned}$$

Letting $b \to 1^-$, we obtain $w'(0) \leq \lambda + 1$. Since $g''(0) = w'(0)z(0)$, by (i), we know that the right side of (ii) holds.

Remark 4.2. To the best of our knowledge, there is very little analytic study on the shear stress functions involved in the system (1.1)-(1.3), hence the results obtained in Theorem 4.1 are new.

Remark 4.3. We use the Falkner-Skan equation to give further results on (1.1)-(1.3) for $\lambda = 1$.

Consider the following Falkner-Skan problem

$$h''' + hh'' + \frac{1}{2}(1 - h'^2) = 0 \quad on \quad \mathbb{R}^+, \tag{4.2}$$

with boundary conditions

$$h(0) = 0, h'(0) = 0, h'(\infty) = 1. \tag{4.3}$$

It is well-known that (4.2)-(4.3) has a unique solution satisfying $h''(\eta) > 0$ on $\mathbb{R}^+$(see Theorems 6.1 and 8.1 in [19]). Let $f(\eta) = g(\eta) = \frac{1}{\sqrt{2}} h(\sqrt{2}\eta)$, then it is easy to verify that (f, g) satisfies

(1.1)-(1.3). By putting $w(s) = s$ in (2.8), we can obtain upper bounds of f'' and g'' as follows:

$$1 \le \|f''\| = \|g''\| \le \frac{8\sqrt{2}-4}{3}.$$

This upper bound of $\|g''\|$ is better than that obtained in Theorem 4.2 (ii).

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