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SHRINKING PROJECTION METHODS FOR SYSTEMS OF MIXED VARIATIONAL INEQUALITIES OF BROWDER TYPE, SYSTEMS OF MIXED EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS

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Abstract. In this paper, a countable family of hemi-relatively nonexpansive mappings, a system of mixed equilibrium problems and a system of mixed variational inequalities of Browder type are considered based on a shrinking projection method. Strong convergence of iterative sequences is obtained in a strictly convex and uniformly smooth Banach space. As an application, the problem of finding zeros of maximal monotone operators is studied.

Keywords. Countable family of hemi-relatively nonexpansive mappings; System of mixed equilibrium problems; System of mixed variational inequalities of Browder type; Uniformly closed; Fixed point.

1. Introduction

Let E be a Banach space and E^* the dual space of E. Let C be a nonempty closed convex subset of E. Let J be the normalized duality mapping from E into 2^{E^*} defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is known that the duality mapping J has the following properties: (1) If E is smooth, then J is single-valued; (2) If E is strictly convex, then J is one-to-one; (3) If E is reflexive, then J is surjective; (4) If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E; (5) If E^* is uniformly

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convex, then J is uniformly continuous on bounded subsets of E and J is singe-valued and also one-to-one (see [1-4]).

Let f be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper extended real-valued function. The mixed equilibrium problems is to find $x \in C$ such that

$$f(x,y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall\, y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $MEP(f)$, that is

$$MEP(f) = \{x \in C : f(x,y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall\, y \in C\}.$$

In particular, if $\varphi \equiv 0$, the problem (1.1) is reduced into the equilibrium problem for finding $x \in C$ such that

$$f(x,y) \geq 0, \quad \forall\, y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $EP(f)$. Equilibrium problems, which were introduced in [19] in 1994, have had a great impact and influence in the development of several branches of pure and applied sciences. It has been shown that equilibrium problem theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, physics, image reconstruction, ecology, transportation, network, elasticity and optimization. Numerous problems in physics, optimization and economics reduce to finding a solution of the problem (1.2).

Let $A : C \to E^*$ be an operator. We consider the mixed variational inequality of Browder type [5]: Find $x \in C$ such that

$$\langle y - x, Ax \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.3}$$

A point $x_0 \in C$ is called a solution of the mixed variational inequality of Browder type (1.3) if $\langle Ax_0, y - x_0 \rangle + \varphi(y) - \varphi(x_0) \geq 0$ for any $y \in C$. The solutions set of the mixed variational inequality of Browder type (1.3) is denoted by $MVI(C,A)$. Obviously, when $\varphi \equiv 0$, the mixed variational inequality of Browder type reduces to the Hartmann-Stampacchia variational inequality (see [6,7]) for finding $x \in C$ such that

$$\langle y - x, Ax \rangle \geq 0, \quad \forall y \in C. \tag{1.4}$$

Analogously, the set of solutions of (1.4) is denoted by $VI(C,A)$. The variational inequality (1.4) has been intensively considered due to its various applications in operations research, economic equilibrium and engineering design. When A has some monotonicity, many iterative methods for solving the variational inequality (1.1) have been proposed; see [8-16].

Let E be a smooth Banach space. The Lyapunov functional $\phi : E \times E \to \mathbb{R}^+$ is defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \forall\, x,\ y \in E. \tag{1.5}$$

It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \le \phi(x,y) \le (\|x\| + \|y\|)^2 \quad \forall\, x,\ y \in E. \tag{1.6}$$

We also know that $\phi(x,y) = 0$ if and only if $x = y$ (see [17]). Moreover, if E is a Hilbert space, (1.5) reduces to $\phi(x,y) = \|x-y\|^2$, for any $x,\ y \in E$.

Let E be a reflexive, strictly convex and smooth Banach space and C a nonempty, closed and convex subset of E. The generalized projection mapping $\Pi_C : E \to C$ introduced by Alber [8] is a map that assigns an arbitrary point $x \in E$ to the minimizer $\bar{x}$ of $\phi(\cdot, x)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min\{\phi(y,x) : y \in C\}. \tag{1.7}$$

Let C be a nonempty, closed convex subset of E, and let T be a mapping from C into itself. Recall that $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall\, x,\ y \in C.$$

A point $x \in C$ is a fixed point of T provided $Tx = x$. The set of fixed points of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : Tx = x\}$. A point $p \in C$ is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed point of T is denoted by $\widehat{F}(T)$. A mapping T from C to itself is called relatively nonexpansive if $\widehat{F}(T) = F(T) \neq \emptyset$ and $\phi(p, Tx) \le \phi(p,x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of relatively nonexpansive mapping was studied in [18-20]. A point $p \in C$ is called a strong asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of strong asymptotic fixed points of T is denoted by $\widetilde{F}(T)$. T is said to be weak relatively nonexpansive if $\widetilde{F}(T) = F(T) \neq \emptyset$ and $\phi(p,Tx) \leq \phi(p,x)$ for all $x \in C$ and $p \in F(T)$. T is said to be hemi-relatively nonexpansive if $F(T) \neq \emptyset$ and $\phi(p,Tx) \leq \phi(p,x)$ for all $x \in C$ and $p \in F(T)$. The class of hemi-relatively nonexpansive mappings which doesn't require the strong restriction $\widehat{F}(T) = F(T) \neq \emptyset$ or $\widetilde{F}(T) = F(T) \neq \emptyset$, is more general than the class of relatively nonexpansive mappings and the class of weak relatively nonexpansive mappings.

In 2005, Matsushita and Takahashi [21] obtained strong convergence theorem for a single relatively nonexpansive mapping in a uniformly convex and uniformly smooth Banach space E as follows:

Theorem MT. *Let E beprecise a uniformly convex and uniformly smooth Banach space and C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that* $0 \leq \alpha_n < 1$ *and* $\limsup_{n\to\infty} \alpha_n < 1$. *Suppose that $\{x_n\}$ is given by*

$$\begin{cases} x_0 = x \in C, \\ y_n = \Pi_C(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ C_n = \{z \in C : \phi(z,y_n) \leq \phi(z,x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where J is the duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

For solving the equilibrium problem (1.2), a bifunction f often be assumed to satisfy the following conditions:

(A_1) $f(x,x) = 0$ for all $x \in C$;

(A_2) f is monotone, that is, $f(x,y) + f(y,x) \leq 0$ for all $x,y \in C$;

(A_3) For all $x,\ y,\ z \in C$,

$$\limsup_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x,y);$$

(A_4) For all $x \in C$, $f(x,\cdot)$ is convex and lower semicontinuous.

For example, let A be a continuous and monotone operator of C into E^* and define

$$f(x,y) = \langle Ax, y-x\rangle, \quad \forall x,y \in C.$$

Then f satisfies (A1)-(A4).

In order to find a common element in the set of fixed points of a relatively nonexpansive mapping and the set of solutions of the equilibrium problem (1.2), Takahashi and Zembayshi [22] proved the following theorem:

Theorem TZ. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let f be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying $(A_1)-(A_4)$ and let T be a relatively nonexpansive mapping from C into itself such that $F(T)\cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ u_n \in C \quad \text{such that } f(u_n,y) + \frac{1}{r_n}\langle y-u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall\, y \in C, \\ C_n = \{z \in C : \phi(z,u_n) \leq \phi(z,x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n\cap Q_n} x, \quad \forall n \in \mathbb{N}\cup\{0\}, \end{cases}$$

where J is the duality mapping on E. $\{\alpha_n\} \subset [0,1]$ satisfies $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\{r_n\} \subset [a,\infty)$ for some $a > 0$. Then $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap EP(f)}x$, where $\Pi_{F(T)\cap EP(f)}$ is the generalized projection of E onto $F(T)\cap EP(f)$.

More recently, Su, Xu and Zhang [23] have modified and improved Theorem MT from a single relatively nonexpansive mapping to two countable families of weak relatively nonexpansive mappings. Precisely, they obtained the following theorem:

Theorem SXZ. *Let E be a uniformly convex and uniformly smooth real Banach space, let C be a nonempty closed convex subset of E. Let $\{T_n\}$, $\{S_n\}$ be two countable families of weak relatively nonexpansive mappings from C into itself such that $F := (\bigcap_{n=0}^{\infty} F(T_n)) \cap (\bigcap_{n=0}^{\infty} F(S_n)) \neq \emptyset$.*

Define a sequence $\{x_n\}$ *in C by the following algorithm:*

$$\begin{cases} x_0 \in C\ chosenarbitrarily, \\ z_n = J^{-1}(\beta_n^{(1)}Jx_n + \beta_n^{(2)}JT_nx_n + \beta_n^{(3)}JS_nx_n), \\ y_n = J^{(-1)}(\alpha_nJx_n + (1-\alpha_n)Jz_n), \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z,y_n) \le \phi(z,x_n)\}, \\ C_0 = \{z \in C : \phi(z,y_0) \le \phi(z,x_0)\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\}, \\ Q_0 = C, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0) \end{cases}$$

with the conditions: (i) $\liminf_{n\to\infty}\beta_n^{(1)}\beta_n^{(2)} > 0$; *(ii)* $\liminf_{n\to\infty}\beta_n^{(1)}\beta_n^{(3)} > 0$; *(iii)* $0 \le \alpha_n \le \alpha < 1$ *for some* $\alpha \in (0,1)$. *Then* $\{x_n\}$ *converges strongly to* $\Pi_F x_0$*, where* Π_F *is the generalized projection from C onto F.*

In this paper, inspired and motivated by the works mentioned above, we introduce an iterative algorithm for finding a common element in the set of fixed points of a countable family of hemi-relatively nonexpansive mappings, the set of solutions of a countable family mixed equilibrium problems (1.1) and the set of solutions of a countable family mixed variational inequalities of Browder type (1.3). We shall discuss our problem in some strictly convex Banach space which is more general than some uniformly convex Banach. Therefore, our results improve Theorem MT, Theorem TZ, Theorem SXZ and Habtu Zegeye, Naseer Shahzad's results in [24].

2. Preliminaries

A Banach space E is said to be strictly convex if $\frac{x+y}{2} < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{\to\infty}\|\frac{x_n+y_n}{2}\| = 1$.

Let $U_E=\{x\in E:\|x\|=1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided $\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$ exists for each $x,y\in U_E$. It is also said to be uniformly smooth if the limit is attained uniformly for $x,y\in U_E$.

It is well known that, if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E, and if E is uniformly smooth if and only if E^* is uniformly convex.

A Banach space E is said to have the Kadec-Klee property , if for a sequence $\{x_n\}$ of E satisfying that $x_n\rightharpoonup x\in E$ and $\|x_n\|\to\|x\|$, $x_n\to x$. It is known that if E is uniformly convex, then E has the Kadec-Klee property (see [4, 25] for more details).

Let C be a closed convex subset of E, and let $\{T_n\}_{n=0}^{\infty}$ be a countable family of mappings from C into itself. We denote by F the set of common fixed point of $\{T_n\}_{n=0}^{\infty}$. That is $F=\bigcap_{n=0}^{\infty}F(T_n)$, where $F(T_n)$ denotes the set of fixed points of T_n, for all $n\in\mathbb{N}\cup\{0\}$. Recall that $\{T_n\}_{n=0}^{\infty}$ is said to be uniformly closed, if $p\in\bigcap_{n=1}^{\infty}F(T_n)$ whenever $\{x_n\}\subset C$ converges strongly to p and $\|x_n-T_nx_n\|\to 0$ as $n\to\infty$, (see [26] for more details). A point $p\in C$ is said to be an asymptotic fixed point of $\{T_n\}_{n=0}^{\infty}$ if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|T_nx_n-x_n\|=0$. The set of asymptotic fixed points of $\{T_n\}_{n=0}^{\infty}$ will be denoted by $\hat{F}(\{T_n\}_{n=0}^{\infty})$. A point $p\in C$ is said to be a strong asymptotic fixed point of $\{T_n\}_{n=0}^{\infty}$ if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty}\|T_nx_n-x_n\|=0$. The set of strong asymptotic fixed points of $\{T_n\}_{n=0}^{\infty}$ will be denoted by $\tilde{F}(\{T_n\}_{n=0}^{\infty})$.

Using the definition of (strong) asymptotic fixed point of $\{T_n\}_{n=0}^{\infty}$, Su, Xu and Zhang [23] introduced the following definitions:

Definition 2.1. *Countable family of mappings $\{T_n\}$ is said to be a countable family of relatively nonexpansive mappings if $\hat{F}(\{T_n\}_{n=0}^{\infty})=F(\{T_n\}_{n=0}^{\infty})\neq\emptyset$ and $\phi(p,T_nx)\leq\phi(p,x)$, for all $p\in F(T_n)$, $x\in C$, $n\in\mathbb{N}\cup\{0\}$.*

Definition 2.2. *Countable family of mappings $\{T_n\}$ is said to be a countable family of weak relatively nonexpansive mappings if $\tilde{F}(\{T_n\}_{n=0}^{\infty})=F(\{T_n\}_{n=0}^{\infty})\neq\emptyset$ and $\phi(p,T_nx)\leq\phi(p,x)$, for all $p\in F(T_n)$, $x\in C$, $n\in\mathbb{N}\cup\{0\}$.*

Now, we introduce the definition of countable family of hemi-relatively nonexpansive mapping which is more general than countable family of relatively nonexpansive mapping and countable family of weak relatively nonexpansive mapping.

Definition 2.3. *Countable family of mappings* $\{T_n\}$ *is said to be a countable family of hemi-relatively nonexpansive mappings if* $F(\{T_n\}_{n=0}^{\infty}) \neq \emptyset$ *and* $\phi(p, T_n x) \leq \phi(p, x)$*, for all* $p \in F(T_n)$*,* $x \in C$*,* $n \in \mathbb{N} \cup \{0\}$*.*

Remark 2.4. *From Definitions 2.1-2.3, one has the following facts:*

(1) The Definitions of relatively nonexpansive mapping, weak relatively nonexpansive mapping and hemi-relatively nonexpansive mapping are special cases of Definition 2.1, Definition 2.2 and Definition 2.3 as $T_n \equiv T$ *for all* $n \in \mathbb{N} \cup \{0\}$*.*

(2) The class of countable family of hemi-relatively nonexpansive mappings, which doesn't need the restriction $\hat{F}(\{T_n\}_{n=0}^{\infty}) = F(\{T_n\}_{n=0}^{\infty})$ $(\tilde{F}(\{T_n\}_{n=0}^{\infty}) = F(\{T_n\}_{n=0}^{\infty}))$*, is more general than the class of countable family of relatively nonexpansive mappings (the class of countable family of weak relatively nonexpansive mappings).*

Next we give an example which is a countable family of hemi-relatively nonexpansive mappings but not a countable family of relatively nonexpansive mappings.

Example 2.5. *Let* E *be any smooth Banach space and* $x_0 = (1+\frac{1}{n})^n \breve{x}_0 \neq 0$ *be any element of* E*. Define a countable family of mappings* $T_n : E \to E$ *as follows: For all* $n \geq 1$*,*

$$T_n(x) = \begin{cases} (\frac{1}{2} + \frac{1}{2^{n+1}})x_0, & \text{if } x = (\frac{1}{2} + \frac{1}{2^n})x_0, \\ -x, & \text{if } x \neq (\frac{1}{2} + \frac{1}{2^n})x_0. \end{cases}$$

Then $\{T_n\}_{n=1}^{\infty}$ *is a countable family of hemi-relatively nonexpansive mappings but not a countable family of relatively nonexpansive mappings.*

Proof. First, it is obvious that T_n has a unique fixed point 0, that is, $F(T_n) = \{0\}$ for all $n \geq 1$. In addition, one easily sees that

$$\|T_n x\| \leq \|x\|, \quad \forall\, x \in E,\ n \geq 1.$$

This implies that

$$\|T_nx\|^2 - \|x\|^2 \leq 2\langle 0, JT_nx - Jx\rangle = 2\langle p, JT_nx - Jx\rangle.$$

for all $p \in \bigcap_{n=1}^{\infty} F(T_n)$. It follows from the above inequality that

$$\|p\|^2 - 2\langle p, JT_nx\rangle + \|T_nx\|^2 \leq \|p\|^2 - 2\langle p, Jx\rangle + \|x\|^2,$$

for all $p \in \bigcap_{n=1}^{\infty} F(T_n)$ and $x \in E$. That is

$$\phi(p, T_nx) \leq \phi(p, x),$$

for all $p \in \bigcap_{n=1}^{\infty} F(T_n)$ and $x \in E$. Hence $\{T_n\}_{n=1}^{\infty}$ is a countable family of hemi-relatively nonexpansive mappings. On the other hand, letting

$$x_n = (\frac{1}{2} + \frac{1}{2^n})x_0, \quad \forall n \geq 1,$$

from the definition of T_n, one has

$$T_nx_n = (\frac{1}{2} + \frac{1}{2^{n+1}})x_0, \quad \forall n \geq 1,$$

which implies that $\|x_n - T_nx_n\| \to 0$ and $x_n \to e\check{x}_0$ ($x_n \rightharpoonup e\check{x}_0$) as $n \to \infty$. That is $e\check{x}_0 \in \hat{F}(\{T_n\}_{n=0}^{\infty})$ but $e\check{x}_0 \notin F(\{T_n\}_{n=0}^{\infty})$, which shows that$\{T_n\}_{n=1}^{\infty}$ is not a countable family of relatively nonexpansive mappings. □

In what follows, we shall need the following lemmas.

Lemma 2.6. [8] *Let C be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth real Banach space E and let $x \in E$. Then for each $y \in C$,*

$$\phi(y, \Pi_C x) + \phi(\Pi_C, x) \leq \phi(y, x).$$

Lemma 2.7. [8] *Let C be a convex subset of a smooth real Banach space E. Let $x \in E$ and $x_0 \in C$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle z - x_0, Jx_0 - Jx\rangle \geq 0, \quad \forall\, z \in C.$$

Lemma 2.8. [27] *If E is a reflexive, strictly convex and smooth Banach space, then $\Pi_C = J^{-1}$.*

Lemma 2.9. *Let E be a uniformly convex Banach space, $r > 0$ be a positive number and $B_r(0)$ be a closed ball of E. Then there exists a continuous, strictly increasing and convex function $g : [0,2r) \to \mathbb{R}$ with $g(0) = 0$ such that*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x-y\|) \tag{2.7}$$

for all $x,\ y \in B_r$ and $t \in [0,1]$, where $B_r = \{z \in B : \|z\| \leq r\}$

Lemma 2.10. [28] *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A_1) - (A_4)$ and let φ be a semi-continuous and convex function from C to $\mathbb{R}$. For all $r > 0$ and $x \in E$, define the mapping $T_r : E \to C$ as follows:*

$$T_r(x) = \{z \in C : f(x,y) + \frac{1}{r}\langle y-z, Jz-Jx\rangle + \varphi(y) - \varphi(z) \geq 0, \quad \forall\, y \in C\}.$$

Then the following hold:

(B_1) *T_r is single-valued;*

(B_2) *$F(T_r) = MEP(f)$;*

(B_3) *$MEP(f)$ is closed and convex;*

(B_4) *$\phi(p, T_r x) + \phi(T_r x, x) \leq \phi(p,x)$, for $p \in F(T_r)$ and $x \in E$.*

We remark here that Lemma 2.10 is a special case of Lemma 1.5 [28]. Similarly, the following lemma also can be obtained by Lemma 1.5 of Zhang [28].

Lemma 2.11. [28] *Let C be a closed convex subset of a uniformly smooth, strictly convex, and reflexive Banach space E. Let A be a continuous monotone mapping from C to E^*, and let φ be a semi-continuous and convex function from C to $\mathbb{R}$. For all $r > 0$ and $x \in E$, define the mapping $F_r : E \to C$ as follows:*

$$F_r(x) = \{z \in C : \langle y-z, Az\rangle + \frac{1}{r}\langle y-z, Jz-Jx\rangle + \varphi(y) - \varphi(z) \geq 0, \quad \forall\, y \in C\}.$$

Then the following hold:

(C_1) *F_r is single-valued;*

(C_2) *$F(F_r) = MVI(C,A)$;*

(C_3) *$MVI(C,A)$ is closed and convex;*

(C_4) $\phi(p, F_r x) + \phi(F_r x, x) \leq \phi(p, x)$, *for* $p \in F(F_r)$.

3. Main results

Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex real Banach space E with dual space E^*. The system of mixed equilibrium problems (1.1) contains a family of the mixed equilibrium problems (1.1), that is

$$f_j(x,y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall\, y \in C,\ j \in \mathbb{N} \cup \{0\}.$$

The solutions set of the system of mixed equilibrium problems (1.1) is denoted by $\bigcap_{j=0}^{\infty} MEPS(f_j)$. The system of mixed variational inequalities of Browder type (1.3) contains a family of the mixed variational inequalities of Browder type (1.3), that is

$$\langle y - x, A_i x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C,\ i \in \mathbb{N} \cup \{0\}.$$

The solutions set of the system of mixed variational inequalities of Browder type (1.3) is denoted by $\bigcap_{i=0}^{\infty} MVI(C, A_i)$.

Now, we give our main results in this paper.

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex real Banach space E which enjoys the Kadec-Klee property. For each $i,\ j \in \mathbb{N} \cup \{0\}$, let $A_i : C \to E^*$ be an continuous monotone, $f_j : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (A_1)-(A_4), $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function. Suppose $\{S_n\}$ be a countable family of uniformly closed hemi-relatively nonexpansive mappings from C into itself such that $F := \bigcap_{n=0}^{\infty} F(S_n) \cap \bigcap_{i=0}^{\infty} MVIS(A_i, C) \cap \bigcap_{j=0}^{\infty} MEPS(f_j) \neq \emptyset$. For a point $x_0 \in C$ chosen arbitrarily, $C_0 = C$, let $\{x_n\}$ be a sequence generated by the following iterative* *scheme:*

$$\begin{cases} v_{n,i} \in C \\ \text{such that } \langle y - v_{n,i}, A_i v_{n,i}\rangle + \frac{1}{r_n}\langle y - v_{n,i}, Jv_{n,i} - Jx_n\rangle + \varphi(y) - \varphi(v_{n,i}) \ge 0,\ \forall y \in C, \\ y_{n,i} = \Pi_C(\alpha_n Jx_n + (1-\alpha_n)JS_n v_{n,i}), \\ u_{n,i,j} \in C \\ \text{such that } f_j(u_{n,i,j}, y) + \frac{1}{r_n}\langle y - u_{n,i,j}, Ju_{n,i,j} - Jy_{n,i}\rangle + \psi(y) - \psi(u_{n,i,j}) \ge 0,\ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \sup_{i,\ j\ge 0} \phi(z, u_{n,i,j}) \le \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \ge 1, \end{cases} \tag{3.1}$$

where the sequences $\{\alpha_n\} \subset [0,1]$ *and* $\{r_n\} \subset [a,\infty)$ *for some* $a > 0$ *satisfy the restriction* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. *Then the sequence* $\{x_n\}$ *converges strongly to a point* $\Pi_{\mathcal{F}} x_0$, *where* $\Pi_{\mathcal{F}}$ *is the generalized projection from* C *onto* $\mathcal{F}$.

Proof. We divide the proof into six steps.

Step 1. $\Pi_F x_0$ and $\Pi_{C_{n+1}} x_0$ are well defined.

From Lemma 2.10 (B_4), Lemma 2.11 (C_3) and Lemma 2.12, we know that

- for each $j \in \cup\{0\}$, $MEPS(f_j)$ is closed and convex;
- for each $i \in \mathbb{N} \cup \{0\}$, $MVIS(A_i, C)$ is closed and convex;
- for each $n \in \mathbb{N} \cup \{0\}$, $F(S_n)$ is closed and convex.

The above facts imply that $\mathcal{F} := \bigcap_{n=0}^{\infty} F(S_n) \cap \bigcap_{i=0}^{\infty} MVIS(A_i, C) \cap \bigcap_{j=0}^{\infty} MEPS(f_j) \ne \emptyset$ is a nonempty, closed and convex subset of C. Therefore, $\Pi_F x_0$ is well defined.

Next, we prove that $\Pi_{C_{n+1}} x_0$ is well defined. In fact, from the definition of the function ϕ, we may show that

$$\begin{aligned} C_{n+1} &= \{z \in C_n : \sup_{i,j\ge 0} \phi(z, u_{n,i,j}) \le \phi(z, x_n)\} \\ &= \bigcap_{i,j\ge 0} \{z \in C_n : \|z\|^2 - 2\langle z, Ju_{n,i,j}\rangle + \|u_{n,i,j}\|^2 \le \|z\|^2 - 2\langle z, Jx_n\rangle + \|x_n\|^2\} \\ &= \bigcap_{i,j\ge 0} \{z \in C : \|u_{n,i,j}\|^2 - \|x_n\|^2 - 2\langle z, Ju_{n,i,j} - Jx_n\rangle \le 0\} \cap C_n. \end{aligned}$$

This implies that C_n is closed and convex for each $n \in \mathbb{N} \cup \{0\}$. So, $\Pi_{C_{n+1}} x_0$ is also well defined.

Step 2. $\mathcal{F} \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$.

From the assumption, one sees that $\mathcal{F} \subset C_0 = C$. Suppose that $\mathcal{F} \subset C_n$ for some fixed $n \in \mathbb{N}$. Using induction method, for $w \in \mathcal{F} \subset C_n$, from Lemma 2.8 and the properties of ϕ, $v_{n,i} = F_{r_n}^{(i)} x_n$ and $u_{n,i,j} = T_{r_n}^{(j)} y_{n,i}$, one has

$$\begin{aligned}
\phi(w, u_{n,i,j})) &= \phi(w, T_{r_n}^{(j)} y_{n,i}) \leq \phi(w, y_{n,i}) \\
&\leq \phi(w, \Pi_C(\alpha_n J x_n + (1-\alpha_n) J S_n v_{n,i})) \\
&= \|w\|^2 - 2\langle w, J\Pi_C(\alpha_n J x_n + (1-\alpha_n) J S_n v_{n,i})\rangle + \|\alpha_n J x_n + (1-\alpha_n) J S_n v_{n,i})\|^2 \\
&\leq \|w\|^2 - 2\alpha_n \langle w, J x_n\rangle - 2(1-\alpha_n)\langle w, J S_n v_{n,i})\rangle + \alpha_n \|J x_n\|^2 + (1-\alpha_n)\|J S_n v_{n,i})\|^2 \\
&= \alpha_n \phi(w, x_n) + (1-\alpha_n)\phi(w, S_n v_{n,i}) \\
&\leq \alpha_n \phi(w, x_n) + (1-\alpha_n)\phi(w, v_{n,i}) \\
&= \alpha_n \phi(w, x_n) + (1-\alpha_n)\phi(w, F_{r_n}^{(i)} x_n) \\
&\leq \alpha_n \phi(w, x_n) + (1-\alpha_n)\phi(w, x_n) \\
&= \phi(w, x_n).
\end{aligned}$$

It follows that $w \in C_{n+1}$, hence, $\mathcal{F} \in C_n$ for all $n \in \mathbb{N} \cup \{0\}$.

Step 3. $p \in \bigcap_{n=0}^{\infty} F(S_n)$.

Since C_n is nonempty, closed and convex for each $n \in \mathbb{N} \cup \{0\}$, D is a nonempty, closed and convex subset of C. Let $p = \Pi_D J x_0$, where p is the unique element that satisfies $\inf_{x \in D} \phi(x, x_0) = \phi(p, x_0)$. On the other hand, from the construction of C_n, one sees that $C \supset C_1 \supset C_2 \supset \cdots$. Since $x_n = \Pi_{C_n} x_0$ and $x_{n+1} \in C_n$ by Lemma 2.6, one has that

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0) \leq \cdots \leq \phi(p, x_0). \tag{3.2}$$

Hence, from (1.6), it follows that the sequence $\{x_n\}$ is bounded. By reflexivity of E, we may assume that $x_n \rightharpoonup x^* \in E$. Since C_n is closed and convex, it is weakly closed. Thus, $x^* \in C_n$ for all $n \in \mathbb{N} \cup \{0\}$. So, $x^* \in D$. Moreover, by using the weak lower semi-continuity of the norm on E and (3.2), one sees that

$$\begin{aligned}
\phi(p, x_0) \leq \phi(x^*, x_0) &\leq \liminf_{n\to\infty} \phi(x_n, x_0) \\
&\leq \limsup_{n\to\infty} \phi(x_n, x_0) \leq \inf_{x \in D} \phi(x, x_0) = \phi(p, x_0).
\end{aligned} \tag{3.3}$$

which implies that

$$\lim_{n\to\infty}\phi(x_n,x_0)=\phi(p,x_0)=\phi(x^*,x_0)=\inf_{x\in D}\phi(x,x_0).$$

By the definition of p, one sees that $p=x^*$. Furthermore, by the definition of ϕ, one has

$$\lim_{n\to\infty}(\|x_n\|^2-2\langle x_n,Jx_0\rangle+\|x_0\|^2)=\|p\|^2-2\langle p,Jx_0\rangle+\|x_0\|^2,$$

which shows that $\lim_{n\to\infty}\|x_n\|=\|p\|$. In view of the Kadec-Klee property of E, one obtains that

$$\|x_n-p\|\to 0,\ as\ n\to\infty, \tag{3.4}$$

where $p=\Pi_D x_0$.

On the other hand, since $x_{n+1}\in C_{n+1}$, from the construction of C_{n+1} and (3.4), one has

$$\phi(x_{n+1},u_{n,i,j})\le\phi(x_{n+1},x_n)\to 0,\ as\ n\to\infty.$$

Thus, from (1.6), one obtains that

$$\|u_{n,i,j}\|\to\|p\|, \tag{3.5}$$

and hence

$$\|Ju_{n,i,j}\|\to\|Jp\|, \tag{3.6}$$

This implies that for each $i,j\in\mathbb{N}\cup\{0\}$, $\{Ju_{n,i,j}\}_{n=0}^{\infty}$ is bounded. Due to reflexivity of E, one knows that E^* is also reflexive. Thus, for each $i,j\in\mathbb{N}\cup\{0\}$, we may assume that $Ju_{n,i,j}\rightharpoonup x^*_{i,j}\in E^*$. Furthermore, reflexivity of E implies that there exists $x_{i,j}\in E$ such that $x^*_{i,j}=Jx_{i,j}$. Then, it follows from the definition of ϕ that

$$\begin{aligned}\phi(x_{n+1},u_{n,i,j})&=\|x_{n+1}\|^2-2\langle x_{n+1},Ju_{n,i,j}\rangle+\|u_{n,i,j}\|^2\\&=\|x_{n+1}\|^2-2\langle x_{n+1},Ju_{n,i,j}\rangle+\|Ju_{n,i,j}\|^2.\end{aligned} \tag{3.7}$$

Taking the limit inferior on both sides of (3.7) over n and using weak lower semi-continuity of $\|\cdot\|$, one has

$$\begin{aligned}0 &\geq \|p\|^2 - 2\langle p, x^*_{i,j}\rangle + \|x^*_{i,j}\|^2 \\ &= \|p\|^2 - 2\langle p, Jx_{i,j}\rangle + \|Jx_{i,j}\|^2 \\ &= \|p\|^2 - 2\langle p, Jx_{i,j}\rangle + \|x_{i,j}\|^2 \\ &= \phi(p, x_{i,j}),\end{aligned}$$

that is, for each $i, j \in \mathbb{N}\cup\{0\}$, $p = x_{i,j}$, which in turn implies for each $i, j \in \mathbb{N}\cup\{0\}$ that $x^*_{i,j} = Jp$. It follows for each $i, j \in \mathbb{N}\cup\{0\}$ that $Ju_{n,i,j} \rightharpoonup Jp$. Now, from (3.6) and the Kadec-Klee property of E^*, one sees that $Ju_{n,i,j} \to Jp$. Then the demicontinuity of J^{-1} implies that $u_{n,i,j} \rightharpoonup p$. Then, from (3.5) and the Kadec-Klee property of E, one has that for each $i, j \in \mathbb{N}\cup\{0\}$,

$$u_{n,i,j} \to p, \ as\ n \to \infty. \tag{3.8}$$

Next, we show that for each $i \in \mathbb{N}\cup\{0\}$, $v_{n,i} \to p$, as $n \to \infty$.

Indeed, from Step 2, one has

$$\phi(w, u_{n,i,j})) \leq \alpha_n \phi(w, x_n) + (1-\alpha_n)\phi(w, v_{n,i}) = \phi(w, x_n). \tag{3.9}$$

Taking limit on every term of (3.9), by applying (3.4) and (3.8), one has that

$$\lim_{n\to\infty} \phi(w, v_{n,i}) = \phi(w, p), \ for\ each\ i \in \mathbb{N}\cup\{0\}.$$

Hence, by (C_4) of Lemma 2.11, it follows from $v_{n,i} = F^{(i)}_{r_n} x_n$ that

$$\begin{aligned}\phi(v_{n,i}, x_n) &= \phi(F^{(i)}_{r_n} x_n, x_n) \\ &\leq \phi(w, x_n) - \phi(w, F^{(i)}_{r_n} x_n) \\ &= \phi(w, x_n) - \phi(w, v_{n,i}) \to 0, \quad as\ n \to \infty.\end{aligned}$$

Deal with $v_{n,i}$ similar to $u_{n,i,j}$ from (3.5) to (3.8), one also has that for each $i \in \mathbb{N}\cup\{0\}$,

$$v_{n,i} \to p, \ as\ n \to \infty. \tag{3.10}$$

Since $\{x_n\}$ is convergent, then $\{x_n\}$ is bounded; so are $\{v_{n,i}\}$, $\{S_n v_{n,i}\}$ for each $i \in \mathbb{N}\cup\{0\}$. And since E is uniformly smooth, E^* is uniformly convex. In view of Lemma 2.9, one sees for all $w \in F$ and each $i, j \in \mathbb{N} \cup \{0\}$ that

$$\begin{aligned}
\phi(w, u_{n,i,j}) &= \phi(w, T_{n,i}^{(j)} y_{n,i}) \leq \phi(w, y_{n,i}) \\
&\leq \phi(w, \Pi_C(\alpha_n Jx_n + (1-\alpha_n) JS_n v_{n,i})) \\
&= \|w\|^2 - 2\langle w, \alpha_n Jx_n + (1-\alpha_n) JS_n v_{n,i}\rangle + \|\alpha_n Jx_n + (1-\alpha_n) JS_n v_{n,i}\|^2 \\
&\leq \|w\|^2 - 2\alpha_n \langle w, Jx_n\rangle - 2(1-\alpha_n)\langle w, JS_n v_{n,i}\rangle + \alpha_n \|Jx_n\|^2 + (1-\alpha_n)\|JS_n v_{n,i}\|^2 \\
&\quad - \alpha_n(1-\alpha_n) g(\|Jx_n - JS_n v_{n,i}\|) \\
&= \alpha_n \phi(w, x_n) + (1-\alpha_n)\phi(w, S_n v_{n,i}) - \alpha_n(1-\alpha_n) g(\|Jx_n - JS_n v_{n,i}\|) \\
&\leq \alpha_n \phi(w, x_n) + (1-\alpha_n)\phi(w, v_{n,i}) - \alpha_n(1-\alpha_n) g(\|Jx_n - JS_n v_{n,i}\|) \\
&\leq \alpha_n \phi(w, x_n) + (1-\alpha_n)\phi(w, x_n) - \alpha_n(1-\alpha_n) g(\|Jx_n - JS_n v_{n,i}\|) \\
&= \phi(w, x_n) - \alpha_n(1-\alpha_n) g(\|Jx_n - JS_n v_{n,i}\|),
\end{aligned}$$

which implies that

$$\alpha_n(1-\alpha_n) g(\|Jx_n - JS_n v_{n,i}\|) \leq \phi(w, x_n) - \phi(w, u_{n,i,j}). \tag{3.11}$$

On the other hand, from (3.4) and (3.8), one knows that

$$\lim_{n\to\infty} [\phi(w, x_n) - \phi(w, u_{n,i,j})] = 0,$$

and since $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, the inequality (3.11) implies that

$$\lim_{n\to\infty} g(\|Jx_n - JS_n v_{n,i}\|) = 0,$$

hence, from the property of g, one has

$$\lim_{n\to\infty} \|Jx_n - JS_n v_{n,i}\| = 0.$$

Furthermore, since J^{-1} is uniformly norm-to-norm continuous on bounded sets, one obtains that

$$\lim_{n\to\infty} \|x_n - S_n v_{n,i}\| = 0. \tag{3.12}$$

In fact, from the property of triangle inequality, one has that

$$\lim_{n\to\infty} \|v_{n,i} - S_n v_{n,i}\| \leq \lim_{n\to\infty} \|v_{n,i} - x_n\| + \lim_{n\to\infty} \|x_n - S_n v_{n,i}\|.$$

Noticing that (3.4), (3.10) and (3.12), one has

$$\lim_{n\to\infty}\|v_{n,i}-S_nv_{n,i}\|=0.$$

Since $v_{n,i}\to p$ for each $i\in\mathbb{N}\cup\{0\}$ and $\{S_n\}_{n=0}^{\infty}$ is uniformly closed, $p\in\bigcap_{n=0}^{\infty}F(S_n)$.

Step 4. $p\in MVIS(A_i,C)$, for each $i\in\mathbb{N}\cup\{0\}$.

In view of the definition of $F_{r_n}x$, one has

$$\langle y-v_{n,i},A_iv_{n,i}\rangle+\langle y-v_{n,i},\frac{Jv_{n,i}-Jx_n}{r_n}\rangle+\varphi(y)-\varphi(v_{n,i})\geq 0 \tag{3.13}$$

for any $y\in C,\ i\in\mathbb{N}\cup\{0\}$. Setting $v_t=tv+(1-t)p$, for all $t\in(0,1]$ and $v\in C$, one knows that $v_t\in C$. From (3.13), it follows that

$$\begin{aligned}\langle v_t-v_{n,i},A_iv_t\rangle+\varphi(v_t)-\varphi(v_{n,i})&\geq\langle v_t-v_{n,i},A_iv_t\rangle+\varphi(v_t)-\varphi(v_{n,i})-\langle v_t-v_{n,i},A_iv_{n,i}\rangle\\&\quad-\langle v_t-v_{n,i},\frac{Jv_{n,i}-Jx_n}{r_n}\rangle-\varphi(v_t)+\varphi(v_{n,i})\\&=\langle v_t-v_{n,i},A_iv_t-A_iv_{n,i}\rangle-\langle v_t-v_{n,i},\frac{Jv_{n,i}-Jx_n}{r_n}\rangle.\end{aligned} \tag{3.14}$$

From the continuity of J, (3.4) and (3.10), one sees that $\frac{Jv_{n,i}-Jx_n}{r_n}\to 0$ as $n\to\infty$. And since A_i is monotone for each $i\in\mathbb{N}\cup\{0\}$, one also obtains that $\langle v_t-v_{n,i},A_iv_t-A_iv_{n,i}\rangle\geq 0$. Therefore, by taking limit on the both side of (3.14) one has that

$$0\leq\lim_{n\to\infty}[\langle v_t-v_{n,i},A_iv_t\rangle+\varphi(v_t)-\varphi(v_{n,i})]=\langle v_t-p,A_iv_t\rangle+\varphi(v_t)-\varphi(p).$$

Due to the convexity of φ, one also sees that

$$\langle v-p,A_iv_t\rangle+\varphi(v)-\varphi(p)\geq 0,\ \forall\ v\in C.$$

Letting $t\to 0$, the above inequality implies that

$$\langle v-p,A_ip\rangle+\varphi(v)-\varphi(p)\geq 0,\ \forall\ v\in C.$$

This implies that, for each $i\in\mathbb{N}\cup\{0\}$, $p\in MVIS(A_i,C)$.

Step 5. $p\in MEPS(f_j)$, for each $j\in\mathbb{N}\cup\{0\}$.

Noticing that Step 2, one has

$$\phi(w,u_{,n,i,j})\leq\phi(w,y_{n,i})\leq\phi(w,x_n). \tag{3.15}$$

Taking limit on every term of (3.15), by using (3.4) and (3.8), one sees that

$$\lim_{n\to\infty}\phi(w,y_{n,i})=\phi(w,p),\quad \forall\, i\in\mathbb{N}\cup\{0\}.$$

Hence, by (B_4) of Lemma 2.10, it follows from $u_{n,i,j}=T_{r_n}^{(j)}y_{n,i}$ that

$$\begin{aligned}\phi(u_{n,i,j},y_{n,i})&=\phi(T_{r_n}^{(j)}y_{n,i},y_{n,i})\\ &\le \phi(w,y_{n,i})-\phi(w,T_{r_n}^{(j)}y_{n,i})\\ &=\phi(w,x_n)-\phi(w,u_{n,i,j})\to 0,\quad as\ n\to\infty.\end{aligned}$$

Trying to imitate $u_{n,i,j}$ from (3.5) to (3.8), one also has that for each $i\in\mathbb{N}\cup\{0\}$,

$$y_{n,i}\to p,\ as\ n\to\infty. \tag{3.16}$$

We define a bifunction $G_j: C\times C\to\mathbb{R}$ for $j\in\mathbb{N}\cup\{0\}$ by

$$G_j(x,y)=f_j(x,y)+\varphi(y)-\varphi(x)\quad \forall\, x,\ y\in C.$$

One knows from Lemma 2.10 that the bifunction $G_j: C\times C\to\mathbb{R}$ for $j\in\mathbb{N}\cup\{0\}$ also satisfies $(A_1)-(A_4)$. Therefore, the system of mixed equilibrium problem (1.1) is equivalent to the following system of equilibrium problem: find $x\in C$ such that

$$G_j(x,y)\ge 0,\quad \forall\, y\in C.$$

Since for each $i,\ j\in\mathbb{N}\cup\{0\}$, $u_{n,i,j}\to p,\ y_{n,i}\to p$, one has

$$\lim_{n\to\infty}\|u_{n,i,j}-y_{n,i}\|=0,$$

and since J is uniformly norm-to-norm continuous on bounded sets, one also gets

$$\lim_{n\to\infty}\|Ju_{n,i,j}-Jy_{n,i}\|=0.$$

From $r_n\ge a$, one obtains

$$\lim_{n\to\infty}\frac{\|Ju_{n,i,j}-Jy_{n,i}\|}{r_n}=0. \tag{3.17}$$

By $u_{n,i,j}=T_{r_n}^{(j)}y_{n,i}$, one has

$$G_j(u_{n,i,j},y)+\frac{1}{r_n}\langle y-u_{n,i,j},Ju_{n,i,j}-Jy_{n,i}\rangle\ge 0,\quad \forall\, y\in C.$$

From (A_2), one gets

$$\begin{aligned}\|y-u_{n,i,j}\|\frac{\|Ju_{n,i,j}-Jy_{n,i}\|}{r_n} &\geq \frac{1}{r_n}\langle y-u_{n,i,j},Ju_{n,i,j}-Jy_{n,i}\rangle \\ &\leq -G_j(u_{n,i,j},y) \\ &\leq G_j(y,u_{n,i,j}), \quad \forall\, y\in C.\end{aligned}$$

Taking $n\to\infty$ in above inequality, it follows from (A_4) and (3.17) that, for each $j\in\mathbb{N}\cup\{0\}$,

$$G_j(y,p)\leq 0, \quad \forall\, y\in C.$$

For all $0<t<1$ and $y\in C$, define $y_t=ty+(1-t)p$. Since $y,\ p\in C$, one obtains $y_t\in C$, which yields that $G_j(y_t,p)\leq 0$, for each $j\in\mathbb{N}\cup\{0\}$. It follows from (A_1) that

$$0=G_j(y_t,y_t)\leq tG_j(y_t,y)+(1-t)G_j(y_t,p)\leq tG_j(y_t,y),$$

that is,

$$G_j(y_t,y)\geq 0, for\ each\ j\in\mathbb{N}\cup\{0\}.$$

Letting $t\downarrow 0$, from (A_3), it follows that $G_j(p,y)\geq 0$, for all $y\in C$, which implies that $p\in\bigcap_{j=0}^{\infty}EPS(G_j)$; that is, $p\in\bigcap_{j=0}^{\infty}MEPS(f_j)$.

Step 6. $p=\Pi_{\mathcal{F}}x_0$.

Indeed, from $x_n=\Pi_{C_n}x_0$, one has that

$$\langle Jx_0-Jx_n,x_n-q\rangle\geq 0, \quad \forall\, q\in\mathcal{F}. \tag{3.18}$$

Taking $n\to\infty$ in (3.18), one obtains that

$$\langle Jx_0-Jp,p-q\rangle\geq 0, \quad \forall\, q\in\mathcal{F}. \tag{3.18}$$

Therefore, by $p\in\mathcal{F}$ and Lemma 2.7, one gets that $p=\Pi_{\mathcal{F}}x_0$. The proof is completed.

Remark 3.2. Theorem 3.1 improves Theorem 3.15 of Su, Xu, Zhang [23] in the following senses:

(1) The class of countable family of hemi-relatively nonexpansive mappings is more general than the class of countable family of weak relatively nonexpansive mappings; see Su, Xu, Zhang [16] for more details.

(2) Banach spaces enjoying strictly convex are more general than Banach spaces enjoying uniformly convex.

(3) The algorithm (3.1) of Theorem 3.15 in Su, Xu, Zhang [23] is related to one problem, that is, the fixed point. But our algorithm in Theorem 3.1 is related to three problems, that is, the fixed point, the system of mixed equilibrium problems and the system of mixed variational inequalities of Browder type.

When $T_n = I$ in (3.1), we can obtain the new modified Mann iteration for the system of mixed equilibrium problems and the system of mixed variational inequalities of Browder type as follows:

Corollary 3.3. *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex real Banach space E which enjoys the Kadec-Klee property. For each $i,\ j \in \mathbb{N} \cup \{0\}$, let $A_i : C \to E^*$ be an continuous monotone, $f_j : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (A_1)-(A_4), $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function such that $\mathcal{F} := \bigcap_{i=0}^{\infty} MVIS(A_i, C) \cap \bigcap_{j=0}^{\infty} MEPS(f_j) \neq \emptyset$. For a point $x_0 \in C$ chosen arbitrarily, $C_0 = C$, let $\{x_n\}$ be a sequence generated by the following iterative scheme:*

$$
\begin{cases}
v_{n,i} \in C \\
\text{such that } \langle y - v_{n,i}, A_i v_{n,i} \rangle + \frac{1}{r_n} \langle y - v_{n,i}, J v_{n,i} - J x_n \rangle + \varphi(y) - \varphi(v_{n,i}) \geq 0,\ \forall y \in C, \\
y_{n,i} = \Pi_C(\alpha_n J x_n + (1 - \alpha_n) J v_{n,i}), \\
u_{n,i,j} \in C \\
\text{such that } f_j(u_{n,i,j}, y) + \frac{1}{r_n} \langle y - u_{n,i,j}, J u_{n,i,j} - J y_{n,i} \rangle + \psi(y) - \psi(u_{n,i,j}) \geq 0,\ \forall y \in C, \\
C_{n+1} = \{z \in C_n : \sup_{i,\ j \geq 0} \phi(z, u_{n,i,j}) \leq \phi(z, x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1,
\end{cases}
\tag{3.1}
$$

where the sequences $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$ satisfy the restriction $\liminf_{n \to \infty} \alpha_n (1 - \alpha_n) > 0$. Then the sequence $\{x_n\}$ converges strongly to a point $\Pi_{\mathcal{F}} x_0$, where $\Pi_{\mathcal{F}}$ is the generalized projection from C onto $\mathcal{F}$.

4. Applications to maximal monotone operators

In this section, we apply the our main results to proving some strong convergence theorem concerning maximal monotone operators in a Banach space E.

Let B be a multi-valued operator from E to E^* with domain $D(B) = \{z \in E : Bz \neq \emptyset\}$ and range $R(B) = \{z \in E : z \in D(B)\}$. An operator B is said to be monotone if

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0, \quad \forall x_1, x_2 \in D(B), y_1 \in Bx_1, y_2 \in Bx_2.$$

A monotone operator B is said to be maximal if it's graph $G(B) = \{(x,y) : y \in Bx\}$ is not properly contained in the graph of any other monotone operator. It is well known that if B is a maximal monotone operator, then $B^{-1}0$ is closed and convex.

The following result is also well known.

Lemma 4.1. [29] *Let E be a reflexive, strictly convex and smooth Banach space and B be a monotone operator from E to E^*. Then B is maximal if and only if $R(J + rB) = E^*$ for all $r > 0$.*

Let E be a reflexive, strictly convex and smooth Banach space and B be a maximal monotone operator from E to E^*. Using Lemma 4.1 and the strict convexity of E, it follows that, for all $r > 0$ and $x \in E$, there exists a unique $x_r \in D(B)$ such that

$$Jx \in Jx_r + rBx_r.$$

If $J_r x = x_r$, then we can define a single valued mapping $J_r : E \to D(B)$ by $J_r = (J + rB)^{-1}J$ and such a J_r is called the resolvent of B. We know that $B^{-1}0 = F(J_r)$ for all $r > 0$ (see [4, 28, 30] for more details).

First, we give an important lemma for this section, and the following lemma can be as an example about a countable family of hemi-relatively nonexpansive mappings.

Lemma 4.2. *Let E be a strictly convex and uniformly smooth Banach space, B be a maximal monotone operator from E to E^* such that $B^{-1}0$ is nonempty, and let $\{r_n\}$ be a sequence of positive real numbers which is bounded away from 0 such that $J_{r_n} = (I + r_n B)^{-1}$. Then $\{J_{r_n}\}$ is a uniformly closed countable family of hemi-relatively nonexpansive mappings.*

Proof. One has $\bigcap_{n=0}^{\infty} F(J_{r_n}) = B^{-1}0 \neq \emptyset$. Firstly, we show J_{r_n} is uniformly closed. Let $\{z_n\}$ be a sequence such that $z_n \to p$ and $\lim_{n\to\infty} \|z_n - J_{r_n} z_n\| = 0$. Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\frac{1}{r_n}(Jz_n - JJ_{r_n}z_n) \to 0, \quad as\ n \to \infty.$$

It follows from

$$\frac{1}{r_n}(Jz_n - JJ_{r_n}z_n) \in BJ_{r_n}z_n$$

and the monotonicity of B that

$$\langle w - J_{r_n}z_n, w^* - \frac{1}{r_n}(Jz_n - JJ_{r_n}z_n)\rangle \geq 0$$

for all $w \in D(B)$ and $w^* \in Bw$. Letting $n \to \infty$, one has $\langle w - p, w^*\rangle \geq 0$ for all $w \in D(B)$ and $w^* \in Bw$. Therefore, from the maximality of B, one obtains $p \in B^{-1}0 = F(J_{r_n})$. Hence, J_{r_n} is uniformly closed.

In addition, for any $w \in E$ and $p \in \bigcap_{n=0}^{\infty} F(J_{r_n})$, from the monotonicity of B, one has

$$\begin{aligned}
\phi(p, J_{r_n}w) &= \|p\|^2 - 2\langle p, JJ_{r_n}w\rangle + \|J_{r_n}w\|^2 \\
&= \|p\|^2 + 2\langle p, Jw - JJ_{r_n}w - Jw\rangle + \|J_{r_n}w\|^2 \\
&= \|p\|^2 + 2\langle p, Jw - JJ_{r_n}w\rangle - 2\langle p, Jw\rangle + \|J_{r_n}w\|^2 \\
&= \|p\|^2 - 2\langle J_{r_n}w - p - J_{r_n}w, Jw - JJ_{r_n}w - Jw\rangle - 2\langle p, Jw\rangle + \|J_{r_n}w\|^2 \\
&= \|p\|^2 - 2\langle J_{r_n}w - p, Jw - JJ_{r_n}w - Jw\rangle + 2\langle J_{r_n}w, Jw - JJ_{r_n}w\rangle - 2\langle p, Jw\rangle + \|J_{r_n}w\|^2 \\
&\leq \|p\|^2 + 2\langle J_{r_n}w, Jw - JJ_{r_n}w\rangle - 2\langle p, Jw\rangle + \|J_{r_n}w\|^2 \\
&= \|p\|^2 - 2\langle p, Jw\rangle + \|w\|^2 - \|J_{r_n}w\|^2 + 2\langle J_{r_n}w, Jw\rangle - \|w\|^2 \\
&= \phi(p, w) - \phi(J_{r_n}w, w) \\
&\leq \phi(p, w),
\end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. This implies that $\{J_{r_n}\}$ is a countable family of hemi-relatively nonexpansive mappings. Hence, $\{J_{r_n}\}$ is a countable family of uniformly closed hemi-relatively nonexpansive mappings.

We consider the problem of strong convergence concerning maximal monotone operators in a Banach space. Such a problem has been also studied in [30, 31]. Using Theorem 3.1, we obtain the following result:

Theorem 4.3. *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex real Banach space E which enjoys the Kadec-Klee property. For each $i,\ j \in \mathbb{N} \cup \{0\}$, let $A_i : C \to E^*$ be an continuous monotone, $f_j : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (A_1)-(A_4), $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function. Assume that B is a maximal monotone operators from E to E^* such that $\digamma := B^{-1}0 \cap \bigcap_{i=0}^{\infty} MVIS(A_i, C) \cap \bigcap_{j=0}^{\infty} MEPS(f_j) \neq \emptyset$. For a point $x_0 \in C$ chosen arbitrarily, $C_0 = C$, let $\{x_n\}$ be a sequence generated by the following iterative scheme:*

$$
\begin{cases}
v_{n,i} \in C \\
\text{such that } \langle y - v_{n,i}, A_i v_{n,i} \rangle + \frac{1}{r_n} \langle y - v_{n,i}, J v_{n,i} - J x_n \rangle + \varphi(y) - \varphi(v_{n,i}) \geq 0,\ \forall y \in C, \\
y_{n,i} = \Pi_C(\alpha_n J x_n + (1 - \alpha_n) J J_{r_n} v_{n,i}), \\
u_{n,i,j} \in C \\
\text{such that } f_j(u_{n,i,j}, y) + \frac{1}{r_n} \langle y - u_{n,i,j}, J u_{n,i,j} - J y_{n,i} \rangle + \psi(y) - \psi(u_{n,i,j}) \geq 0,\ \forall y \in C, \\
C_{n+1} = \{z \in C_n : \sup_{i,\ j \geq 0} \phi(z, u_{n,i,j}) \leq \phi(z, x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 1,
\end{cases}
$$

where $J_{r_n} = (I + r_n B)^{-1}$, the sequences $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$ satisfy the restriction $\liminf_{n \to \infty} \alpha_n (1 - \alpha_n) > 0$. Then the sequence $\{x_n\}$ converges strongly to a point $\Pi_\digamma x_0$, where $\Pi_\digamma$ is the generalized projection from C onto $\digamma$.

Proof. From Lemma 4.2, we know that J_{r_n} is a countable family of uniformly closed hemi-relatively nonexpansive mappings. Furthermore, applying Theorem 3.1, one sees that the sequence $\{x_n\}$ converges strongly to a point $\Pi_\digamma x_0$.

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