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# UNIQUENESS OF A POSITIVE SOLUTION AND EXISTENCE OF A SIGN-CHANGING SOLUTION FOR $(p,q)$-LAPLACE EQUATION

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**Abstract.** In this paper, we deal with the existence of a sign-changing solution and the uniqueness of a positive solution for quasilinear elliptic equations of the form

$$-\mu\Delta_p u - \Delta_q u = \lambda m_q(x)|u|^{q-2}u \quad \text{in } \Omega$$

with $1 < q < p < \infty$ and $\mu > 0$, under the Dirichlet boundary condition, where $\Omega$ is a bounded domain in $\mathbb{R}^N$ and $m_q$ is a weight function in $L^\infty(\Omega)$ admitting sign-change.

**Keywords.** Indefinite weight; Nonlinear eigenvalue problems; $(p,q)$-Laplacian; Mountain pass theorem.

## 1. Introduction and statements

In this paper, we consider the following quasilinear elliptic equation:

$$(GEV;q,\lambda,\mu) \qquad \begin{cases} -\mu\Delta_p u - \Delta_q u = \lambda m_q(x)|u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with $C^2$ boundary $\partial\Omega$, $\lambda \in \mathbb{R}$, $1 < q < p < \infty$, $\mu > 0$. Moreover, $\Delta_r$ denotes the usual $r$-Laplacian (i.e. $\Delta_r u = \mathrm{div}\,(|\nabla u|^{r-2}\nabla u)$ for $r \in (1,\infty)$) and $m_q \in L^\infty(\Omega)$ such that the Lebesgue measure of $\{x \in \Omega;, m_q(x) > 0\}$ is positive.

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In this paper, we say that $u \in W_0^{1,p}(\Omega)$ is a solution of $(GEV;q,\lambda,\mu)$ if there hold for all $\varphi \in W_0^{1,p}(\Omega)$

$$\mu \int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi \, dx + \int_\Omega |\nabla u|^{q-2}\nabla u \nabla \varphi \, dx = \lambda \int_\Omega m_q(x)|u|^{q-2}u\varphi \, dx.$$

However, it is proved that any solutions of our equation are of class $C_0^{1,\alpha}(\overline{\Omega})$ (some $\alpha \in (0,1)$) due to the regularity result (cf. [10] and [11], see Remark 2.1 also).

Letting $\mu \to 0^+$, our equation $(GEV;q,\lambda,\mu)$ turns into the following homogeneous equation $(EV;q,\lambda)$ known as the usual weighted eigenvalue problem for the $q$-Laplacian:

$$(EV;q,\lambda) \qquad \begin{cases} -\Delta_q u = \lambda m_q(x)|u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It is said that $\lambda$ is an eigenvalue of $-\Delta_r$ with weight function $m_r$ (for $r \in (1,\infty)$) if the equation

$$(EV;r,\lambda) \qquad \begin{cases} -\Delta_r u = \lambda m_r(x)|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a non-trivial solution which is called as an eigenfunction corresponding to $\lambda$. We denote the set of all eigenvalues of $-\Delta_r$ with weight function $m_r$ by $\sigma(-\Delta_r, m_r)$. In particular, in the case of $m_r \equiv 1$, we write $\sigma(-\Delta_r)$ instead of $\sigma(-\Delta_r, 1)$. Similarly, also in the non-homogeneous case, we say that $\lambda$ is a *generalized eigenvalue* of $-\Delta_q - \mu\Delta_p$ with weight function $m_q$ if $(GEV;q,\lambda,\mu)$ has a non-trivial solution.

Recently, many authors have studied $(p,q)$-Laplace equations (cf. [6], [12], [20], [24], [25] and [26]). However, there are few results on generalized eigenvalue problems of the $(p,q)$-Laplacian. In [1] and [2], Benouhiba and Belyacine considered the equation

$$-\Delta_p u - \Delta_q u = \lambda g(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N$$

under several assumptions on $g \geq 0$. They showed the existence of principal eigenvalue and a *continuous* family of generalized eigenvalues $\lambda$.

In [4, Theorem 4.2], Cingolani and Degiovanni proved the existence of a *non-trivial* solution for

$$-\Delta_p u - \mu\Delta u = \lambda|u|^{p-2}u + g(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

in the case of $p > 2$, $g \in C^1$ and $\lambda \notin \sigma(-\Delta_p)$.

The present author ([23]) completely described the generalized eigenvalues $\lambda$ for which the following equation has at least one *positive* solution:

$$(GEV;r,\lambda,\mu) \qquad \begin{cases} -\Delta_r u - \mu\Delta_{r^*} u = \lambda m_r(x)|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < r \neq r^* < \infty$ and $m_r \in L^\infty(\Omega) \setminus \{0\}$. Our equation $(GEV;q,\lambda,\mu)$ is the case of $r = q < p = r^*$.

Under the Neumann boundary condition, Mihăilescu [14] gave a set of all generalized eigenvalues $\lambda$ for which the equation

$$-\Delta_p u - \Delta u = \lambda u \quad \text{in} \quad \Omega,$$

has a *non-trivial* solution, where $p > 2$.

Regarding generalized eigenvalue problems for non-homogeneous operators, we refer to [5], [7], [16], [17], [19] and [22].

Here, let us recall the weighted eigenvalue problem of the $r$-Laplacian for $r \in (1,\infty)$. It is well known that the first (smallest) positive eigenvalue $\lambda_1(r,m_r)$ of $-\Delta_r$ with weight function $m_r$ is obtained by the following Rayleigh quotient:

$$\lambda_1(r,m_r) := \inf\left\{ \frac{\int_\Omega |\nabla u|^r\,dx}{\int_\Omega m_r|u|^r\,dx}\,;\, u \in W_0^{1,r}(\Omega),\ \int_\Omega m_r|u|^r\,dx > 0 \right\}.$$

Because there exist *no non-negative* eigenvalues provided $m_r \leq 0$, we set

$$\lambda_1(r,-m_r) = +\infty \quad \text{if } m_r \geq 0.$$

Moreover, $\lambda_1(r,m_r)$ has a positive eigenfunction $\varphi_1(r,m_r) \in C_0^{1,\alpha_r}(\overline{\Omega})$ (for some $\alpha_r \in (0,1)$). According to the standard argument using Rayleigh quotient, it is proved that if $-\lambda_1(r,-m_r) < \lambda < \lambda_1(r,m_r)$ holds, then $(EV;r,\lambda)$ has no non-trivial solutions (refer to [9, Proposition 4.1]). Moreover, it is also known that the homogeneous equation $(EV;r,\lambda)$ has no *positive* (or negative) solutions provided $\pm\lambda \neq \lambda_1(r,\pm m_r)$ (refer to [8, Section 6.2]). On the contrary, the present author ([23]) proved that equation $(GEV;r,\lambda,\mu)$ has no *non-trivial* solutions if $-\lambda_1(r,-m_r) \leq \lambda \leq \lambda_1(r,m_r)$, but it has *at least one* positive solution if $\pm\lambda > \lambda_1(r,\pm m_r)$.

First purpose of this paper is to prove the uniqueness of a positive solution. This is stated as follows.

**Theorem 1.1.** *If* $\pm\lambda > \lambda_1(q, \pm m_q)$ *holds respectively, then for any* $\mu > 0$, $(GEV; q, \lambda, \mu)$ *has a unique positive solution* $u_\mu \in \mathrm{int} C_0^1(\overline{\Omega})_+ := \{u \in C_0^1(\overline{\Omega})\,;\, u > 0 \text{ in } \Omega,\ \partial u/\partial \nu < 0 \text{ on } \partial\Omega\}$, *where* $\nu$ *is the outward unit normal vector. In particular,* $u_\mu = \mu^{1/(q-p)} u_1$ *holds, where* $u_1$ *is the positive solution of* $(GEV; q, \lambda, 1)$.

To state our second result, we recall the second (positive) eigenvalue $\lambda_2(r, m_r)$ of $-\Delta_r$ with weight function $m_r$. It is defined by

$$\lambda_2(r, m_r) = \min\{\lambda > \lambda_1(r, m_r)\,;\, \lambda \in \sigma(-\Delta_r, m_r)\,\}.$$

(note that $\lambda_1(r, m_r)$ is *isolated* and $\sigma(-\Delta_r, m_r)$ is closed). By the definitions of $\lambda_1(r, m_r)$ and $\lambda_2(r, m_r)$, if $\lambda_1(r, \pm m_r) < \pm\lambda < \lambda_2(r, \pm m_r)$ or $0 \le \pm\lambda < \lambda_1(r, \pm m_r)$ holds respectively, then $(EV; r, \lambda)$ has no non-trivial solutions. This assertion was generalized to a non-existence of a *sign-changing* solution for our problem. In more detail, it was proved in [23] that for any $\mu > 0$, $(GEV; r, \lambda, \mu)$ has *no sign-changing* solutions provided $-\lambda_2(r, -m_r) \le \lambda \le \lambda_2(r, m_r)$. So, second purpose is to provide the existence of a sign-changing solution as follows.

**Theorem 1.2.** *If* $\pm\lambda > \lambda_2(q, \pm m_q)$ *holds respectively, then for any* $\mu > 0$, $(GEV; q, \lambda, \mu)$ *has at least one sing-changing solution in* $C_0^{1,\alpha}(\overline{\Omega})$ *(for some* $\alpha \in (0,1)$*).*

Throughout this paper, $\|u\|_r$ denotes the usual norm of $L^r(\Omega)$ for $u \in L^r(\Omega)$ $(1 \le r \le \infty)$ and $\|u\|$ one of $W_0^{1,p}(\Omega)$ given by $\|u\| := \|\nabla u\|_p$. In addition, we write a positive or negative part of $u$ by $u_\pm := \max\{\pm u, 0\}$, that is, $u = u_+ - u_-$.

## 2. Uniqueness of a positive solution

First, we focus the following remark.

**Remark 2.1.** If $u$ is a non-trivial solution of $(GEV; q, \lambda, \mu)$ with $\mu > 0$, then $u \in L^\infty(\Omega)$ by the Moser iteration process (cf. Appendix in [15]). Hence, the regularity result up to the boundary in [11, Theorem 1] and [10, p. 320] ensures that $u \in C_0^{1,\alpha}(\overline{\Omega})$ with some $\alpha \in (0,1)$. Moreover, [18, Theorem 5.4.1] guarantees that $u > 0$ in $\Omega$ provided $u \not\equiv 0$ and $u \ge 0$. As a result, $u$ is a positive solution of $(GEV; q, \lambda, \mu)$ if $u$ is non-trivial and non-negative. According to [18, Theorem 5.5.1], we see that $\partial u/\partial \nu < 0$ on $\partial\Omega$ holds, where $\nu$ is the outward unit normal vector. Thus, $u \in \mathrm{int}\, C_0^1(\overline{\Omega})_+$.

The following result is known and it plays an important role for the proof of the simplicity of the first eigenvalue. See [8, Lemma 6.4.3] for the proof.

**Lemma 2.2** ([8, Lemma 6.4.3])**.** *Define*

$$I(u,v) := \langle -\Delta_q u, \frac{u^q - v^q}{u^{q-1}} \rangle - \langle -\Delta_q v, \frac{u^q - v^q}{v^{q-1}} \rangle$$

*for* $(u,v) \in D(I) := \Big\{ (u,v) \in W_0^{1,q}(\Omega) \times W_0^{1,q}(\Omega)\,;\, u > 0,\ v > 0,\ u/v, v/u \in L^\infty(\Omega) \Big\}$*. Then,* $I(u,v) \geq 0$ *for any* $(u,v) \in D(I)$ *and* $I(u,v) = 0$ *if and only if* $u = tv$ *for some* $t \geq 0$*.*

For the uniqueness of a positive solution for our problem, we prepare the following result (see [6] for the proof).

**Lemma 2.3** ([6, Lemma 2.1.])**.** *Let* $\nu \geq 0$ *and* $w_1, w_2 \in L^\infty(\Omega)$ *satisfy* $w_i \geq 0$ *a.e. in* $\Omega$*,* $w_i^{1/q} \in W^{1,p}(\Omega)$*,* $\Delta_p w_i^{1/q} \in L^\infty(\Omega)$ *for* $i = 1,2$*, and* $w_1 = w_2$ *on* $\partial\Omega$*. If* $w_1/w_2, w_2/w_1 \in L^\infty(\Omega)$*, then there holds*

$$\int_\Omega \left( -\frac{\Delta_p w_1^{1/q} + \nu \Delta_q w_1^{1/q}}{w_1^{(q-1)/q}} + \frac{\Delta_p w_2^{1/q} + \nu \Delta_q w_2^{1/q}}{w_2^{(q-1)/q}} \right) (w_1 - w_2)\, dx \geq 0.$$

Now we start to prove Theorem 1.1.

**Proof of Theorem 1.1.** Since it is proved in [23, Theorem 5.] that for any $\mu > 0$, $(GEV; q, \lambda, \mu)$ has at least one positive solution provided $\lambda > \lambda_1(q, m_q)$ or $\lambda < -\lambda_1(q, -m_q)$ (see Lemma 3.2 also), we shall show only the uniqueness of a positive solution.

Fix any $\mu > 0$ and $\pm\lambda > \lambda_1(q, \pm m_q)$, respectively. Let $u$ and $v$ be positive solutions of $(GEV; q, \lambda, \mu)$. According to Remark 2.1, we already know that $u, v \in \mathrm{int}\, C_0^1(\overline{\Omega})_+$. This ensures that $u/v$, $v/u \in L^\infty(\Omega)$. Here, we claim that

$$I(u,v) = \left\langle -\Delta_q u, \frac{u^q - v^q}{u^{q-1}} \right\rangle - \left\langle -\Delta_q v, \frac{u^q - v^q}{v^{q-1}} \right\rangle = 0. \tag{2.1}$$

If we prove this claim, then Lemma 2.2 guarantees the existence of $t > 0$ satisfying $u = tv$ because $u$ and $v$ are positive. Thus, by inserting $u = tv$ to $(GEV; q, \lambda, \mu)$ (note $t > 0$), we easily see that $v$ satisfies

$$-t^{p-q}\mu\Delta_p v - \Delta_q v = \lambda m_q v^{q-1} \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega \tag{2.2}$$

because $\Delta_r$ is $(r-1)$ homogeneous. By using (2.2) and the fact that $v$ is also a positive solution of $(GEV; q, \lambda, \mu)$, we have

$$-(1-t^{p-q})\mu\Delta_p v = 0 \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega.$$

Therefore, if $t \neq 1$ occurs, then $v = 0$, whence we have a contradiction (note $\mu > 0$). Consequently, $t = 1$ holds, and so $u = v$.

Now, we shall prove our claim (2.1). Fix any $\rho > 0$. By applying Lemma 2.3 with $w_1 = u^q$, $w_2 = v^q$ and $\nu = \rho/\mu$, we obtain

$$\begin{aligned}
0 &\leq \mu \left( \langle -\Delta_p u - \nu\Delta_q u, \frac{u^q - v^q}{u^{q-1}} \rangle - \langle -\Delta_p v - \nu\Delta_q v, \frac{u^q - v^q}{v^{q-1}} \rangle \right) \\
&= \langle -\mu\Delta_p u - \rho\Delta_q u, \frac{u^q - v^q}{u^{q-1}} \rangle - \langle -\mu\Delta_p v - \rho\Delta_q v, \frac{u^q - v^q}{v^{q-1}} \rangle \\
&= \langle -\mu\Delta_p u - \Delta_q u, \frac{u^q - v^q}{u^{q-1}} \rangle - \langle -\mu\Delta_p v - \Delta_q v, \frac{u^q - v^q}{v^{q-1}} \rangle + (\rho - 1) I(u,v) \\
&= (\rho - 1) I(u,v), \qquad (2.3)
\end{aligned}$$

where we use our assumption that both $u$ and $v$ are positive solutions of $(GEV; q, \lambda, \mu)$ in the last equality. Since (2.3) holds for any $\rho > 0$, we obtain $I(u,v) = 0$. Hence, our claim is shown.

Finally, we shall see that a positive solution $u_\mu$ of $(GEV; q, \lambda, \mu)$ equals $\mu^{1/(q-p)} u_1$, where $u_1$ is the positive solution of $(GEV; q, \lambda, 1)$. Indeed, multiplying $(GEV; q, \lambda, \mu)$ by $s^{q-1}$ for $s > 0$, $v = s u_\mu$ is a solution of

$$-s^{q-p}\mu\Delta_p v - \Delta_q v = \lambda m v^{q-1} \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega.$$

Thus, choosing $s_0 > 0$ such that $s_0^{q-p}\mu = 1$, we see that $s_0 u_\mu$ is a positive solution of $(GEV; q, \lambda, 1)$. By the uniqueness of a positive solution, we have $u_1 = s_0 u_\mu$, whence $u_\mu = u_1/s_0 = \mu^{1/(q-p)} u_1$ holds. □

## 3. Proof of Theorem 1.2

If we can obtain a sign-changing solution $u_1$ of $(GEV;q,\lambda,1)$, then $u_\mu = \mu^{1/(q-p)}u_1$ is one of $(GEV;q,\lambda,\mu)$ multiplying $(GEV;q,\lambda,1)$ by $\mu^{(q-1)/(q-p)}$. Hence, it is sufficient to consider only the case of $\mu = 1$ for the proof of Theorem 1.2.

3.1. **Setting functionals and global minimizers.** We define functionals $I_\lambda$ and $I_\lambda^\pm$ on $W_0^{1,p}(\Omega)$ by

$$I_\lambda(u) := \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \frac{\lambda}{q}\int_\Omega m_q|u|^q\,dx$$

and

$$I_\lambda^\pm(u) := \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \frac{\lambda}{q}\int_\Omega m_q u_\pm^q\,dx$$

for $u \in W_0^{1,p}(\Omega)$, where $u_\pm := \max\{\pm u, 0\}$. It is easily shown that $I_\lambda$ and $I_\lambda^\pm$ are $C^1$ functionals on $W_0^{1,p}(\Omega)$ because of $p > q > 1$.

**Remark 3.1.** If $u \in W_0^{1,p}(\Omega)$ is a non-trivial critical point of $I_\lambda^+$ (resp. $I_\lambda^-$), then $u$ is a positive (resp. negative) solution of $(GEV;q,\lambda,1)$. Indeed, by taking $-u_-$ (resp. $u_+$) as test function, we have

$$0 = \langle (I_\lambda^\pm)'(u), \pm u_\pm\rangle = \|\nabla u_\pm\|_p^p + \|\nabla u_\pm\|_q^q,$$

whence $u_- \equiv 0$ (resp. $u_+ \equiv 0$). Thus, $u$ satisfies

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\,dx + \int_\Omega |\nabla u|^{q-2}\nabla u\nabla v\,dx = \lambda\int_\Omega m_q|u|^{q-2}uv\,dx$$

for any $v \in W_0^{1,p}(\Omega)$. This means that $u$ is a non-negative (resp. non-positive) solution of $(GEV;q,\lambda,1)$. Moreover, as the reason stated in Remark 2.1, we have $u \in \operatorname{int} C_0^1(\overline{\Omega})_+$ (resp. $u \in \operatorname{int}(-C_0^1(\overline{\Omega})_+)$).

**Lemma 3.2.** *Assume $\pm\lambda > \lambda_1(q, \pm m_q)$. Then,*

$$\inf_{u\in W_0^{1,p}(\Omega)} I_\lambda(u) < 0 \quad \text{and} \quad \inf_{u\in W_0^{1,p}(\Omega)} I_\lambda^+(u) = \inf_{u\in W_0^{1,p}(\Omega)} I_\lambda^-(u) < 0 \tag{3.4}$$

*hold and all infimum in* (3.4) *are attained. In particular, $I_\lambda^+$ and $I_\lambda^-$ have a unique global minimizer.*

**Proof.** It is sufficient to treat only the case of $\lambda > \lambda_1(q,m_q)$ because when $\lambda < 0$ we can argue with $-\lambda$ and $-m_q$. First, we note that $I_\lambda$ and $I_\lambda^\pm$ are weakly lower semi-continuous on $W_0^{1,p}(\Omega)$ since $m_q \in L^\infty(\Omega)$ and the embedding of $W_0^{1,p}(\Omega)$ into $L^q(\Omega)$ is compact.

Here, we shall show that $I_\lambda$ is coercive on $W_0^{1,p}(\Omega)$. For every $u \in W_0^{1,p}(\Omega)$, we obtain

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \frac{\lambda}{q}\|m_q\|_\infty \|u\|_q^q \\ &\geq \frac{1}{p}\|\nabla u\|_p^p - \frac{\lambda}{q}\|m_q\|_\infty \|u\|_p^q |\Omega|^{1-q/p} \\ &\geq \frac{1}{p}\|\nabla u\|_p^p - \frac{\lambda \|m_q\|_\infty |\Omega|^{1-q/p}}{q\lambda_1(p,1)^{q/p}}\|\nabla u\|_p^q \end{aligned} \tag{3.5}$$

by the Hölder's inequality. Because the right hand side in (3.5) replacing $\|\nabla u\|_p$ with $t \geq 0$ is coercive and bounded from below on $[0,\infty)$ by $p > q$, this implies that $I_\lambda$ is coercive and bounded from below on $W_0^{1,p}(\Omega)$.

Similarly, we can prove that $I_\lambda^\pm$ is also coercive and bounded from below on $W_0^{1,p}(\Omega)$ because $\|u_\pm\|_q \leq \|u\|_q$ for every $u \in W_0^{1,p}(\Omega)$. Consequently, by the standard argument [13, Theorem 1.1.], we can obtain global minimizers of $I_\lambda$ and $I_\lambda^\pm$.

Because we are considering the case $\lambda > \lambda_1(q,m_q)$, the following inequality leads to all infimum in (3.4) are negative:

$$I_\lambda^-(-t\varphi_1) = I_\lambda^+(t\varphi_1) = I_\lambda(t\varphi_1) = t^q\left(\frac{t^{p-q}}{p}\|\nabla\varphi_1\|_p^p + \frac{\lambda_1(q,m_q)-\lambda}{q}\right) < 0$$

for sufficiently small $t > 0$, where we take a positive eigenfunction $\varphi_1$ corresponding to $\lambda_1(q,m_q)$ such that $\int_\Omega m_q\varphi_1^q\,dx = 1$.

Recall that non-trivial critical points of $I_\lambda^+$ (resp. $I_\lambda^-$) correspond to positive (resp. negative) solutions of $(GEV;q,\lambda,1)$ (see Remark 3.1 ). Note also that $(GEV;q,\lambda,1)$ is odd and $I_\lambda^+(u) = I_\lambda^-(-u)$ for all $u \in W_0^{1,p}(\Omega)$. Hence, $I_\lambda^\pm$ has no non-trivial critical points other than a global minimizer since we already know that a positive (or negative) solution of $(GEV;q,\lambda,1)$ is unique due to Theorem 1.1. Thus, $\min_{W_0^{1,p}(\Omega)} I_\lambda^+ = \min_{W_0^{1,p}(\Omega)} I_\lambda^-$ holds.

3.2. **Palais–Smale condition.** It is well known that the Palais–Smale condition implies the compactness of the critical set at any level $c \in \mathbb{R}$, and it plays an important role in minimax argument. Here, we recall the definition of the Palais–Smale condition.

**Definition 3.3.** A $C^1$ functional $I$ on a Banach space $X$ is said to satisfy the Palais–Smale condition at $c \in \mathbb{R}$ if any sequence $\{u_n\} \subset X$ satisfying

$$I(u_n) \to c \quad \text{and} \quad \|I'(u_n)\|_{X^*} \to 0 \quad \text{as } n \to \infty$$

has a convergent subsequence. We say that $I$ satisfies the Palais–Smale condition if $I$ satisfies the Palais–Smale condition at any $c \in \mathbb{R}$. Moreover, we say also that $I$ satisfies the bounded Palais–Smale condition if any bounded (Palais–Smale) sequence $\{u_n\}$ satisfying $\lim_{n\to\infty} \|I'(u_n)\|_{X^*} = 0$, has a convergent subsequence.

The following lemma called as the second deformation lemma is useful to prove our existence results.

**Lemma 3.4** ([3, Theorem 3.2.]). *Let $I$ be a $C^1$ functional on a Banach space $X$ and suppose that $I$ satisfies the Palais–Smale condition at any level $c \in [a,b]$ and $I$ has no critical values in $(a,b)$. Assume that $K(I) \cap I^{-1}(\{a\})$ consists only of isolated points (including the case of empty set), where $K(I)$ denotes the critical set of $I$, that is, $K(I) := \{u \in X\,;\, I'(u) = 0\}$. Denote the set $\{u \in X\,;\, I(u) \le c\}$ by $I^c$ for every $c \in \mathbb{R}$. Then, there exists an $\eta \in C([0,1] \times X, X)$ satisfying the following:*

(i) *$I(\eta(t,u))$ is nonincreasing in $t$ for every $u \in X$,*
(ii) *$\eta(t,u) = u$ for any $u \in I^a$, $t \in [0,1]$,*
(iii) *$\eta(0,u) = u$ and $\eta(1,u) \in I^a$ for any $u \in I^b \setminus \big(K(I) \cap I^{-1}(\{b\})\big)$,*

*that is, $I^a$ is a strong deformation retract of $I^b \setminus \big(K(I) \cap I^{-1}(\{b\})\big)$.*

**Lemma 3.5.** *For any $\lambda \in \mathbb{R}$, $I_\lambda^\pm$ and $I_\lambda$ satisfy the Palais–Smale condition, where $I_\lambda^\pm$ and $I_\lambda$ are functionals defined in Section 3.1.*

**Proof.** By the inequality (3.5) as in the proof of Lemma 3.2, we see that any Palais–Smale sequence of $I_\lambda$ and $I_\lambda^\pm$ at any level $c$ is bounded in $W_0^{1,p}(\Omega)$. By the standard argument, it is known that $I_\lambda$ and $I_\lambda^\pm$ satisfies the bounded Palais–Smale condition (note $m_q \in L^\infty(\Omega)$). Hence, our conclusion holds. For readers's convenience, we give a sketch only for $I_\lambda$. Let $\{u_n\}$ be a *bounded* sequence such that $\|I_\lambda'(u_n)\|_{W_0^{1,p}(\Omega)^*} \to 0$ as $n \to \infty$. Since $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, by choosing a subsequence, we may assume that

$$u_n \rightharpoonup u_0 \quad \text{in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u_0 \quad \text{in } L^q(\Omega)$$

for some $u_0 \in W_0^{1,p}(\Omega)$. Then, we have

$$\begin{aligned} o(1) &= \langle I'_\lambda(u_n), u_n - u_0 \rangle \\ &= \langle -\Delta_p u_n, u_n - u_0 \rangle + \langle -\Delta_q u_n, u_n - u_0 \rangle + o(1), \end{aligned}$$

(where we use $m_q \in L^\infty(\Omega)$ and $u_n \to u_0$ in $L^q(\Omega)$) and hence

$$\begin{aligned} o(1) &= \langle -\Delta_p u_n - \Delta_p u_0, u_n - u_0 \rangle + \langle -\Delta_q u_n - \Delta_q u_0, u_n - u_0 \rangle \\ &\geq (\|\nabla u_n\|_p - \|\nabla u_0\|_p)(\|\nabla u_n\|_p^{p-1} - \|\nabla u_0\|_p^{p-1}) \\ &\quad + (\|\nabla u_n\|_q - \|\nabla u_0\|_q)(\|\nabla u_n\|_q^{q-1} - \|\nabla u_0\|_q^{q-1}) \geq 0 \end{aligned}$$

holds, where we use the Hölder's inequality. This leads to $\lim_{n\to\infty} \|\nabla u_n\|_p = \|\nabla u_0\|_p$. Because $W_0^{1,p}(\Omega)$ is uniformly convex, $u_n$ converges to $u_0$ in $W_0^{1,p}(\Omega)$.

3.3. **Proof of Theorem 1.2.** For the proof of Theorem 1.2, we produce a continuous path near the origin with negative energy.

**Lemma 3.6.** *Assume $\lambda > \lambda_2(q, m_q)$. Then, there exists $s_0 > 0$ satisfying*

$$\max_{t \in [0,1]} I_\lambda \left( (1-t)s_0(\varphi_2)_+ - ts_0(\varphi_2)_- \right) < 0,$$

*where $\varphi_2$ is the eigenfunction corresponding to $\lambda_2(q, m_q)$.*

**Proof.** Let $\varphi_2 \in C_0^{1,\alpha}(\overline{\Omega})$ (for some $\alpha \in (0,1)$) be a sign-changing solution of $(EV; q, \lambda)$ with $\lambda = \lambda_2(q, m_q)$, namely, $\varphi_2$ satisfies

$$-\Delta_q \varphi_2 = \lambda_2(q, m_q) m_q(x) |\varphi_2|^{q-2} \varphi_2 \quad \text{in } \Omega, \quad \varphi_2 = 0 \quad \text{on } \partial\Omega.$$

By taking $\pm(\varphi_2)_\pm$ as test function, we have

$$0 < \|\nabla(\varphi_2)_\pm\|_q^q = \lambda_2(q, m_q) \int_\Omega m_q (\varphi_2)_\pm^q \, dx,$$

respectively. Therefore, these imply that for any $t \in [0,1]$ and $s > 0$

$$\begin{aligned}
&I_\lambda\left((1-t)s(\varphi_2)_+ - ts(\varphi_2)_-\right) \\
&= (1-t)^q s^q \left(\frac{(1-t)^{p-q}s^{p-q}}{p}\|\nabla(\varphi_2)_+\|_p^p - \frac{\lambda-\lambda_2(q,m_q)}{q}\int_\Omega m_q(\varphi_2)_+^q\,dx\right) \\
&\quad + t^q s^q \left(\frac{t^{p-q}s^{p-q}}{p}\|\nabla(\varphi_2)_-\|_p^p - \frac{\lambda-\lambda_2(q,m_q)}{q}\int_\Omega m_q(\varphi_2)_-^q\,dx\right) \\
&\le (1-t)^q s^q \left(\frac{s^{p-q}}{p}\|\nabla(\varphi_2)_+\|_p^p - \frac{\lambda-\lambda_2(q,m_q)}{q}\int_\Omega m_q(\varphi_2)_+^q\,dx\right) \\
&\quad + t^q s^q \left(\frac{s^{p-q}}{p}\|\nabla(\varphi_2)_-\|_p^p - \frac{\lambda-\lambda_2(q,m_q)}{q}\int_\Omega m_q(\varphi_2)_-^q\,dx\right).
\end{aligned}$$

Because of $p-q>0$ and $\lambda-\lambda_2(q,m_q)>0$, choosing a sufficiently small $s_0>0$ such that

$$s_0^{p-q} < \frac{p(\lambda-\lambda_2(q,m_q))}{q}\min\left\{\frac{\int_\Omega m_q(\varphi_2)_+^q\,dx}{\|\nabla(\varphi_2)_+\|_p^p}, \frac{\int_\Omega m_q(\varphi_2)_-^q\,dx}{\|\nabla(\varphi_2)_-\|_p^p}\right\},$$

our conclusion is shown.

Now we start to prove Theorem 1.2.

**Proof of Theorem 1.2.** As stated in the first part of Section 3, it is sufficient to handle the case of $\mu=1$. Moreover, we treat only the case of $\lambda > \lambda_2(q,m_q)$ because when $\lambda<0$ we can argue with $-\lambda$ and $-m_q$.

At first, we recall that there exists a unique positive solution $v_0 \in \mathrm{int}\left(C_0^1(\overline{\Omega})_+\right)$ of $(GEV;q,\lambda,1)$ and it is a global minimizer of $I_\lambda^+$ with $I_\lambda^+(v_0)<0$ according to Lemma 3.2 and Theorem 1.1. Moreover, since equation $(GEV;q,\lambda,1)$ is odd and by the uniqueness of a positive solution, $-v_0 \in -\mathrm{int}\left(C_0^1(\overline{\Omega})_+\right)$ is a unique negative solution of $(GEV;q,\lambda,1)$ and

$$I_\lambda^-(-v_0) = \min_{W_0^{1,p}(\Omega)} I_\lambda^- = \min_{W_0^{1,p}(\Omega)} I_\lambda^+ = I_\lambda^+(v_0) < 0.$$

To find a sign-changing solution, it suffices to obtain a *non-trivial* critical point of $I_\lambda$ other than $v_0$ and $-v_0$ because Theorem 1.1 guarantees the uniqueness of a positive (and negative) solution of $(GEV;q,\lambda,1)$. Moreover, we may assume that global minimum points of $I_\lambda$ are exactly equal to $v_0$ or $-v_0$ (see Lemma 3.2 for the existence of a global minimizer). This means that

$$I_\lambda^-(-v_0) = I_\lambda^+(v_0) = I_\lambda(v_0) = I_\lambda(-v_0) = \min_{W_0^{1,p}(\Omega)} I_\lambda < 0,$$

where the last inequality has shown in Lemma 3.2. Indeed, if $u$ is a global minimizer of $I_\lambda$, then $u$ is a non-trivial critical point of $I_\lambda$ because of $\min_{W_0^{1,p}(\Omega)} I_\lambda < 0$. Hence, $u$ is a non-trivial solution of $(GEV;q,\lambda,1)$. Furthermore, if $u$ is different from both $v_0$ and $-v_0$, then $u$ is not a constant sign solution by the uniqueness of a positive (and negative) solution. Thus $u$ changes sign.

Here, we recall that $I_\lambda$ satisfies the Palais–Smale condition by Lemma 3.5. Because we are assuming that $I_\lambda$ has no global minimizers other than $v_0$ and $-v_0$, we may assume that there exists an $r > 0$ such that $r < \|v_0\|$ and

$$I_\lambda(-v_0) = I_\lambda(v_0) < \inf_{\partial B_r(v_0)} I_\lambda < 0, \tag{3.6}$$

where $B_r(v_0) = \{z \in W_0^{1,p}(\Omega); \|z - v_0\| < r\}$. Indeed, if $\inf_{\partial B_r(v_0)} I_\lambda = I_\lambda(v_0)$ holds, by using the quantitative deformation theorem, we can show that $I_\lambda$ has another non-trivial critical point (global minimum point) $z_0 \in \partial B_r(v_0)$ (see Appendix in [21] for details). So, assuming (3.6), we define

$$\Gamma := \left\{ \gamma \in C\left([0,1], W_0^{1,p}(\Omega)\right); \gamma(0) = -v_0,\ \gamma(1) = v_0 \right\}$$

and

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)).$$

Then

$$c \geq \inf_{\partial B_r(v_0)} I_\lambda > I_\lambda(v_0) = I_\lambda(-v_0) \tag{3.7}$$

holds. Thus, the mountain pass theorem guarantees that $c$ is a critical value of $I_\lambda$.

Let us show that $c < 0$ holds to prove the existence of a non-trivial critical point of $I_\lambda$ other than $v_0$ and $-v_0$ (note (3.7)), which implies the existence of a sign-changing solution. For our purpose, it suffices to produce a $\gamma_0 \in \Gamma$ such that $\max_{t\in[0,1]} I_\lambda(\gamma_0(t)) < 0$.

Recall that from Lemma 3.6, there exists $s_0 > 0$ satisfying

$$\max_{t\in[0,1]} I_\lambda\left(s_0(1-t)(\varphi_2)_+ - s_0 t(\varphi_2)_-\right) < 0. \tag{3.8}$$

Since $I_\lambda^\pm$ has no critical values in an open interval $\left(I_\lambda^\pm(\pm v_0), 0\right)$ and $\pm v_0$ is the unique global minimum point of $I_\lambda^\pm$ by Lemma 3.2 respectively, Lemma 3.4 yields the existence of $\xi, \eta \in C([0,1], W_0^{1,p}(\Omega))$ satisfying

$$\begin{cases} \xi(0) = s_0(\varphi_2)_+ \quad \text{and} \quad \xi(1) = v_0, \\ I_\lambda^+(\xi(t)) \le I_\lambda^+(\xi(0)) = I_\lambda^+(s_0(\varphi_2)_+) < 0 \quad \text{for every } t \in [0,1]. \end{cases} \tag{3.9}$$

and

$$\begin{cases} \eta(0) = -s_0(\varphi_2)_- \quad \text{and} \quad \eta(1) = -v_0 \\ I_\lambda^-(\eta(t)) \le I_\lambda^-(\eta(0)) = I_\lambda^-(-s_0(\varphi_2)_-) < 0 \quad \text{for every } t \in [0,1]. \end{cases} \tag{3.10}$$

Note that

$$\xi(0)_+ = s_0(\varphi_2)_+,\ \xi(1)_+ = v_0,\ -(\eta(0))_- = -s_0(\varphi_2)_-, \quad -(\eta(1))_- = -v_0$$

and

$$I_\lambda^+(u) \ge I_\lambda^+(u_+) = I_\lambda(u_+), \quad I_\lambda^-(u) \ge I_\lambda^-(-u_-) = I_\lambda(-u_-) \tag{3.11}$$

for every $u \in W_0^{1,p}(\Omega)$. These inequalities are easily shown by using the elementary fact that $\|\nabla u\|_r^r = \|\nabla u_+\|_r^r + \|\nabla u_-\|_r^r \ge \|\nabla u_\pm\|_r^r$ for all $u \in W_0^{1,r}(\Omega)$ with $r \in (1,\infty)$. Setting

$$\gamma_0(t) := \begin{cases} -(\eta(1-3t))_- & \text{if } 0 \le t \le 1/3, \\ s_0(3t-1)(\varphi_2)_+ - s_0(2-3t)(\varphi_2)_- & \text{if } 1/3 \le t \le 2/3, \\ \xi(3t-2)_+ & \text{if } 2/3 \le t \le 1, \end{cases}$$

we have $\gamma_0 \in \Gamma$ such that $\max_{t\in[0,1]} I_\lambda(\gamma_0(t)) < 0$ by (3.8), (3.9) (3.10) and (3.11). □

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