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EXISTENCE OF POSITIVE SOLUTIONS OF FRACTIONAL BOUNDARY VALUE PROBLEMS INVOLVING BOUNDED LINEAR OPERATORS

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Abstract. This paper is concerned with boundary value problems for nonlinear fractional differential equations with a nonlinear term involving a bounded linear operator and satisfying boundary conditions containing a bounded linear functional. The explicit expression for an equivalent integral operator for the BVP is given. A recent fixed point theorem is used to obtain the existence of at least three positive solutions. The paper also provides an example as an application of the existence theorem.

Keywords. Nonlinear fractional differential equations; Bounded linear operators; Riemann-Liouville fractional derivative; Green functions; Positive solutions.

1. Introduction

Differential equations of fractional orders appear more and more frequently in various research areas and applications in diverse fields of science and engineering. Readers are referred to the monographs [2, 12, 17, 18] for the theory and applications of fractional calculus. In particular, the existence of positive solutions of boundary value problems for fractional differential equations has been studied by many researchers, and we cite as recent examples the papers [1, 4, 6, 7, 10, 14, 15, 16, 19]. Some of these works consider BVP's involving fractional derivatives in the nonlinear term, *e.g.*, [1, 16]. Some deal with problems having fractional

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derivatives or integral boundary conditions ([4, 16]). Hence, it seems natural to think of generalizing derivatives and integrals to bounded linear operators. Ordinary differential equations with bounded linear operators have been studied in [9, 10, 11]. In this paper, we consider a BVP for fractional order differential equations with bounded linear operators appearing in the nonlinear term and in the boundary conditions. In particular, we show the interchangeability of a bounded linear functional and a Riemann integral. A comprehensive analysis of the Green's function and the explicit equivalent expression for the operator corresponding to the original BVP are given. This paper directly improves the results in [15]. We also use some ideas from [1, 8, 9, 16, 19, 20].

We begin by mentioning a few boundary value problems for differential equations with Riemann-Liouville fractional derivative that have been studied in the literature. El-Shahed [5] considered the problem

$$\mathscr{D}_{0^+}^{p}x(t)+\lambda a(t)f(x)(t)=0,\ 0<t<1,\ 2<p\leq 3,$$
$$x(0)=x'(0)=x'(1)=0,$$

and Liang [14] studied the more general case

$$\mathscr{D}_{0^+}^{p}x(t)+f(t,x(t))=0,\ 0<t<1,\ 2<p\leq 3,$$
$$x(0)=x'(0)=0,\quad x'(1)=\int_0^{\eta}x(s)ds.$$

In [15], Liang and Zhang considered the multipoint problem

$$\mathscr{D}_{0^+}^{p}x(t)+f(t,x(t))=0,\ 0<t<1,\ 2<p\leq 3,$$
$$x(0)=x'(0)=0,\ x'(1)=\sum_{i=1}^{m}q_ix'(\xi_i),$$

where $0<\xi_1<\cdots<\xi_m<1$ with $0<\sum_{i=1}^{m}q_i\xi_i^{p-1}<1$.

Here, we consider a more general problem. We are concerned with the existence of positive solutions of a fractional differential equation involving a bounded linear operator in the nonlinear term and a bounded linear functional in the boundary condition, namely,

$$\mathscr{D}_{0^+}^{p}x(t)+f(t,x(t),(Nx)(t))=0,\ 0<t<1, \tag{1}$$

$$x(0)=x'(0)=0,\ x'(1)=L(x), \tag{2}$$

where $2 < p \leq 3$, $\mathscr{D}_{0+}^{p}$ is the standard Riemann-Liouville fractional derivative of order p, N and L are, respectively, a bounded linear operator and a bounded linear functional defined on a Banach space E to be specified later, and $f : [0,1] \times [0,\infty) \times \mathbb{R} \to [0,\infty)$ is continuous. By a *positive solution* of problem (1), (2), we mean a function $x : [0,1] \to [0,\infty)$ such that $x(t) > 0$ for $t \in (0,1)$ and x satisfies (1) and (2).

The boundary condition $x'(1) = L(x)$ includes conditions such as $x'(1) = \mu \int_0^1 x(s) d\Lambda s$, which is a Stieltjes integral with a signed measure $\Lambda(s)$, and the multipoint condition

$$x'(1) = \sum_{i=1}^{m} \lambda_i \mathscr{D}_{0+}^{q} x(\xi_i),$$

where $0 < \xi_1 < \xi_2 < \ldots < \xi_m < 1$, $q \in [0, p-1)$, and $\mu, \lambda_i \geq 0$. With the bounded linear operator $N(x)$, we include cases such as $(Nx)(t) = \mathscr{D}_{0+}^{\delta} x(t)$ and $(Nx)(t) = \mathscr{I}_{0+}^{\gamma} x(t)$, where $\delta \in [0, p-1)$, $\gamma \geq 0$, and $\mathscr{I}_{0+}^{\gamma}$ is the Riemann-Liouville fractional integral of order γ.

The paper is organized as follows. Section 2 contains some preliminaries on the fractional calculus and the fixed point theorem to be used in the paper. Section 3 provides some basic lemmas about the Green's function, the equivalent integral operator for the BVP, and the interchangeability of a bounded linear functional and a Riemann integral. In Section 4, we establish criteria for the existence of three positive solutions to (1), (2). An example is provided to show the application of our results.

2. Preliminaries

In this section, we introduce some basic notations, definitions, and preliminary facts from fractional calculus; see [2, 12, 17, 18] for additional details.

Definition 1. *The Riemann-Liouville fractional integral of order $p \geq 0$ for a function $h : (0,\infty) \to \mathbb{R}$ is defined by*

$$\mathscr{I}_{0+}^{p} h(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} h(s) ds, \; t \geq 0,$$

provided the right side exists pointwise on $(0,\infty)$, where Γ is the Gamma function.

Definition 2. *The Riemann-Liouville fractional derivative of order $p \geq 0$ for a function $h : (0,\infty) \to \mathbb{R}$ is defined by*

$$\mathscr{D}_{0^+}^{p} h(t) = D^n(\mathscr{I}_{0^+}^{n-p} h(t)) = \frac{1}{\Gamma(n-p)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-p-1} h(s)ds,\ t \geq 0,$$

where $n = \lfloor p \rfloor + 1$ and $\lfloor p \rfloor$ is the integer part of p. If p is an integer, then $\mathscr{D}_{0^+}^{p} h(t) = \frac{d^p h(t)}{dt^p}$.

Lemma 1. *The Riemann-Liouville fractional derivative of order $p \geq 0$ of the power function t^λ with $\lambda > -1$ is given by*

$$\mathscr{D}_{0^+}^{p} t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+n-p)} \frac{d^n(t^{n-p+\lambda})}{dt^n} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-p+1)} t^{\lambda-p},$$

where $n = \lfloor p \rfloor + 1$. In particular,

$$\mathscr{D}_{0^+}^{p} t^{p-m} = 0 \text{ for all } m = 1,2,\ldots,n.$$

Lemma 2. *Let $p > 0$. The fractional differential equation $\mathscr{D}_{0^+}^{p} x(t) = 0$, where $x \in C(0,1) \cap L(0,1)$, has unique solutions given by*

$$x(t) = \sum_{i=1}^{\lfloor p \rfloor + 1} c_i t^{p-i},$$

where $c_i \in \mathbb{R}$, $i = 1,2,\ldots,n$, $n = \lfloor p \rfloor + 1$.

Lemma 3. *Assume that $x \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $p > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then,*

$$\mathscr{I}^p \mathscr{D}_{0^+}^{p} x(t) = x(t) + \sum_{i=1}^{\lfloor p \rfloor + 1} c_i t^{p-i},$$

where $c_i \in \mathbb{R}$, $i = 1,2,\ldots,n$, $n = \lfloor p \rfloor + 1$.

The following fixed point theorem in a cone due to Bai and Ge [3] (Lemma 4 below) can be viewed as a generalization of the Leggett-Williams fixed point theorem. We intend to use this theorem to obtain the existence of at least three positive solutions of problem (1), (2).

Let E be a Banach space and $P \subset E$ be a cone. Let u, $v : P \to [0,+\infty)$ be two nonnegative continuous convex functions satisfying

$$\|x\| \leq M \max\{u(x), v(x)\}, \quad x \in P,\ M > 0, \tag{3}$$

and

$$\Omega = \{x \in P : u(x) < k,\ v(x) < Q\} \text{ for } k > 0 \text{ and } Q > 0. \tag{4}$$

Then, Ω is a bounded nonempty open subset of P.

Let $k > c > 0$ and $Q > 0$ be given, α, $\beta : P \to [0,\infty)$ be two nonnegative continuous convex functions satisfying (3) and (4), and γ be a nonnegative continuous concave function on the cone P. Define the bounded convex sets

$$P(\alpha,k;\beta,Q) = \{x \in P : \alpha(x) < k, \beta(x) < Q\}$$

and

$$P(\alpha,k;\beta,Q;\gamma,c) = \{x \in P : \alpha(x) < k, \beta(x) < Q, \gamma(x) > c\}.$$

Lemma 4. *Let E be a Banach space, $P \subset E$ be a cone, and $k_2 \geq d > b > k_1 > 0$ and $Q_2 \geq Q_1 > 0$ be given. Assume that α, β are nonnegative continuous convex functions on P, such that* (3) *and* (4) *are satisfied, and γ is an nonnegative continuous concave function on P such that $\gamma(x) \leq \alpha(x)$ for all $x \in \overline{P}(\alpha,k_2;\beta,Q_2)$. Let $T : \overline{P}(\alpha,k_2;\beta,Q_2) \to \overline{P}(\alpha,k_2;\beta,Q_2)$ be a completely continuous operator and assume that:*

- (C1) $\{x \in \overline{P}(\alpha,d;\beta,Q_2;\gamma,b) : \gamma(x) > b\} \neq \emptyset$ *and* $\gamma(Tx) > b$ *for* $x \in \overline{P}(\alpha,d;\beta,Q_2;\gamma,b)$*;*
- (C2) $\alpha(Tx) < k_1$, $\beta(Tx) < Q_1$ *for all* $x \in \overline{P}(\alpha,k_1;\beta,Q_1)$*;*
- (C3) $\gamma(Tx) > b$ *for all* $x \in \overline{P}(\alpha,k_2;\beta,Q_2;\gamma,b)$ *with* $\alpha(Tx) > d$.

Then, T has at least three fixed points x_1, x_2, $x_3 \in \overline{P}(\alpha,k_2;\beta,Q_2)$. Furthermore,

$$x_1 \in \overline{P}(\alpha,k_1;\beta,Q_1),$$

$$x_2 \in \{\overline{P}(\alpha,d;\beta,Q_2;\gamma,b) : \gamma(x) > b\},$$

$$x_3 \in \overline{P}(\alpha,d;\beta,Q_2) \setminus \{\overline{P}(\alpha,d;\beta,Q_2;\gamma,b) \cup \overline{P}(\alpha,k_1;\beta,Q_1)\}.$$

3. Lemmas

Throughout the paper, we make use of the following conditions.

(A1) The nonlinear function $f:[0,1]\times[0,\infty)\times\mathbb{R}\to[0,\infty)$ is continuous and $f(t,0,0)\not\equiv 0$ on any subinterval of $[0,1]$.

(A2) Assume that $2<p\le 3$ and fix $q\in[0,p-1)$. Define $E:=\{x: x,\mathscr{D}^q_{0^+}x\in C[0,1]\}$, which is a Banach space with the norm $\|x\|_E=\max\{\|x\|_\infty,\|\mathscr{D}^q_{0^+}x\|_\infty\}$, where $\|y\|_\infty:=\max\limits_{t\in[0,1]}|y(t)|$ for any $y\in C[0,1]$.

(A3) $N:E\to E$ and $L:E\to\mathbb{R}$ are bounded linear operators.

Remark 1. We note that for $\gamma>0$, $I^\gamma x(1)$ is an example of a bounded linear functional defined on E. It is easy to see that $\|I^\gamma x(1)\|=\frac{1}{\Gamma(\gamma+1)}$.

The Green's function for $\mathscr{D}^p_{0^+}x(t)=0$ satisfying $x(0)=x'(0)=x'(1)=0$ is given by (see [15])

$$G(t,s)=\frac{1}{\Gamma(p)}\begin{cases}(1-s)^{p-2}t^{p-1}-(t-s)^{p-1}, & 0\le s\le t\le 1,\\ (1-s)^{p-2}t^{p-1}, & 0\le t\le s\le 1.\end{cases}\tag{5}$$

The following lemma provides some important properties of the Green's function $G(t,s)$.

Lemma 5. *Suppose* (A2) *and* (A3) *hold. Then:*

(i) $G:[0,1]\times[0,1]\to\mathbb{R}$ *is nonnegative and continuous;*

(ii) $G(0,s)=G(t,0)=G(t,1)=0$ *for any* $t,s\in[0,1]$*;*

(iii) $G(t,s)$ *is strictly increasing in* t *for any fixed* $s\in(0,1)$*;*

(iv) $\max\limits_{t\in[0,1]}G(t,s)=G(1,s)=\frac{1}{\Gamma(p)}\left((1-s)^{p-2}-(1-s)^{p-1}\right)=\frac{1}{\Gamma(p)}s(1-s)^{p-2}$ *for any* $s\in[0,1]$*;*

(v) $t^{p-1}G(1,s)\le G(t,s)\le t^{p-2}G(1,s)$ *if* $t\le s$*, and* $t^{p-2}G(1,s)\le G(t,s)\le(p-1)t^{p-2}G(1,s)$ *if* $t\ge s$*.*

Proof. From the definition of G, it is easy to see that (i) and (ii) hold.

To prove (iii), notice that for any fixed $s\in(0,1)$,

$$\frac{\partial G(t,s)}{\partial t}=\frac{1}{\Gamma(p-1)}\begin{cases}(1-s)^{p-2}t^{p-2}-(t-s)^{p-2}, & 0\le s\le t\le 1,\\ (1-s)^{p-2}t^{p-2}, & 0\le t\le s\le 1.\end{cases}$$

Clearly, if $0 < t \le s$, then $\frac{\partial G(t,s)}{\partial t} > 0$; and if $1 > t \ge s$, then

$$\frac{\partial G(t,s)}{\partial t} = \frac{1}{\Gamma(p-1)}\left((t-ts)^{p-2} - (t-s)^{p-2}\right) > 0.$$

Therefore, $G(t,s)$ is strictly increasing in its first component t for any fixed $s \in (0,1)$, so (iii) and (iv) hold.

To show (v) holds, first note that (v) is true if $s = 0$ or 1. For $s \neq 0,1$, define $c(t) = \frac{G(t,s)}{G(1,s)}$. Then,

$$c(t) = \begin{cases} \frac{(1-s)^{p-2}t^{p-1} - (t-s)^{p-1}}{s(1-s)^{p-2}}, & 0 \le s \le t \le 1, \\ \frac{t^{p-1}}{s}, & 0 \le t \le s \le 1. \end{cases}$$

We begin by showing that $t^{p-1}G(1,s) \le G(t,s) \le t^{p-2}G(1,s)$, that is, $t^{p-1} \le c(t) \le t^{p-2}$ for $t \le s$. Since $t^{p-1} \le \frac{t^{p-1}}{s} = \frac{t}{s} \cdot t^{p-2} \le t^{p-2}$, we have $t^{p-1} \le c(t) \le t^{p-2}$.

Next, we show $t^{p-2} \le c(t) \le (p-1)t^{p-2}$ for $s \le t \le 1$. Notice that

$$\begin{aligned} c(t) &= \frac{(1-s)^{p-2}t^{p-1} - (t-s)^{p-1}}{s(1-s)^{p-2}} \\ &= \frac{(1-s)^{p-2}t^{p-2}(t-s+s) - (t-s)^{p-1}}{s(1-s)^{p-2}} \\ &= \frac{(1-s)^{p-2}t^{p-2}(t-s) + (1-s)^{p-2}t^{p-2}s - (t-s)^{p-1}}{s(1-s)^{p-2}} \\ &= \frac{(t-ts)^{p-2}(t-s) - (t-s)^{p-1}}{s(1-s)^{p-2}} + t^{p-2} \\ &\ge \frac{(t-s)^{p-2}(t-s) - (t-s)^{p-1}}{s(1-s)^{p-2}} + t^{p-2} \\ &= t^{p-2} \end{aligned}$$

since $(t-ts)^{p-2} \ge (t-s)^{p-2}$. On the other hand, we have

$$\begin{aligned} c(t) &= \frac{1}{s}\left(t^{p-1} - \frac{(t-s)^{p-1}}{(1-s)^{p-2}}\right) \\ &= \frac{t^{p-2}}{s}\left(t - t\frac{(1-\frac{s}{t})^{p-1}}{(1-s)^{p-2}}\right) \\ &= t^{p-2}\left(\frac{t}{s} - \frac{t}{s}\frac{(1-\frac{s}{t})^{p-1}}{(1-s)^{p-2}}\right). \end{aligned}$$

Then, to prove $c(t) \leq (p-1)t^{p-2}$, it suffices to show

$$\frac{t}{s} - \frac{t}{s}\frac{(1-\frac{s}{t})^{p-1}}{(1-s)^{p-2}} \leq p-1,$$

that is,

$$1 - \frac{(1-\frac{s}{t})^{p-1}}{(1-s)^{p-2}} \leq (p-1)\frac{s}{t}, \tag{6}$$

Now set $z := 1-\frac{s}{t}$. Then, $0 \leq z \leq 1-s$, and (6) is equivalent to

$$1 - \frac{z^{p-1}}{(1-s)^{p-2}} \leq (p-1)(1-z),$$

that is,

$$0 \leq z^{p-1} - (p-1)(1-s)^{p-2}z - (2-p)(1-s)^{p-2}.$$

Denoting the right hand side of this inequality by $d(z)$, all we need to do is to show $d(z) \geq 0$ for any $z \in [0,1-s]$. Notice that at $z=0$, that is, $t=s$, $d(0) = (p-2)(1-s)^{p-2} \geq 0$; and at $z = 1-s$, that is, $t=1$,

$$d(1-s) = (1-s)^{p-1} - (p-1)(1-s)^{p-1} - (2-p)(1-s)^{p-2} = (p-2)s(1-s)^{p-2} \geq 0.$$

Also,

$$d'(z) = (p-1)z^{p-2} - (p-1)(1-s)^{p-2} = (p-1)(z^{p-2} - (1-s)^{p-2}) \leq 0,$$

which means $d(z)$ is non-increasing on the interval $[0,1-s]$. Therefore, $d(z) \geq 0$ for $0 \leq z \leq 1-s$, that is, $c(t) \leq (p-1)t^{p-2}$ for $s \leq z \leq 1$. This completes the proof of the lemma. □

Remark 2. For a fixed $s \in [0,1]$, we denote the Riemann-Liouville derivative of order q of $G(t,s)$ with respect to t by ${}_t\mathscr{D}^p_{0^+}G(t,s)$. We then have the following expression for ${}_t\mathscr{D}^p_{0^+}G(t,s)$:

$$ {}_t\mathscr{D}^q_{0^+}G(t,s) = \frac{1}{\Gamma(p-q)}\begin{cases} (1-s)^{p-2}t^{p-1-q} - (t-s)^{p-1-q}, & 0 \leq s \leq t \leq 1, \\ (1-s)^{p-2}t^{p-1-q}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{7}$$

Lemma 6. *Suppose* $(A2)$ *and* $(A3)$ *hold. If* $0 \leq q < p-2$*, then the following properties of* ${}_t\mathscr{D}^q_{0^+}G(t,s)$ *hold:*

(i) ${}_t\mathscr{D}^p_{0^+}G(t,s) : [0,1]\times[0,1] \to \mathbb{R}$ *is nonnegative and continuous;*

(ii) ${}_t\mathscr{D}^p_{0^+}G(0,s) = {}_t\mathscr{D}^p_{0^+}G(t,0) = {}_t\mathscr{D}^p_{0^+}G(t,1) = 0$ *for any* $t,s \in [0,1]$*;*

(iii) ${}_t\mathscr{D}^q_{0^+}G(t,s)$ *is strictly increasing and concave up in* t *for* $t \le s$, *and concave down in* t *for* $t \ge s$;

(iv) $\max\limits_{t\in[0,1]} {}_t\mathscr{D}^q_{0^+}G(t,s) = {}_t\mathscr{D}^q_{0^+}G(t_0,s) = \frac{s(1-s)^{p-2}t_0^{p-2-q}}{\Gamma(p-q)}$ *for any* $s \in (0,1]$, *where* $s \le t_0 = \frac{s}{1-(1-s)^{\frac{p-2}{p-2-q}}} \le 1$;

(v) $t^{p-1-q}{}_t\mathscr{D}^q_{0^+}G(t_0,s) \le {}_t\mathscr{D}^q_{0^+}G(t,s) \le {}_t\mathscr{D}^q_{0^+}G(t_0,s)$ *if* $t \le s$, *and*
$\min\{\frac{1-(1-s)^{1-q}}{s}, s^{p-1-q}\}{}_t\mathscr{D}^q_{0^+}G(t_0,s) \le {}_t\mathscr{D}^q_{0^+}G(t,s) \le {}_t\mathscr{D}^q_{0^+}G(t_0,s)$ *if* $t \ge s$.

Proof. That (i) and (ii) hold can easily be seen from the expression for ${}_t\mathscr{D}^q_{0^+}G(t,s)$.

To see that (iii) holds, notice for any fixed $s \in [0,1]$,

$$\frac{\partial({}_t\mathscr{D}^q_{0^+}G(t,s))}{\partial t} = \frac{1}{\Gamma(p-q-1)}\begin{cases}(1-s)^{p-2}t^{p-2-q}-(t-s)^{p-2-q}, & 0 \le s \le t \le 1,\\ (1-s)^{p-2}t^{p-2-q}, & 0 \le t \le s \le 1,\end{cases} \tag{8}$$

and

$$\frac{\partial^2({}_t\mathscr{D}^q_{0^+}G(t,s))}{\partial t^2} = \frac{1}{\Gamma(p-q-2)}\begin{cases}(1-s)^{p-2}t^{p-3-q}-(t-s)^{p-3-q}, & s<t,\\ (1-s)^{p-2}t^{p-3-q}, & t<s.\end{cases} \tag{9}$$

Clearly, if $0 < t < s$, then $\frac{\partial({}_t\mathscr{D}^q_{0^+}G(t,s))}{\partial t} \ge 0$ and $\frac{\partial^2({}_t\mathscr{D}^q_{0^+}G(t,s))}{\partial t^2} \ge 0$, so ${}_t\mathscr{D}^q_{0^+}G(t,s)$ is strictly increasing and concave up in t for $t \in [0,s]$. Since

$$\begin{aligned}(1-s)^{p-2}t^{p-3-q}-(t-s)^{p-3-q} &\le (1-s)^{p-3-q}t^{p-3-q}-(t-s)^{p-3-q}\\ &= (t-ts)^{p-3-q}-(t-s)^{p-3-q}\\ &\le 0,\end{aligned}$$

for $t > s$, we have $\frac{\partial^2({}_t\mathscr{D}^q_{0^+}G(t,s))}{\partial t^2} \le 0$ for $t \in [s,1]$, which means ${}_t\mathscr{D}^q_{0^+}G(t,s)$ is concave down in t for $t \in [s,1]$.

To prove (iv), by (iii), it suffices to find the critical point $t_0 \in [s,1]$ of ${}_t\mathscr{D}^q_{0^+}G(t,s)$ for any fixed $s \in (0,1)$. Let $\frac{\partial({}_t\mathscr{D}^q_{0^+}G(t,s))}{\partial t} = 0$, that is, $(1-s)^{p-2}t_0^{p-2-q} - (t_0-s)^{p-2-q} = 0$. Then, $(1-s)^{p-2} = (1-\frac{s}{t_0})^{p-2-q}$. Solving for t_0 gives $t_0 = \frac{s}{1-(1-s)^{\frac{p-2}{p-2-q}}}$. Since $0 \le (1-s)^{\frac{p-2}{p-2-q}} \le 1-s$, we have $s \le t_0 \le 1$. Therefore, $\max\limits_{t\in[0,1]} {}_t\mathscr{D}^q_{0^+}G(t,s) = {}_t\mathscr{D}^q_{0^+}G(t_0,s)$. Also, $t_0 - s = t_0(1-\frac{s}{t_0}) =$

$t_0(1-s)^{\frac{p-2}{p-2-q}}$, which means

$$\begin{aligned}\Gamma(p-q)\cdot {}_t\mathscr{D}^q_{0+}G(t_0,s) &= (1-s)^{p-2}t_0^{p-1-q}-(t_0-s)^{p-1-q}\\ &= (1-s)^{p-2}t_0^{p-1-q}-t_0^{p-1-q}(1-s)^{\frac{(p-2)(p-1-q)}{p-2-q}}\\ &= (1-s)^{p-2}t_0^{p-1-q}(1-(1-s)^{\frac{p-2}{p-2-q}})\\ &= (1-s)^{p-2}t_0^{p-1-q}\,\frac{s}{t_0}\\ &= s(1-s)^{p-2}t_0^{p-2-q}.\end{aligned}$$

From the expression for ${}_t\mathscr{D}^q_{0+}G(t_0,s)$ and that for ${}_t\mathscr{D}^q_{0+}G(t,s)$, it is easy to see that the first part of (v) with $t\le s$ holds. If $t\ge s$, ${}_t\mathscr{D}^q_{0+}G(t,s)$ is concave down in t, so

$${}_t\mathscr{D}^q_{0+}G(t,s)\ge \min\{{}_t\mathscr{D}^q_{0+}G(1,s),{}_t\mathscr{D}^q_{0+}G(s,s)\}\ge \min\left\{\frac{1-(1-s)^{1-q}}{s},s^{p-1-q}\right\}{}_t\mathscr{D}^q_{0+}G(t_0,s),$$

since ${}_t\mathscr{D}^q_{0+}G(1,s)=\frac{1}{\Gamma(p-q)}(1-s)^{p-2}-(1-s)^{p-1-q}=\frac{(1-s)^{p-2}}{\Gamma(p-q)}(1-(1-s)^{1-q})$. This completes the proof of the lemma. □

Lemma 7. *Suppose (A2) and (A3) hold. If $p-2\le q\le 1$, then the following properties of ${}_t\mathscr{D}^q_{0+}G(t,s)$ hold:*

(i) ${}_t\mathscr{D}^q_{0+}G(t,s):[0,1]\times[0,1]\to\mathbb{R}$ *is nonnegative and continuous;*

(ii) ${}_t\mathscr{D}^q_{0+}G(0,s)={}_t\mathscr{D}^q_{0+}G(t,0)={}_t\mathscr{D}^q_{0+}G(t,1)=0$ *for any $t,s\in[0,1]$;*

(iii) ${}_t\mathscr{D}^q_{0+}G(t,s)$ *is strictly increasing and concave down in t for $t\in[0,s]$;* ${}_t\mathscr{D}^q_{0+}G(t,s)$ *is strictly decreasing and concave up in t for $t\in[s,1]$;*

(iv) $\max\limits_{t\in[0,1]}{}_t\mathscr{D}^q_{0+}G(t,s)={}_t\mathscr{D}^q_{0+}G(s,s)$ *for any $s\in[0,1]$.*

Proof. If $q=p-2$, then

$${}_t\mathscr{D}^q_{0+}G(t,s)=\begin{cases}(1-s)^{p-2}t-(t-s), & 0\le s\le t\le 1,\\ (1-s)^{p-2}t, & 0\le t\le s\le 1,\end{cases}$$

and for any fixed $s\in(0,1)$,

$$\frac{\partial({}_t\mathscr{D}^q_{0+}G(t,s))}{\partial t}=\begin{cases}(1-s)^{p-2}-1, & s<t,\\ (1-s)^{p-2}, & t<s.\end{cases}$$

Hence, all the conclusions are true.

If $p-2<q\leq 1$, parts (i) and (ii) are clearly true. Notice that for any fixed $s\in(0,1)$,

$$\frac{\partial({}_t\mathscr{D}^q_{0^+}G(t,s))}{\partial t}=\frac{1}{\Gamma(p-q-1)}\begin{cases}(1-s)^{p-2}t^{p-2-q}-(t-s)^{p-2-q}<0, & s<t<1,\\ (1-s)^{p-2}t^{p-2-q}>0, & 0<t<s,\end{cases}$$

and

$$\Gamma(p-q-2)\frac{\partial^2({}_t\mathscr{D}^q_{0^+}G(t,s))}{\partial t^2}=\begin{cases}(1-s)^{p-2}t^{p-3-q}-(t-s)^{p-3-q}<0, & s<t<1,\\ (1-s)^{p-2}t^{p-3-q}>0, & 0<t<s.\end{cases}$$

Thus, (iii) and (iv) are true, and this completes the proof. □

Lemma 8. *Suppose* (A2) *and* (A3) *hold. If* $1<q<p-1$*, then:*

(i) ${}_t\mathscr{D}^q_{0^+}G(t,s):[0,1]\times[0,1]\to\mathbb{R}$ *is continuous;*

(ii) ${}_t\mathscr{D}^q_{0^+}G(0,s)={}_t\mathscr{D}^q_{0^+}G(t,0)={}_t\mathscr{D}^q_{0^+}G(t,1)=0$ *for any* $t,s\in[0,1]$*;*

(iii) ${}_t\mathscr{D}^q_{0^+}G(t,s)$ *is strictly increasing and concave down in* t *for* $t\in[0,s]$*, and* ${}_t\mathscr{D}^q_{0^+}G(t,s)$ *is strictly decreasing and concave up in* t *for* $t\in[s,1]$*;*

(iv) $\max_{t\in[0,1]}|{}_t\mathscr{D}^q_{0^+}G(t,s)|={}_t\mathscr{D}^q_{0^+}G(s,s)$ *for any* $s\in[0,1]$.

Proof. The proofs of parts (i), (ii), and (iii) are similar to those of the corresponding parts of Lemma 7.

To prove (iv), it suffices to show that ${}_t\mathscr{D}^q_{0^+}G(s,s)\geq|{}_t\mathscr{D}^q_{0^+}G(1,s)|$. Now $\Gamma(p-q)\,{}_t\mathscr{D}^q_{0^+}G(s,s)=(1-s)^{p-2}s^{p-1-q}$ and $-\Gamma(p-q)\,{}_t\mathscr{D}^q_{0^+}G(1,s)=(1-s)^{p-1-q}-(1-s)^{p-2}\geq 0$, so

$$\begin{aligned}\Gamma(p-q)({}_t\mathscr{D}^q_{0^+}G(s,s)-|{}_t\mathscr{D}^q_{0^+}G(1,s)|) &= (1-s)^{p-2}s^{p-1-q}-(1-s)^{p-1-q}+(1-s)^{p-2}\\ &= (1-s)^{p-2}(s^{p-1-q}-(1-s)^{1-q}+1).\end{aligned}$$

Let $c(s):=s^{p-1-q}-(1-s)^{1-q}+1$ and notice that $c(s)\in C[0,1]\cap C^2(0,1)$, $c(0)=0$, and $c(1)=2>0$. We next show $c''(s)\leq 0$ for $s\in(0,1)$. It is easy to see that for any $s\in(0,1)$,

$$c'(s)=(p-1-q)s^{p-2-q}+(1-q)(1-s)^{-q}$$

and

$$c''(s)=(p-1-q)(p-2-q)s^{p-3-q}+(1-q)q(1-s)^{-q-1}\leq 0.$$

Therefore, $c(s) \geq 0$ for any $s \in [0,1]$, and so $\max_{t\in[0,1]} |{}_t\mathscr{D}^q_{0+}G(t,s)| = {}_t\mathscr{D}^q_{0+}G(s,s)$ for any $s \in [0,1]$. □

Remark 3. Not all the properties of ${}_t\mathscr{D}^q_{0+}G(s,s)$ derived in Lemmas 5–8 will be directly used in the proof of the main theorem in our paper, but they are useful in applications and for future reference.

Definition 3. ([13]) *For a function $h : [0,1] \times [0,1] \to \mathbb{R}$, $h(t,s)$ is defined to be Riemann integrable with respect to s on $[0,1]$ uniformly in $t \in [0,1]$, if for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for any partition $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ with a finite sequence of numbers $s_1, s_2, \ldots, s_n$ satisfying $\Delta x_i := x_i - x_{i-1} < \delta(\varepsilon)$ and $s_i \in [x_{i-1}, x_i]$ for $i = 1,2,\ldots,n$, we have*

$$\left|\sum_{i=1}^{n} h(t,s_i)\Delta x_i - \int_0^1 h(t,s)ds\right| < \varepsilon \text{ for any } t \in [0,1].$$

Lemma 9. ([10]) *If $h : [0,1] \times [0,1] \to \mathbb{R}$ is continuous, then $h(t,s)$ is Riemann integrable with respect to s on $[0,1]$ uniformly in $t \in [0,1]$.*

In what follows, we will make use of the following notation. The bounded linear operator L applied to $G(t,s)$ with respect to t will be denoted by $L^t(G(t,s))$.

Lemma 10. *Suppose (A2) and (A3) hold and assume that $g : [0,1] \times [0,1] \to \mathbb{R}$ and ${}_t\mathscr{D}^q_{0+}g(t,s)$ are continuous on $[0,1] \times [0,1]$. Then,*

$$L^t\left(\int_0^1 g(t,s)ds\right) = \int_0^1 L^t(g(t,s))ds. \tag{10}$$

Proof. Claim 1: $\int_0^1 L^t(g(t,s))ds$ is well defined.

Since L is a bounded linear functional defined on E and g and ${}_t\mathscr{D}^q_{0+}g(t,s)$ are uniformly continuous on $[0,1] \times [0,1]$, we have

$$|L^t(g(t,s_1)) - L^t(g(t,s_2))| \leq \|L^t\| \cdot \|g(t,s_1) - g(t,s_2)\|_E,$$

which implies that $L^t(g(t,s))$ is continuous in $s \in [0,1]$. Hence, $L^t(g(t,s))$ is Riemann integrable with respect to s on $[0,1]$.

Claim 2: $\int_0^1 {}_t\mathscr{D}^q_{0+}(g(t,s))ds = {}_t\mathscr{D}^q_{0+}\left(\int_0^1 g(t,s)ds\right)$.

Since ${}_t\mathscr{D}^q_{0^+}g(t,s)$ is continuous on $[0,1]\times[0,1]$, by the Lebesgue Convergence Theorem and the definition of differentiation of an integer order, we have

$$\int_0^1 \frac{\partial^n}{\partial t^n}\left(\int_0^t \frac{g(r,s)}{(t-r)^{q-n+1}}dr\right)ds = \frac{d^n}{dt^n}\left(\int_0^1\int_0^t \frac{g(r,s)}{(t-r)^{q-n+1}}drds\right). \tag{11}$$

Therefore, by Fubini's Theorem and (11),

$$\begin{aligned}
{}_t\mathscr{D}^q_{0^+}\left(\int_0^1 g(t,s)ds\right) &= \frac{1}{\Gamma(n-q)}\frac{d^n}{dt^n}\left(\int_0^t (t-r)^{n-q-1}\int_0^1 g(r,s)dsdr\right)\\
&= \frac{1}{\Gamma(n-q)}\frac{d^n}{dt^n}\left(\int_0^t\int_0^1 (t-r)^{n-q-1}g(r,s)dsdr\right)\\
&= \frac{1}{\Gamma(n-q)}\frac{d^n}{dt^n}\left(\int_0^1\int_0^t (t-r)^{n-q-1}g(r,s)drds\right)\\
&= \int_0^1 {}_t\mathscr{D}^q_{0^+}(g(t,s))ds.
\end{aligned}$$

Claim 3: $L^t\left(\int_0^1 g(t,s)ds\right)$ is well defined.

By Claim 2, $\int_0^1 g(t,s)ds\in E$. Therefore, $L^t\left(\int_0^1 g(t,s)ds\right)$ is well defined.

Claim 4: $L^t\left(\int_0^1 g(t,s)ds\right)=\int_0^1 L^t(g(t,s))ds$.

It suffices to show that for any $\varepsilon>0$, there exists $\sigma(\varepsilon)>0$ such that for any partition $0=x_0<x_1<\cdots<x_{l-1}<x_l=1$ with $s_1,s_2,\ldots,s_l$ satisfying $\Delta x_i<\sigma(\varepsilon)$ and $s_i\in[x_{i-1},x_i]$ for $i=1,2,\ldots,l$, we have

$$\left|\sum_{i=1}^{l} L^t(g(t,s_i))\Delta x_i - L^t\left(\int_0^1 g(t,s)ds\right)\right|<\varepsilon. \tag{12}$$

By Lemma 9, $g(t,s)$ and ${}_t\mathscr{D}^q_{0^+}g(t,s)$ are both Riemann integrable with respect to s on $[0,1]$ uniformly in $t\in[0,1]$. Then, for the above $\varepsilon>0$, there exists $\sigma(\varepsilon)>0$ such that for any partition $0=x_0<x_1<\cdots<x_{n-1}<x_n=1$ with $s_1,s_2,\ldots,s_n$ satisfying $\Delta x_i<\sigma(\varepsilon)$ and $s_i\in[x_{i-1},x_i]$ for $i=1,2,\ldots,n$, we have

$$\left|\sum_{i=1}^{n} g(t,s_i)\Delta x_i - \int_0^1 g(t,s)ds\right|<\frac{\varepsilon}{\|L\|},\quad \text{for any}\quad t\in[0,1], \tag{13}$$

and

$$\left|\sum_{i=1}^{n} {}_t\mathscr{D}^q_{0^+}g(t,s_i)\Delta x_i - \int_0^1 {}_t\mathscr{D}^q_{0^+}g(t,s)ds\right|<\frac{\varepsilon}{\|L\|},\quad \text{for any}\quad t\in[0,1]. \tag{14}$$

By (13), (14), and Claim 2, we have

$$\left\| \sum_{i=1}^{n} g(t,s_i)\Delta x_i - \int_0^1 g(t,s)ds \right\|_E < \frac{\varepsilon}{\|L\|}, \text{ for any } t \in [0,1]. \tag{15}$$

Therefore,

$$\begin{aligned} \left| \sum_{i=1}^{n} L^t(g(t,s_i))\Delta x_i - L^t\left(\int_0^1 g(t,s)ds\right) \right| &= \left| L^t\left(\sum_{i=1}^{n} g(t,s_i)\Delta x_i - \int_0^1 g(t,s)ds\right) \right| \\ &\le \|L\| \cdot \left\| \sum_{i=1}^{n} g(t,s_i)\Delta x_i - \int_0^1 g(t,s)ds \right\|_E \\ &< \varepsilon. \end{aligned}$$

This proves the lemma. □

4. Existence criteria

Now, consider the linear fractional differential equations with non-homogeneous boundary conditions for some $h \in C([0,1],[0,\infty))$:

$$_t\mathscr{D}^q_{0^+}x(t) + h(t) = 0,\ 0 < t < 1, \tag{16}$$

$$x(0) = x''(0) = 0,\ x(1) = L(x). \tag{17}$$

Lemma 11. *Suppose (A2) and (A3) hold and $L^t(t^{p-1}) \neq p-1$. Then the boundary value problem (16), (17) has a unique solution $x \in E$ that can be expressed as*

$$x(t) = \int_0^1 (G(t,s) + H(t,s))h(s)ds, \tag{18}$$

where

$$H(t,s) = \frac{L^t(G(t,s))}{p-1-L(t^{p-1})}t^{p-1} \tag{19}$$

and

$$\max_{t\in[0,1]} |H(t,s)| = |H(1,s)|, \quad H(t,s) = t^{p-1}H(1,s).$$

Proof. From Lemma 3, we have

$$x(t) = -\frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}h(s)ds + c_1 t^{p-1} + c_2 t^{p-2} + c_3 t^{p-3}.$$

Now $x(0) = x'(0) = 0$ implies $c_2 = c_3 = 0$. From the fact that $x'(1) = L(x)$, it follows that

$$L(x) = -\int_0^t (p-1)\frac{(t-s)^{p-2}}{\Gamma(p)}h(s)ds + c_1(p-1)t^{p-2}\Big|_{t=1},$$

that is,

$$L(x) = -\int_0^1 (p-1)\frac{(1-s)^{p-2}}{\Gamma(p)}h(s)ds + c_1(p-1).$$

Thus,

$$c_1 = \frac{L(x)}{p-1} + \int_0^1 \frac{(1-s)^{p-2}}{\Gamma(p)}h(s)ds,$$

which gives

$$\begin{aligned} x(t) &= -\frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}h(s)ds + \left(\int_0^1 \frac{(1-s)^{p-2}}{\Gamma(p)}h(s)ds\right)t^{p-1} + \frac{L(x)}{p-1}t^{p-1} \\ &= \int_0^1 G(t,s)h(s)ds + \frac{L(x)}{p-1}t^{p-1}. \end{aligned} \tag{20}$$

Applying L to both sides of (20), by Lemma 6 (i), Lemma 7 (i), Lemma 8 (i), and Lemma 10, we have

$$L(x) = \int_0^1 L^t(G(t,s)h(s)ds + \frac{L(x)}{p-1}L(t^{p-1}).$$

Hence,

$$L(x) = \int_0^1 \frac{p-1}{p-1-L(t^{p-1})}L^t(G(t,s)h(s)ds.$$

Therefore, for $t \in [0,1]$,

$$\begin{aligned} x(t) &= \int_0^1 G(t,s)h(s)ds + \int_0^1 \left(\frac{L^t(G(t,s))}{p-1-L(t^{p-1})}\cdot t^{p-1}\right)h(s)ds \\ &= \int_0^1 (G(t,s)+H(t,s))h(s)ds, \end{aligned}$$

which implies the uniqueness of solutions of BVP (16), (17). □

Next, we define the operator $T : E \to E$ by

$$(Tx)(t) = \int_0^1 (G(t,s)+H(t,s))f(s,x(s),(Nx)(s))ds. \tag{21}$$

By Lemma 11, the fixed points of T coincide with the solutions of problem (1), (2). Next, we will apply the fixed point theorem given in Lemma 4 to obtain the existence of at least three positive solutions of (1), (2).

We define the cone $P = \{x \in E : x(t) \geq 0,\ 0 \leq t \leq 1,\ x \text{ is nondecreasing on } [0,1]\}$ and non-negative continuous functionals α, β, $\gamma : P \to [0,\infty)$ by

$$\alpha(x) = \|x\|_\infty, \quad \beta(x) = \|\mathscr{D}^q_{0^+}x\|_\infty, \quad \gamma(x) = \min_{t\in[\xi,1]} |x(t)|.$$

Then, α and β are convex functionals on P, γ is concave on P, and $\gamma(x) \leq \alpha(x)$ for any $x \in P$. Also, $\|x\|_E = \max\{\alpha(x), \beta(x)\}$ for any $x \in P$ and $P(\alpha, k; \beta, Q)$ is a nonempty bounded open subset of P for any $k > 0$ and $Q > 0$.

Theorem 1. *Assume* (A_1), (A_2), *and* (A_3) *hold,* $L^t(G(t,s)) \geq 0$ *for* $s \in [0,1]$, *and* $L^t(t^{p-1}) \in [0, p-1)$. *In addition, assume there exist* $k_2 \geq d > b > k_1 > 0$ *and* $Q_2 > Q_1 > 0$, *where* $d = \xi^{1-p}b$, *such that* f *satisfies the following conditions:*

(H1) $f(t,x(t),(Nx)(t)) \leq U_2(t)$ *for* $t \in [0,1]$, $x \in P$ *with* $\|x\|_\infty \leq k_2$, *and* $\|\mathscr{D}^q_{0^+}x\|_\infty \leq Q_2$ *such that* $\int_0^1 (G(1,s)+H(1,s))U_2(s)ds \leq k_2$ *and* $\int_0^1 (\max_{t\in[0,1]} |{}_t\mathscr{D}^q_{0^+}G(t,s)| + {}_t\mathscr{D}^q_{0^+}H(1,s))U_2(s)ds \leq Q_2$;

(H2) $f(t,x(t),(Nx)(t)) \geq U_0(t)$ *for any* $t \in [\xi,1]$, $x \in P$ *with* $b \leq x(t) \leq d$ *for* $t \in [\xi,1]$, *and* $\|\mathscr{D}^q_{0^+}x\|_\infty \leq Q_2$ *such that* $\int_\xi^1 (G(\xi,s)+H(\xi,s))U_0(s)ds > b$;

(H3) $f(t,x(t),(Nx)(t)) \leq U_1(t)$ *for any* $t \in [0,1]$, $x \in P$ *with* $\|x\|_\infty \leq k_1$, *and* $\|\mathscr{D}^q_{0^+}x\|_\infty \leq Q_1$ *such that* $\int_0^1 (G(1,s)+H(1,s))U_1(s)ds \leq k_1$ *and*

$$\int_0^1 (\max_{t\in[0,1]} |{}_t\mathscr{D}^q_{0^+}G(t,s)| + {}_t\mathscr{D}^q_{0^+}H(1,s))U_1(s)ds \leq Q_1.$$

Then, the problem (1), (2) *has at least three positive solutions* x_1, x_2, x_3 *such that*

$$0 \leq x_1(t) \leq k_1, \quad 0 \leq {}_t\mathscr{D}^q_{0^+}x_1(t) \leq Q_1 \quad \text{for} \quad t \in [0,1];$$

$$0 \leq x_2(t) \leq k_2, \quad 0 \leq {}_t\mathscr{D}^q_{0^+}x_2(t) \leq Q_2 \quad \text{for} \quad t \in [0,1], \quad x_2(t) > b \quad \text{for} \quad t \in [\xi,1];$$

$$0 \leq x_3(t) \leq k_2, \quad 0 \leq {}_t\mathscr{D}^q_{0^+}x_3(t) \leq Q_2 \quad \text{for} \quad t \in [0,1];$$

and there exists $\eta_1 \in [\xi,1]$ *such that* $x_3(t) < b$ *for* $t \in [0,\eta]$, *and there exists* $\eta_2 \in [0,1]$ *such that either* $x_3(\eta_2) > k_1$ *or* ${}_t\mathscr{D}^q_{0^+}x_3(\eta_2) > Q_1$.

Proof. Claim 1: $T : \overline{P}(\alpha, k_2; \beta, Q_2) \to \overline{P}(\alpha, k_2; \beta, Q_2)$.

First, T maps P into P because of Lemmas 5–8, the properties of $H(t,s)$, and condition (A1). Notice that for any $x \in \overline{P}(\alpha, k_2; \beta, Q_2)$, we have $0 \le x(t) \le k_2$ and $|{}_t\mathscr{D}^q_{0^+}x(t)| \le Q_2$ for any $t \in [0,1]$, which implies $|(Nx)(t))| \le \|N\|\|x\|_E \le \|N\| \max\{k_2, Q_2\}$ for any $t \in [0,1]$. Hence, by condition (H1), $f(t,x(t),(Nx)(t)) \le U_2(t)$ for all $t \in [0,1]$. Therefore, by (H1),

$$0 \le (Tx)(t) \le \int_0^1 (G(t,s)+H(t,s))U_2(s)ds \le \int_0^1 (G(1,s)+H(1,s))U_2(s)ds \le k_2,$$

and

$$|{}_t\mathscr{D}^q_{0^+}(Tx)(t)| \le \int_0^1 (\max_{t\in[0,1]} |{}_t\mathscr{D}^q_{0^+}G(t,s)| + {}_t\mathscr{D}^q_{0^+}H(1,s))U_2(s)ds \le Q_2,$$

that is, $Tx \in \overline{P}(\alpha, k_2; \beta, Q_2)$.

Claim 2: T is completely continuous on P.

Since f is continuous on $[0,1]\times[0,\infty)\times\mathbb{R}$, and $G(t,s)$ and ${}_t\mathscr{D}^q_{0^+}G(t,s)$ are continuous on $[0,1]\times[0,1]$, by the Ascoli-Arzelá Theorem, a standard argument shows that T is completely continuous.

Claim 3: $\{x \in \overline{P}(\alpha,d;\beta,Q_2;\gamma,b) : \gamma(x) > b\} \neq \emptyset$ and $\gamma(Tx) > b$ for $x \in \overline{P}(\alpha,d;\beta,Q_2;\gamma,b)$.

Let $x_0(t) := \frac{b+d}{2}$ for $t \in [0,1]$. Then, $x_0 \in \{x \in \overline{P}(\alpha,d;\beta,Q_2;\gamma,b) : \gamma(x) > b\}$. For any $x \in \overline{P}(\alpha,d;\beta,Q_2;\gamma,b)$, we have $0 \le x(t) \le d$ and $|{}_t\mathscr{D}^q_{0^+}x(t)| \le Q_2$ for $t \in [0,1]$ and $b \le x(t) \le d$ for $t \in [\xi,1]$. Hence, by (H2),

$$\begin{aligned}
\gamma(Tx) &= (Tx)(\xi) \\
&= \int_0^1 (G(\xi,s)+H(\xi,s))f(s,x(s),(Nx)(s))ds \\
&\ge \int_\xi^1 (G(\xi,s)+H(\xi,s))f(s,x(s),(Nx)(s))ds \\
&\ge \int_\xi^1 (G(\xi,s)+H(\xi,s))U_0(s)ds \\
&> b.
\end{aligned}$$

Claim 4: $\alpha(Tx) < k_1$ and $\beta(Tx) < Q_1$ for all $x \in \overline{P}(\alpha,k_1;\beta,Q_1)$.

For any $x \in \overline{P}(\alpha,k_1;\beta,Q_1)$, we have $0 \le x(t) \le k_1$ and $|{}_t\mathscr{D}^q_{0^+}x(t)| \le Q_1$ for $t \in [0,1]$. By (H3), we have

$$\begin{aligned}
\alpha(Tx) &= (Tx)(1) \\
&= \int_0^1 (G(1,s)+H(1,s))f(s,x(s),(Nx)(s))ds \\
&\le \int_\xi^1 (G(\xi,s)+H(\xi,s))U_2(s)ds \\
&< k_1
\end{aligned}$$

and

$$\begin{aligned}
\beta(Tx) &= \max_{t\in[0,1]} |{}_t\mathscr{D}^q_{0^+}(Tx)(t)| \\
&\le \int_0^1 \max_{t\in[0,1]} |{}_t\mathscr{D}^q_{0^+}G(t,s) + {}_t\mathscr{D}^q_{0^+}H(t,s)| f(s,x(s),(Nx)(s))ds \\
&\le \int_0^1 (\max_{t\in[0,1]} |{}_t\mathscr{D}^q_{0^+}G(t,s)| + {}_t\mathscr{D}^q_{0^+}H(1,s))U_2(s)ds \\
&< Q_1.
\end{aligned}$$

Claim 5: $\gamma(Tx) > b$, for all $x \in \overline{P}(\alpha,k_2;\beta,Q_2;\gamma,b)$ with $\alpha(Tx) > d$.

For any $x \in \overline{P}(\alpha,k_2;\beta,Q_2;\gamma,b)$ satisfying $\alpha(Tx) > d$, we have $0 \le x(t) \le k_2$, $|{}_t\mathscr{D}^q_{0^+}x(t)| \le Q_2$ for $t \in [0,1]$, and $b \le x(t) \le k_2$ for $t \in [\xi,1]$.

Since $\alpha(Tx) > d$, that is, $(Tx)(1) > d$, by Lemma 5, we have $G(t,s) \ge t^{p-1}G(1,s)$ and $H(t,s) = t^{p-1}H(1,s)$ for any $t \in [0,1]$. Hence,

$$\begin{aligned}
\gamma(Tx) &= (Tx)(\xi) \\
&= \int_0^1 (G(\xi,s)+H(\xi,s))f(s,x(s),(Nx)(s))ds \\
&\ge \xi^{p-1}\int_0^1 (G(1,s)+H(1,s))f(s,x(s),(Nx)(s))ds \\
&= \xi^{p-1}(Tx)(1) \\
&> \xi^{p-1}d \\
&= b.
\end{aligned}$$

Therefore, all conditions of Lemma 4 are satisfied. We can conclude that there are three positive solutions x_1, x_2, x_3 of problem (1), (2) such that $x_1, x_2, x_3 \in \overline{P}(\alpha, k_2; \beta, Q_2)$ and

$$x_1 \in \overline{P}(\alpha, k_1; \beta, Q_1),$$

$$x_2 \in \{\overline{P}(\alpha, d; \beta, Q_2; \gamma, b) : \gamma(x) > b\},$$

$$x_3 \in \overline{P}(\alpha, d; \beta, Q_2) \setminus \{\overline{P}(\alpha, d; \beta, Q_2; \gamma, b) \cup \overline{P}(\alpha, k_1; \beta, Q_1)\},$$

i.e.,

$$0 \le x_1(t) \le k_1, \quad 0 \le {}_t\mathscr{D}^q_{0+} x_1(t) \le Q_1 \text{ for } t \in [0,1],$$

$$0 \le x_2(t) \le k_2, \quad 0 \le {}_t\mathscr{D}^q_{0+} x_2(t) \le Q_2 \text{ for } t \in [0,1], \quad x_2(t) > b \text{ for } t \in [\xi, 1],$$

$$0 \le x_3(t) \le k_2, \quad 0 \le {}_t\mathscr{D}^q_{0+} x_3(t) \le Q_2 \text{ for } t \in [0,1],$$

and there exists $\eta_1 \in [\xi, 1]$ such that $x_3(t) < b$ for $t \in [0, \eta]$ and there exists $\eta_2 \in [0,1]$ such that $x_3(\eta_2) > k_1$ or ${}_t\mathscr{D}^q_{0+} x_3(\eta_2) > Q_1$. This completes the proof of the theorem. □

Example 1. Consider the BVP,

$$\mathscr{D}^{2.5}_{0^+} x(t) + f\left(t, x(t), \int_0^t x(s)ds\right) = 0,\ 0 < t < 1, \tag{22}$$

$$x(0) = x'(0) = 0,\ x'(1) = -\int_0^{\frac{1}{2}} x(t)dt + \int_{\frac{1}{2}}^1 x(t)dt, \tag{23}$$

where $f(t,u,v) = \begin{cases} e^{v-1}(999000u^3 + 0.001), & \text{if } 0 \le u \le 0.01, \\ e^{v-1}(1 + (u - 0.01)\sin(3t)), & \text{if } u \ge 0.01. \end{cases}$

We have $p = 2.5$ and $q = 0$. Clearly, (A1) and (A2) are satisfied. We take $E = C[0,1]$ for the Banach space in this example. It is easy to check that $L(x) = -\int_0^{\frac{1}{2}} x(t)dt + \int_{\frac{1}{2}}^1 x(t)dt$ and $(Nx)(t) = \int_0^t x(s)ds$ are bounded linear operators on E, $\|L\| < 1$, and $\|N\| = 1$.

Notice that

$$0 < L^t(t^{p-1}) = L^t(t^{1.5}) = -\int_0^{\frac{1}{2}} t^{1.5}dt + \int_{\frac{1}{2}}^1 t^{1.5}dt = \frac{2}{5}(1 - 2^{-1.5}) \approx 0.259 < 1.5 = p - 1;$$

for $s \in [0, \frac{1}{2}]$,

$$\begin{aligned} 0 &\leq L^t(G(t,s)) \\ &= \frac{1}{\Gamma(2.5)}\left[-\int_0^{\frac{1}{2}}(1-s)^{0.5}t^{1.5}dt + \int_{\frac{1}{2}}^1 (1-s)^{0.5}t^{1.5} - (t-s)^{1.5}dt + \int_s^{\frac{1}{2}}(t-s)^{1.5}dt\right] \\ &= \frac{1}{\Gamma(3.5)}\left[2(0.5-s)^{2.5} + (1-2^{-1.5})(1-s)^{0.5} - (1-s)^{2.5}\right] \\ &\leq \frac{1}{5\Gamma(2.5)} < \frac{1}{5}; \end{aligned}$$

and for $s \in [\frac{1}{2}, 1]$,

$$\begin{aligned} 0 &\leq L^t(G(t,s)) \\ &= \frac{1}{\Gamma(2.5)}\left[-\int_0^{\frac{1}{2}}(1-s)^{0.5}t^{1.5}dt + \int_{\frac{1}{2}}^1 (1-s)^{0.5}t^{1.5}dt - \int_s^1 (t-s)^{1.5}dt\right] \\ &= \frac{1}{\Gamma(3.5)}\left[(1-2^{-1.5})(1-s)^{0.5} - (1-s)^{2.5}\right] \\ &\leq \frac{1}{5\Gamma(2.5)} < \frac{1}{5}. \end{aligned}$$

Choose $\xi = \frac{1}{2}$, $Q_1 = k_1 = 0.001$, $b = 0.01$, $Q_2 = k_2 = 1$, and $d = 2^{1.5}b \approx 0.028$. Let $U_0(t) = e^{0.01t-1.005}$, $U_1(t) = 0.001999e^{0.001t-1}$, and $U_2(t) = e^{t-1}(1+\sin(3t))$. For any $x \in P$ with $\|x\|_E \leq 1$, we have $0 \leq x(t) \leq 1$ for $t \in [0,1]$ and so for $t \in [0,1]$,

$$\begin{aligned} f\left(t, x(t), \int_0^t x(s)ds\right) &= \begin{cases} e^{\int_0^t x(s)ds-1}(999000x^3(t)+0.001), & \text{if } 0 \leq x(t) \leq 0.01, \\ e^{\int_0^t x(s)ds-1}(1+(x(t)-0.01)\sin(3t)), & \text{if } x(t) \geq 0.01, \end{cases} \\ &\leq e^{\int_0^t x(s)ds-1}(1+(x(t)-0.01)\sin(3t)) \\ &\leq e^{t-1}(1+\sin(3t)) = U_2(t). \end{aligned}$$

Now to show (H1) holds, it suffices to prove $\int_0^1 (G(1,s)+H(1,s))e^{t-1}(1+\sin(3t))ds \leq 1$. Notice that $0 \leq \frac{L^t(G(t,s))}{1.5-L^t(t^{1.5})} < \frac{1}{5}$ for any $s \in [0,1]$, so

$$\begin{aligned}
&\int_0^1 (G(1,s)+H(1,s))e^{t-1}(1+\sin(3t))ds \\
&\qquad = \int_0^1 \left(\frac{1}{\Gamma(2.5)}(1-s)^{0.5}s + \frac{L^t(G(t,s))}{1.5-L^t(t^1.5)}\right) e^{t-1}(1+\sin(3t))ds \\
&\qquad \leq \int_0^1 \left((1-s)^{0.5}s+0.2\right) e^{t-1}(1+\sin(3t))ds \\
&\qquad \approx 0.52 < 1.
\end{aligned}$$

For any $x \in P$ with $\|x\|_E \leq 0.001$, we have $0 \leq x(t) \leq 0.001$ for $t \in [0,1]$. Hence, for $t \in [0,1]$,

$$f\left(t, x(t), \int_0^t x(s)ds\right) = e^{\int_0^t x(s)ds-1}(999000x(t)^3+0.001) \leq 0.00100999e^{0.001t-1} = U_1(t),$$

and so

$$\begin{aligned}
\int_0^1 (G(1,s)+H(1,s))0.00100999e^{0.001t-1}ds &\leq \int_0^1 \left((1-s)^{0.5}s+0.2\right)0.00100999e^{0.001t-1}ds \\
&\approx 0.00033647 < 0.001 = k_1.
\end{aligned}$$

Thus, (H3) holds.

Finally, we need to show (H2) is satisfied. Let $x \in P$ with $0.01 = b \leq x(t) \leq d \approx 0.028$ for $t \in [\xi, 1] = [0.5, 1]$. Then for $t \in [0.5, 1]$,

$$\begin{aligned}
f\left(t, x(t), \int_0^t x(s)ds\right) &= e^{\int_0^t x(s)ds-1}(1+(x(t)-0.01)\sin(3t)) \\
&\geq e^{\int_{0.5}^t x(s)ds-1} \\
&\geq e^{0.01t-1.005} = U_0(t),
\end{aligned}$$

and so

$$\begin{aligned}
\int_\xi^1 (G(\xi,s)+H(\xi,s))U_0(s)ds &\geq \int_{0.5}^1 (0.5)^{1.5}\left(\frac{1}{\Gamma(2.5)}(1-s)^{0.5}s\right)e^{0.01s-1.005}ds \\
&\approx 0.011 \\
&> 0.01 = b.
\end{aligned}$$

Hence, (H2) holds.

Therefore, by Theorem 1, the BVP (22), (23) has at least three positive solutions $x_1, x_2, x_3 \in C[0,1]$, such that $0 \le x_1(t) \le 0.001$, $0 \le x_2(t) \le 1$, $0 \le x_3(t) \le 1$ for $t \in [0,1]$, and $x_2(t) > 0.01$ for $t \in [0.5,1]$. Also, there exists $\eta_1 \in [0.5,1]$ such that $x_3(t) < 0.01$ for $t \in [0,\eta_1]$ and there exists $\eta_2 \in (0,1]$ such that $x_3(\eta_2) > 0.001$.

References

[1] B. Ahmad, J. J. Nieto, Anti-periodic fractional boundary value problems with nonlinear term depending on lower order derivative, Fract. Calc. Appl. Anal. 15 (2012), 451–462.

[2] G. A. Anastassiou, Fractional Differentiation Inequalities, Springer, Dordrecht, 2009.

[3] Z. Bai, W. Ge, Existence of three positive solutions for some second-order boundary value problems, Comput. Math. Appl. 48 (2004), 699–707.

[4] A. Cabada, G Wang, Positive solutions of nonlinear fractional differential equations with integral boundary value problems, J. Math. Anal. Appl. 389 (2012), 403–411.

[5] M. El-Shahed, Positive solutions for boundary-value problem of nonlinear fractional differential equation, Abstr. Appl. Anal. 2007 (2007), Article ID 10368.

[6] J. R. Graef, L. Kong, Q. K. Kong, M. Wang, Uniqueness of positive solutions of fractional boundary value problems with non-homogeneous integral boundary conditions, Fract. Calc. Appl. Anal. 15 (2012), 509–528.

[7] J. R. Graef, L. Kong, B. Yang, Positive solutions for a semipositone fractional boundary value problem with a forcing term, Fract. Calc. Appl. Anal. 15 (2012), 8–24.

[8] A. Guezane-Lakoud and L. Zenkoufi, Existence of positive solutions for a third-order multi-point boundary value problem, Appl. Math. 3 (2012), 1008–1013.

[9] J. Henderson, X. Liu, S. Sutherland, Y. Tian, Positive solutions for a second order impulsive BVP with bounded linear operator conditions, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 20 (2013), 571–592.

[10] J. Henderson, X. Liu, Positive solutions of fractional differential equations with bounded linear functional conditions, Nonlinear Stud. to appear.

[11] G. L. Karakostas, Solvability of the Φ-Laplacian with nonlocal boundary conditions, Appl. Math. Comput. 215 (2009), 514–523.

[12] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Math. Stud., vol 204, Elsevier Science B.V., Amsterdam, 2006.

[13] R. E. Langer, J. D. Tamarkin, A notion of uniform integrability, Bull. Amer. Math. Soc. 32 (1926), 335–337.

[14] S. H. Liang, Y. Q. Song, Existence and uniqueness of positive solutions to nonlinear fractional differential equation with integral boundary conditions, Lithuanian Math. J. 52 (2012), 62–76.

[15] S. H. Liang, J. H. Zhang, Existence and uniqueness of positive solutions to m-point boundary value problems for nonlinear fractional differential equation, J. Appl. Math. Comput. 38 (2012), 225–241.

[16] X. P. Liu, J. F. Jin, M. Jia, Multiple positive solutions of integral boundary value problems for fractional differential equations, J. Appl. Math. & Informatics 30 (2012), 305–320.

[17] I. Podlubny, Fractional Differential Equations, Mathematics in Science and Engineering 198, Academic Press, New York, 1999.

[18] S. G. Samko, A. A. Kilbas, O. I. Marichev, Fractional Integrals and Derivatives, Theory and Applications, Gordon and Breach, Yverdon, 1993.

[19] G. Wang, S. K. Ntouyas, L. Zhang, Positive solutions of the three-point boundary value problem for fractional-order differential equations with an advanced argument, Adv. Difference Equ. 2011 (2011), Article ID 2.

[20] J. R. L. Webb, G. Infante, Positive solutions of nonlocal boundary value problems: a unified approach, J. Lond. Math. Soc. 74 (2006), 673–693.