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BIFURCATIONS OF RATIO-DEPENDENT PREDATOR-PREY HOLLING TYPE III SYSTEMS WITH HARVESTING RATES

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Abstract. We study the phase portraits and Hopf bifurcations of the ratio-dependent Holling type III models with constant harvesting rates. We provide the ranges of the parameters involved under which all the equilibria be positive. We show that these positive equilibria can be saddles, topological saddles, saddle-nodes, stable (or unstable) nodes or focuses, weak centers or cusps, depending on the choices of the parameters. We compute the first Lyapunov number to obtain the subcritical and supercritical Hopf bifurcations of the models.

Keywords. Holling type III; Hopf bifurcation; Predator-prey system; Harvesting rates; Limit cycle.

1. Introduction

One of important ecological fields is the dynamics between predators and prey. Various mathematical models governed by differential equations are developed to study the relationship between predators and their prey. Some of these models are those with Holling types I, II, III and IV functional responses and have been intensively investigated, for example in [6, 13, 14,

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17, 21, 22, 24, 27]. The Holling type III model is of the form

$$\begin{cases} \dot{x} = x(r - sx) - \dfrac{c_1 x^2 y}{x^2 + m^2 y^2}, \\ \dot{y} = y\big(-d + \dfrac{c_2 x^2 y}{x^2 + m^2 y^2}\big), \end{cases} \tag{1.1}$$

where r/s is the carrying capacity of the prey, $d > 0$ is the death rate of the predator, $s, c_1, m, c_2 > 0$ present the prey intrinsic growth rate, capturing rate, half capturing saturation constant and conversion rate, respectively. The term $c_1 x^2/(x^2 + m^2 y^2)$ is called the ratio-dependent Holling type III functional response, and is derived from $p(x/y)$, where p is the Holling type III prey-dependent functional response defined by

$$p(x) = c_1 x^2/(m + x^2). \tag{1.2}$$

We refer to [11] and the references therein for the study of the predator-prey systems with prey-dependent functional responses like that given in (1.2). For recent decades, there have been numerous laboratory experiments and observations showing that functional and numerical responses over typical ecological timescales ought to depend on the populations of both predators and prey, especially when predators have to search or share or compete for food, see [2, 3, 4, 5, 9, 12]. In these situations, more realistic and suitable predator-prey systems should rely on the ratio-dependent functional responses. Roughly speaking, the per capita predator growth rate should be a function of the ratio of prey to predator abundance. Hence, the prey-dependent functional response $p(x)$ given in (1.2) would be replaced by the ratio-dependent functional response $p(x/y)$.

The dynamics of predator-prey systems with the ratio-dependent Holling type III functional responses have been studied, for example in [18, 19]. The necessary and sufficient conditions for the equilibria of (1.1) to exist were given in [19]. Note that the origin $(0,0)$ is not an equilibrium of (1.1). It is showed in [19] that (1.1) has two positive equilibria, one is on the positive x-axis and is a saddle or a stable-node [18], and another is in the interior of the first quadrant and its stability and Hopf bifurcations are discussed in [19]. We refer to [8] for the study on existence, uniqueness, global attractivity of an almost periodic positive solution and feedback controls of ratio-dependent Leslie systems with Holling III scheme.

In realistic ecology, the activities of harvesting or stocking often occur in fishery, forestry, and wildlife management. For example, certain number of animals are removed per year by hunting. It leads one to add harvesting rates or stocking rates into some models, see [7, 15, 20, 25, 26, 28].

In this paper, we insert a constant harvesting rate h of prey into the first equation (1.1) and study the bifurcations of such ratio-dependent predator-prey Holling type III systems with constant harvesting rates. We assume that the prey in (1.1) is continuously harvested at the constant rate h by a harvesting agency and the density of predator is not affected directly by the harvesting activity. Under these hypotheses, we see that the density of prey is reduced due to the harvesting rate and the density of predator is reduced indirectly due to the availability of prey to the predators.

By rescaling the system, we obtain the following equivalent system

$$\begin{cases} \dot{u} = u(1-u) - \dfrac{\alpha u^2 v}{u^2+v^2} - \delta := f(u,v), \\ \dot{v} = v\big(-\gamma + \dfrac{\beta u^2}{u^2+v^2}\big) := g(u,v). \end{cases} \tag{1.3}$$

According to Clark [10], the management of renewable resources is based on the maximum sustainable yield (MSY). If the harvesting rate is larger than the MSY, then it leads to extinction of the population. We show that it is true for the system (1.3), that is, the maximum harvesting rate of prey for (1.3) is $1/4$. Hence, if the harvesting rate is greater than $1/4$, then the prey eventually becomes extinct. Except the effect of harvesting rates on the persistence of the population, the harvesting rates also effect the number of equilibria of (1.3). We prove that when the harvesting rate is greater than $1/4$, there are no positive equilibria and when the harvesting rate is smaller than or equals $1/4$, there are up to four equilibria, which is different from the system without harvesting rates.

We apply the well-known qualitative theory on phase portraits of planar systems [1, 23] to study the dynamical properties of (1.3) near each of the equilibria. We exhibit that these equilibria can be saddles, topological saddles, saddle-nodes, stable (unstable) nodes or focuses, weak centers or cusps, depending on the ranges of the parameters. Stability and directions of the Hopf bifurcations of (1.3) are studied by computing the first Lyapunov number. Our results indicate that the system (1.3) has more equilibria and richer dynamics than (1.1).

2. Positive equilibria and phase portraits

In this section, we study the number of positive equilibria and phase portraits of the following ratio-dependent predator-prey Holling type III systems with a constant harvesting rate

$$\begin{cases} \dot{x} = x(r - sx) - \dfrac{c_1 x^2 y}{x^2 + m^2 y^2} - h, \\ \dot{y} = y\big(-d + \dfrac{c_2 x^2 y}{x^2 + m^2 y^2}\big). \end{cases} \tag{2.4}$$

Using the transformation $u = sx/r$, $v = smy/r$ and $\tilde{t} = rt$, we reduce (2.4) into the following equivalent system

$$\begin{cases} \dot{u} = u(1-u) - \dfrac{\alpha u^2 v}{u^2+v^2} - \delta := f(u,v), \\ \dot{v} = v(-\gamma + \dfrac{\beta u^2}{u^2+v^2}) := g(u,v), \end{cases} \tag{2.5}$$

where $\alpha = c_1/(rm)$, $\beta = c_2/r$, $\gamma = d/r$ and $\delta = sh/r^2$. The parameters α, β, γ and δ have the same biological meanings as c_1, c_2, d and h, respectively, namely, α, β, γ and δ represent prey capturing rate, prey conversion rate, death rate of the predator and harvesting rate on prey, respectively.

Recall that $(u,v) \in \mathbb{R}^2$ is an equilibrium of (2.5) if it satisfies $f(u,v) = 0$ and $g(u,v) = 0$. An equilibrium point (u,v) is said to be positive if

$$(u,v) \in K := [0,\infty) \times [0,\infty).$$

It is obvious that (u,v) is a positive equilibrium of (2.5) if and only if $(u,v) \in K$ and (u,v) satisfies one of the following systems.

$$\begin{cases} u(1-u) - \delta = 0, \\ v = 0, \end{cases} \tag{2.6}$$

or

$$\begin{cases} u(1-u) - \dfrac{\alpha u^2 v}{u^2+v^2} - \delta = 0, \\ -\gamma + \dfrac{\beta u^2}{u^2+v^2} = 0, \ v \neq 0. \end{cases} \tag{2.7}$$

If (u,v) is a positive equilibrium of (2.5), then by the first equation of (2.5), $u > 0$. Noting that the first equation of (2.6) is equivalent to $(u-1/2)^2 = 1/4 - \delta$, we obtain the following result.

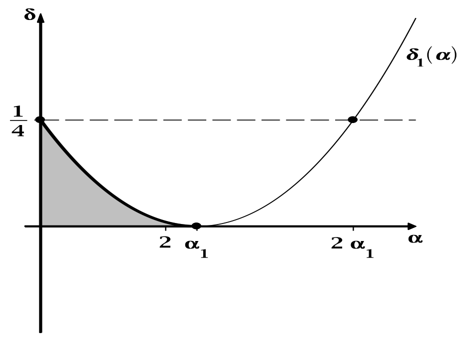

FIGURE 1

Theorem 2.1. (i) *If* $\delta > 1/4$, (2.6) *has no solutions in K.*

(ii) *If* $\delta = 1/4$, (2.6) *has a solution* $(1/2, 0)$ *in K.*

(iii) *If* $0 < \delta < 1/4$, (2.6) *has two solutions* (u_1, v_1) *and* (u_2, v_2) *in K, where*

$$u_1 = \frac{1-\sqrt{1-4\delta}}{2},\ v_1 = 0 \text{ and } u_2 = \frac{1+\sqrt{1-4\delta}}{2},\ v_2 = 0.$$

Now, we consider the solutions of (2.7). For $\beta > \gamma$, let

$$\rho_0 = \sqrt{(\beta-\gamma)/\gamma}, \quad \alpha_1 := \alpha_1(\beta,\gamma) = \frac{\beta}{\gamma\rho_0}$$

and

$$\delta_1 := \delta_{\beta,\gamma}(\alpha) = \frac{(\beta-\alpha\gamma\rho_0)^2}{4\beta^2} = \frac{\gamma^2\rho_0^2}{4\beta^2}(\alpha_1-\alpha)^2. \tag{2.8}$$

Then $\alpha_1 > 2$. When $\alpha = 0$, $\delta_1(\alpha) = \dfrac{1}{4}$ and when $\alpha = \alpha_1$, $\delta_1(\alpha) = 0$. Moreover, when $\alpha = 2\alpha_1$, $\delta_1(\alpha) = \dfrac{1}{4}$. Indeed, by (2.8), we have

$$\delta_1 - \frac{1}{4} = \frac{(\beta-\alpha\gamma\rho_0)^2}{4\beta^2} - \frac{1}{4} = \frac{\alpha\gamma^2\rho_0^2}{4\beta^2}(\alpha - 2\alpha_1).$$

By (2.8), we see that when $\beta > \gamma$, the function δ_1 is a parabola in the $\delta\alpha$-plane, see Figure 1. Let

$$D^1 := \{(\alpha,\delta)) : \delta = \delta_1(\alpha)) \quad \text{for } 0<\alpha<\alpha_1\},$$

$$D^2 := \{(\alpha,\delta) : 0<\delta<\delta_1(\alpha)) \quad \text{for } 0<\alpha<\alpha_1\}$$

$$D_1^0 := K_0 \setminus (D^1 \cup D^2) \quad \text{and } D_2^0 = \{(\alpha,\delta) : \alpha>0, \beta>0\},$$

where $K_0 = (0,\infty)\times(0,\infty)$.

Theorem 2.2. (1) *If either* $0<\beta\le\gamma$ *and* $(\alpha,\delta)\in D_2^0$ *or* $0<\gamma<\beta$ *and* $(\alpha,\delta)\in D_1^0$*, then* (2.7) *has no solutions in* K*.*

(2) *If* $0<\gamma<\beta$ *and* $(\alpha,\delta)\in D^1$*, then* (2.7) *has only one solution* (u^*,v^*) *in* K*, where*

$$u^* = \frac{\gamma\rho_0}{2\beta}(\alpha_1-\alpha) \quad \textit{and } v^* = \rho_0 u^*.$$

(3) *If* $0<\gamma<\beta$ *and* $(\alpha,\delta)\in D^2$*, then* (2.7) *has two solutions* (u_3,v_3) *and* (u_4,v_4) *in* K*, where*

$$u_3 = \frac{\gamma\rho_0}{2\beta}(\alpha_1-\alpha) - \sqrt{\delta_1(\alpha)-\delta} \quad \textit{and } v_3 = \rho_0 u_3$$

and

$$u_4 = \frac{\gamma\rho_0}{2\beta}(\alpha_1-\alpha) + \sqrt{\delta_1(\alpha)-\delta} \quad \textit{and } v_4 = \rho_0 u_4.$$

Proof. By the second equation of (2.7), we have

$$\gamma v^2 = (\beta-\gamma)u^2. \tag{2.9}$$

It follows that if $0<\beta\le\gamma$ and $(\alpha,\delta)\in D_2^0$, then (2.9) has no positive solutions in K. From now on, we assume that $\beta>\gamma$. By (2.9),

$$v = u\sqrt{(\beta-\gamma)/\gamma} = \rho_0 u.$$

Substituting the above function v into the first equation of (2.7) and simplifying the new equation implies

$$\big[u - \frac{\gamma\rho_0}{2\beta}(\alpha_1-\alpha)\big]^2 = \delta_1(\alpha)-\delta. \tag{2.10}$$

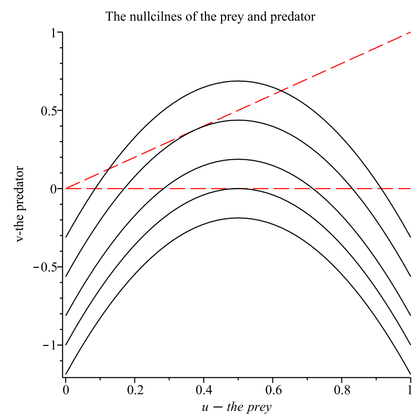

FIGURE 2. is obtained from (2.5) with $f(x,y)=0$ and $g(x,y)=0$ by taking the following values: $\beta=2$, $\gamma=1$, $\alpha=1/2$ (In this case, $\alpha_1=2$ and $\delta_1=9/64$) and δ takes the values $5/64$, $9/64$, $13/64$, $1/4$ and $19/64$, respectively. The curves from the top to bottom correspond to the five values of δ and the dash and solid curves correspond to the nullclines of the predator and prey, respectively.

Let

$$S_1=\{(\alpha,\delta):\delta_1(\alpha))<\delta \quad \text{for } 0<\alpha<\infty\},$$
$$S_2=\{(\alpha,\delta):\delta=\delta_1(\alpha)) \quad \text{for } \alpha_1<\alpha<\infty\} \quad \text{and}$$
$$S_3=\{(\alpha,\delta):0<\delta<\delta_1(\alpha)) \quad \text{for } \alpha_1<\alpha<\infty\}.$$

Then $D_1^0=S_1\cup S_2\cup S_3$. By (2.10), it is readily verified that if $(\alpha,\delta)\in S_i$ for $i=1,2,3$, then (2.7) has no solutions in K. If $0<\gamma<\beta$ and $(\alpha,\delta)\in D^1$, then (2.7) has only one positive solution u^*, and if $0<\gamma<\beta$ and $(\alpha,\delta)\in D^2$, then by (2.10), (2.7) has two positive solutions u_3 and u_4.

By the Figure 1 and Theorems 2.1 and 2.2, it is easy to see that the following result holds. We leave the detailed proof to the reader.

Theorem 2.3. (1) *If* $\alpha,\beta,\gamma>0$ *and* $\delta>1/4$, *then* (2.5) *has no positive equilibria.*

(2) *If* $\alpha,\beta,\gamma>0$ *and* $\delta=1/4$, (2.5) *has one positive equilibrium* $(1/2,0)$.

(3) *If either* $0<\beta\le\gamma$ *and* $(\alpha,\delta)\in D_2^0$ *with* $\delta_1<\delta<1/4$ *or* $0<\gamma<\beta$ *and* $(\alpha,\delta)\in D_1^0$ *with* $\delta_1<\delta<1/4$, *then* (2.5) *has two positive equilibria* $(u_1,0)$ *and* $(u_2,0)$.

(4) *If* $0<\gamma<\beta$ *and* $(\alpha,\delta)\in D^1$*, then* (2.5) *has three positive equilibria* $(u_1,0)$*,* $(u_2,0)$ *and* (u^*,v^*).

(5) *If* $0<\gamma<\beta$ *and* $(\alpha,\delta)\in D^2$*, then* (2.5) *has four positive equilibria* $(u_1,0)$*,* $(u_2,0)$*,* (u_3,v_3) *and* (u_4,v_4).

Remark 2.1. The results of Theorem 2.3 is illustrated by Figure 2. By Theorem 2.3, when the death rate γ of the predator is larger than the conversion rate β of the predator, then (2.5) has no interior equilibria and the predator and prey can not coexist. If the death rate γ of the predator is smaller than the conversion rate β of the predator, (2.5) has up to two interior equilibria.

To study the dynamical behaviors of our planar systems near equilibria, we need some knowledge of the qualitative and bifurcation theories of planar systems [1, 23]. We consider the following planar system

$$\begin{cases} \dot{x}=f(x,y), \\ \dot{y}=g(x,y), \end{cases} \tag{2.11}$$

where $f,g:X\to\mathbb{R}$ are functions having continuous first partial derivatives and X is an open subset in $\mathbb{R}^2$. We denote by $A(x,y)$ the Jacobian matrix of f and g, that is,

$$A(x,y)=\frac{\partial(f,g)}{\partial(x,y)}=\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}, \tag{2.12}$$

by $|A(x,y)|$ and $\mathrm{tr}(A(x,y))$ the determinant and trace of $A(x,y)$, respectively.

Lemma 2.1 ([15, 23, 28])**.** *Let* (x_0,y_0) *be an equilibrium of* (2.11)*. Then the following assertions hold.*

(*i*) *If* $|A(x_0,y_0)|<0$*, then* (x_0,y_0) *is a saddle of* (2.11).

(*ii*) *If* $|A(x_0,y_0)|>0$, $\mathrm{tr}(A(x_0,y_0))\neq 0$ *and*

$$(\mathrm{tr}(A(x_0,y_0)))^2-4|A(x_0,y_0)|\geq 0,$$

then (x_0,y_0) *is a node of* (2.11)*; it is stable if* $\mathrm{tr}(A(x_0,y_0))<0$ *and unstable if* $\mathrm{tr}(A(x_0,y_0))>0$.

(*iii*) *If* $|A(x_0,y_0)|>0$, $\mathrm{tr}(A(x_0,y_0))\neq 0$ *and*

$$(\mathrm{tr}(A(x_0,y_0)))^2-4|A(x_0,y_0)|<0,$$

then (x_0,y_0) *is a focus of* (2.11)*; it is stable if* $\mathrm{tr}(A(x_0,y_0))<0$ *and unstable if*

$$\mathrm{tr}(A(x_0,y_0))>0.$$

(*iv*) *If* $|A(x_0,y_0)|>0$ *and* $\mathrm{tr}(A(x_0,y_0))=0$*, then* (x_0,y_0) *is a center, or a focus of* (2.11).

We need the following result which was first given and proved in [15] and used in [28] to study the dynamical behaviors of Leslie-Gower predator-prey systems with harvesting rates.

Lemma 2.2. *Let* (x_0,y_0) *be an equilibrium point of system* (2.11)*. Assume that* $|A(x_0,y_0)|=0$, $\mathrm{tr}(A(x_0,y_0))\neq 0$ *and* (2.11) *is equivalent to the following system*

$$\begin{cases}\dot{u}=p(u,v),\\ \dot{v}=\rho v+q(u,v)\end{cases}\tag{2.13}$$

with an isolated equilibrium $(0,0)$*, where* $\rho\neq 0$, $p(u,v)=\sum_{i+j=2,i,j\geq 0}^{\infty}a_{ij}u^iv^j$ *and* $q(u,v)=\sum_{i+j=2,i,j\geq 0}^{\infty}b_{ij}u^iv^j$ *are convergent power series. If* $a_{20}\neq 0$*, then* (x_0,y_0) *is a saddle-node of* (2.11)*. For* $a_{20}=a_{11}=a_{02}=0$ *and* $\rho>0$*, if* $a_{30}<0$*, then* (x_0,y_0) *is a topological saddle, and if* $a_{30}>0$*, then* (x_0,y_0) *is an unstable node.*

When $|A(x_0,y_0)|=0$, $\mathrm{tr}(A(x_0,y_0))=0$ and $A(x_0,y_0)\neq 0$, under suitable regular transformations, (2.11) is equivalent to the following form

$$\begin{cases}\dot{x}=y,\\ \dot{y}=a_kx^k[1+h(x)]+b_nx^ny[1+g(x)]+y^2R(x,y)\end{cases}\tag{2.14}$$

with an equilibrium point $(0,0)$, where h,g and R are analytic in a neighborhood of $(0,0)$, $h(0)=g(0)=0$, $k\geq 2$, $a_k\neq 0$ and $n\geq 1$.

The following result is a special case of [23, Theorems 2 and 3 of section 2.11] (also see [1, Theorems 66 and 67, page 357-362]).

Lemma 2.3. *Assume that* $|A(x_0,y_0)|=\mathrm{tr}(A(x_0,y_0))=0$, $A(x_0,y_0)\neq 0$*. Then*

(1) *If* $k=2m+1$ *and* $a_k>0$*, then* (x_0,y_0) *is a topological saddle of* (2.11).

(2) *If* $k=2m$ *and* $n\geq m$, *then* (x_0,y_0) *is a cusp of* (2.11).

To employ Lemmas 2.2 and 2.3, we change (2.5) into an equivalent system with the equilibrium $(0,0)$. Let $(\bar{u},\bar{v})$ be an equilibrium of (2.5). Using the transformation $x=u-\bar{u}$ and $y=v-\bar{v}$, (2.5) can be changed into the system

$$\begin{cases}\dot{x}=(\bar{u}+x)(1-\bar{u}-x)-\dfrac{\alpha(\bar{u}+x)^2(\bar{v}+y)}{(\bar{u}+x)^2+(\bar{v}+y)^2}-\delta:=f_1(x,y),\\ \dot{y}=(\bar{v}+y)\big[-\gamma+\dfrac{\beta(\bar{u}+x)^2}{(\bar{u}+x)^2+(\bar{v}+y)^2}\big]:=g_1(x,y).\end{cases}\tag{2.15}$$

We need the power series representations of f_1 and g_1. To do this, we introduce some symbols to shorten the expressions. Let

$$\begin{aligned}
&a=1-2\bar{u}-\frac{2\alpha\bar{u}\bar{v}^3}{(\bar{u}^2+\bar{v}^2)^2},\ b=-\frac{\alpha\bar{u}^2(\bar{u}^2-\bar{v}^2)}{(\bar{u}^2+\bar{v}^2)^2},\ a_{20}=-1+\frac{\alpha\bar{v}^3(3\bar{u}^2-\bar{v}^2)}{(\bar{u}^2+\bar{v}^2)^3},\\
&a_{11}=-\frac{2\alpha\bar{u}\bar{v}^2(3\bar{u}^2-\bar{v}^2)}{(\bar{u}^2+\bar{v}^2)^3},\ a_{02}=\frac{\alpha\bar{u}^2\bar{v}(3\bar{u}^2-\bar{v}^2)}{(\bar{u}^2+\bar{v}^2)^3},\ a_{30}=-\frac{4\alpha\bar{u}\bar{v}^3(\bar{u}^2-\bar{v}^2)}{(\bar{u}^2+\bar{v}^2)^4},\\
&a_{21}=\frac{\alpha\bar{v}^2(9\bar{u}^4-14\bar{u}^2\bar{v}^2+\bar{v}^4)}{(\bar{u}^2+\bar{v}^2)^4},\ a_{12}=-\frac{4\alpha\bar{u}^3\bar{v}(\bar{u}^2-5\bar{v}^2)}{(\bar{u}^2+\bar{v}^2)^4}-\frac{2\alpha\bar{u}\bar{v}}{(\bar{u}^2+\bar{v}^2)^2},\\
&a_{03}=\frac{\alpha\bar{u}^2(\bar{u}^4-6\bar{u}^2\bar{v}^2+\bar{v}^4)}{(\bar{u}^2+\bar{v}^2)^4},\ c=\frac{2\beta\bar{u}\bar{v}^3}{(\bar{u}^2+\bar{v}^2)^2},\ d=-\gamma+\frac{\beta\bar{u}^2(\bar{u}^2-\bar{v}^2)}{(\bar{u}^2+\bar{v}^2)^2},
\end{aligned}$$

$$\begin{aligned}
&b_{20}=-\frac{\beta\bar{v}^3(3\bar{u}^2-\bar{v}^2)}{(\bar{u}^2+\bar{v}^2)^3},\ b_{11}=\frac{2\beta\bar{u}\bar{v}^2(3\bar{u}^2-\bar{v}^2)}{(\bar{u}^2+\bar{v}^2)^3},\ b_{02}=-\frac{\beta\bar{u}^2\bar{v}(3\bar{u}^2-\bar{v}^2)}{(\bar{u}^2+\bar{v}^2)^3},\\
&b_{30}=\frac{4\beta\bar{u}\bar{v}^3(\bar{u}^2-\bar{v}^2)}{(\bar{u}^2+\bar{v}^2)^4},\ b_{21}=-\frac{\beta\bar{v}^2(9\bar{u}^4-14\bar{u}^2\bar{v}^2+\bar{v}^4)}{(\bar{u}^2+\bar{v}^2)^4},\\
&b_{12}=\frac{4\beta\bar{u}^3\bar{v}(\bar{u}^2-5\bar{v}^2)}{(\bar{u}^2+\bar{v}^2)^4}+\frac{2\beta\bar{u}\bar{v}}{(\bar{u}^2+\bar{v}^2)^2},b_{03}=-\frac{\beta\bar{u}^2(\bar{u}^4-6\bar{u}^2\bar{v}^2+\bar{v}^4)}{(\bar{u}^2+\bar{v}^2)^4}\\
&\xi=\bar{u}^2+\bar{v}^2,\quad \bar{\xi}=\bar{u}^2-\bar{v}^2,\quad R(x,y)=\frac{1}{(\bar{u}+x)^2+(\bar{v}+y)^2}.
\end{aligned}$$

Lemma 2.4. *The power series representations for* f_1 *and* g_1 *given in* (2.15) *are*

$$f_1(x,y)=ax+by+\sum_{i+j=2}^{3}a_{ij}x^iy^j+O_4^{(1)}(x,y),$$

$$g_1(x,y)=cx+dy+\sum_{i+j=2}^{3}b_{ij}x^iy^j+O_4^{(2)}(x,y),$$

where $O_4^{(2)}(x,y) = -O_4^{(1)}(x,y)\beta/\alpha$, $O_4^{(1)}(x,y) = -\alpha R_4(x,y) - \alpha(2\bar{u}\bar{v}x + \bar{u}^2 y)\big(R(x,y) - \frac{1}{\xi} + \frac{2\bar{u}x+2\bar{v}y}{\xi^2} - \frac{3\bar{u}^2-\bar{v}^2}{\xi^3}x^2 - \frac{8\bar{u}\bar{v}}{\xi^3}xy + \frac{\bar{u}^2-3\bar{v}^2}{\xi^3}y^2\big) - \alpha(\bar{v}x^2 + 2\bar{u}xy + x^2 y)(R(x,y) - \frac{1}{\xi} + \frac{2\bar{u}x+2\bar{v}y}{\xi^2})$, $R_4(x,y) = -\frac{\zeta(x,y)^3}{\xi^3} + \frac{(x^2+y^2)^2}{\xi^2} + \frac{(2\bar{u}x+2\bar{v}y)^3}{\xi^3} + \sum_{j=4}^{\infty}\frac{\zeta(x,y)^j}{(-1)^j\xi^j}$
and $\zeta(x,y) = 2\bar{u}x + 2\bar{v}y + x^2 + y^2$.

Proof. We first give the power series for $R(x,y) = \frac{1}{\xi}\frac{1}{(1+\frac{\zeta(x,y)}{\xi})}$ as follows.

$$\begin{aligned}
R(x,y) &= \frac{1}{\xi}\Big[1 - \frac{\zeta(x,y)}{\xi} + (\frac{\zeta(x,y)}{\xi})^2 - (\frac{\zeta(x,y)}{\xi})^3 + \sum_{j=4}^{\infty}(-\frac{\zeta(x,y)}{\xi})^j\Big] \\
&= \frac{1}{\xi}\Big[1 - (\frac{2\bar{u}x+2\bar{v}y}{\xi})^3 - \frac{\zeta(x,y)}{\xi} + \frac{(2\bar{u}x+2\bar{v}y)^2 + 4(\bar{u}x+\bar{v}y)(x^2+y^2)}{\xi^2}\Big] \\
&\quad + R_4(x,y) \\
&= \frac{1}{\xi} - \frac{2\bar{u}x+2\bar{v}y}{\xi^2} + \frac{w_1}{\xi^3}x^2 + \frac{8\bar{u}\bar{v}}{\xi^3}xy - \frac{w_2}{\xi^3}y^2 - \frac{4\bar{u}\bar{\xi}}{\xi^4}x^3 - \frac{4\bar{v}w_1}{\xi^4}x^2y \\
&\quad + \frac{4\bar{u}}{w_2}\xi^4 xy^2 + \frac{4\bar{v}}{\bar{\xi}}\xi^4 y^3 + R_4(x,y),
\end{aligned}$$

where $w_1 = 3\bar{u}^2 - \bar{v}^2$ and $w_2 = \bar{u}^2 - 3\bar{v}^2$. Using the power series of $R(x,y)$ and

$$f_1(x,y) = \bar{u} + x - (\bar{u}+x)^2 - \alpha(\bar{u}+x)^2(\bar{v}+y)R(x,y) - \delta,$$

we obtain

$$\begin{aligned}
f_1(x,y) &= \big(1 - 2\bar{u} - \frac{2\alpha\bar{u}\bar{v}}{\xi} + \frac{2\alpha\bar{u}^3\bar{v}}{\xi^2}\big)x - (\frac{\alpha\bar{u}^2}{\xi} + \frac{2\alpha\bar{u}^2\bar{v}^2}{\xi^2})y - \big(1 + \frac{\alpha\bar{v}}{\xi} - \frac{4\alpha\bar{u}^2\bar{v}}{\xi^2} \\
&\quad + \frac{\alpha\bar{u}^2\bar{v}w_1}{\xi^3})x^2 + (\frac{2\alpha\bar{u}\bar{v}^2}{\xi^2} - \frac{8\alpha\bar{u}^3\bar{v}^2}{\xi^3})xy + (\frac{2\alpha\bar{u}^2\bar{v}}{\xi^2} + \frac{\alpha\bar{u}^2\bar{v}w_2}{\xi^3})y^2 \\
&\quad + (\frac{2\alpha\bar{u}\bar{v}}{\xi^2} - \frac{2\alpha\bar{u}\bar{v}w_1}{\xi^3} + \frac{4\alpha\bar{u}^3\bar{v}\bar{\xi}}{\xi^4})x^3 - (\frac{\alpha}{\xi} + \frac{2\alpha\bar{u}^2}{\xi^2} - \frac{3\alpha\bar{u}^2(\bar{u}^2-5\bar{v}^2)}{\xi^3} \\
&\quad + \frac{4\alpha\bar{u}^2\bar{v}^2(5\bar{u}^2-\bar{v}^2)}{\xi^4})x^2y - (\frac{2\alpha\bar{u}\bar{v}}{\xi^2} + \frac{4\alpha\bar{u}^3\bar{v}w_2}{\xi^4})xy^2 + (\frac{\alpha\bar{u}^2w_2}{\xi^3} \\
&\quad - \frac{4\alpha\bar{u}^2\bar{v}^2\bar{\xi}}{\xi^4})y^3 + O_4^{(1)}(x,y) = ax + by + \sum_{i+j=2}^{3} a_{ij}x^iy^j + O_4^{(1)}(x,y)
\end{aligned}$$

and

$$
\begin{aligned}
g_1(x,y) &= -\gamma(\bar{v}+y)+\beta(\bar{u}+x)^2(\bar{v}+y)R(x,y) \\
&= -\gamma(\bar{v}+y)+\beta(\bar{u}^2\bar{v}+2\bar{u}\bar{v}x+\bar{u}^2y+\bar{v}x^2+2\bar{u}xy+x^2y)\Big[\frac{1}{\xi}-\frac{2\bar{u}x+2\bar{v}y}{\xi^2} \\
&\quad +\frac{w_1}{\xi^3}x^2-\frac{w_2}{\xi^3}y^2-\frac{4}{\xi^4}(\bar{u}\bar{\xi}x^3-\bar{v}(5\bar{u}^2-\bar{v}^2)x^2y+\bar{u}(\bar{u}^2-5\bar{v}^2)xy^2+\bar{v}\bar{\xi}y^3) \\
&\quad +\frac{8\bar{u}\bar{v}}{\xi^3}xy+R_4(x,y)\Big] \\
&= \Big(\frac{2\beta\bar{u}\bar{v}}{\xi}-\frac{2\beta\bar{u}^3\bar{v}}{\xi^2}\Big)x-\Big(\gamma-\frac{\beta\bar{u}^2}{\xi}-\frac{2\beta\bar{u}^2\bar{v}^2}{\xi^2}\Big)y+\Big(\frac{\beta\bar{v}}{\xi}+\frac{\beta\bar{u}^2\bar{v}w_1}{\xi^3} \\
&\quad -\frac{4\beta\bar{u}^2\bar{v}}{\xi^2}\Big)x^2+\Big(\frac{\beta\bar{u}}{\xi}-\frac{4\beta\bar{u}\bar{v}}{\xi^2}-\frac{2\beta\bar{u}^3}{\xi^2}+\frac{8\beta\bar{u}^3\bar{v}^2}{\xi^3}\Big)xy-\Big(\frac{\beta\bar{u}^2\bar{v}w_2}{\xi^3} \\
&\quad +\frac{2\beta\bar{u}^2\bar{v}}{\xi^2}\Big)y^2+\Big(\frac{2\beta\bar{u}\bar{v}w_1}{\xi^3}-\frac{2\beta\bar{u}\bar{v}}{\xi^2}-\frac{4\beta\bar{u}^3\bar{v}\bar{\xi}}{\xi^4}\Big)x^3+\Big(\frac{\beta}{\xi}-\frac{4\beta}{\xi} \\
&\quad -\frac{4\beta\bar{u}^2\bar{v}^2(5\bar{u}^2-\bar{v}^2)}{\xi^4}+\frac{\beta\bar{u}^2(19\bar{u}^2-\bar{v}^2)}{\xi^3}\Big)x^2y+\Big(\frac{4\beta\bar{u}^3\bar{v}(\bar{u}^2-5\bar{v}^2)}{\xi^4} \\
&\quad +\frac{2\beta\bar{u}\bar{v}}{\xi^2}\Big)xy^2-\beta\bar{u}^2\Big(\frac{w_2}{\xi^3}-\frac{4\bar{v}^2\bar{\xi}}{\xi^4}\Big)y^3+O_4^{(2)}(x,y) \\
&= cx+dy+\sum_{i+j=2}^{3}b_{ij}x^iy^j+O_4^{(2)}(x,y).
\end{aligned}
$$

Let $A(u,v)$ be the Jacobian matrix of f and g given in (2.5). Then

$$
A(u,v)=\begin{pmatrix} 1-2u-\dfrac{2\alpha uv^3}{(u^2+v^2)^2} & -\dfrac{\alpha u^2(u^2-v^2)}{(u^2+v^2)^2} \\ \dfrac{2\beta uv^3}{(u^2+v^2)^2} & -\gamma+\dfrac{\beta u^2(u^2-v^2)}{(u^2+v^2)^2} \end{pmatrix}. \tag{2.16}
$$

The following result gives the phase portraits of (2.5) near the equilibria on the u-axis.

Theorem 2.4. *Let $\alpha>0$ and $0<\delta<1/4$. Then the following assertions hold.*

(1) *If $0<\beta<\gamma$, then $(u_1,0)$ is a saddle and $(u_2,0)$ is a stable node.*

(2) *If $\beta>\gamma$,, then $(u_1,0)$ is an unstable node and $(u_2,0)$ is a saddle.*

(3) *If $\beta=\gamma$, then $(u_1,0)$ is a topological saddle and $(u_2,0)$ is an unstable node.*

Proof. Let $(\bar{u},0)$ be an equilibrium of (2.5). By (2.16), we have

$$A(\bar{u},0)=\begin{pmatrix}1-2\bar{u} & -\alpha\\ 0 & \beta-\gamma\end{pmatrix}. \tag{2.17}$$

It follows that

$$A(u_1,0)=\begin{pmatrix}\sqrt{1-4\delta} & -\alpha\\ 0 & \beta-\gamma\end{pmatrix} \text{ and } A(u_2,0)=\begin{pmatrix}-\sqrt{1-4\delta} & -\alpha\\ 0 & \beta-\gamma\end{pmatrix}.$$

The results (1) and (2) follow from Lemma 2.1 (i) and (ii). We prove the result (3). Assume that $\beta=\gamma$. If $(\bar{u},\bar{v})=(u_1,0)$, then $a=\sqrt{1-4\delta}$, $b=-\alpha$, $a_{20}=-1$,

$$c=d=a_{11}=a_{02}=a_{30}=a_{21}=a_{12}=b_{20}=b_{11}=b_{02}=b_{30}=b_{21}=b_{12}=0,$$

$a_{03}=\frac{4\alpha}{(1-\sqrt{1-4\delta})^2}$ and $b_{03}=-\frac{4\beta}{(1-\sqrt{1-4\delta})^2}$. By Lemma 2.4, system (2.15) with $(\bar{u},\bar{v})=(u_1,0)$ becomes

$$\begin{cases}\dot{x}=ax-\alpha y-x^2+a_{03}y^3+O_4^{(1)}(x,y),\\ \dot{y}=b_{03}y^3+O_4^{(2)}(x,y).\end{cases} \tag{2.18}$$

Let $x_1=ax-\alpha y$ and $y_1=y$. Then by (2.18), we have

$$\begin{aligned}\dot{y}_1&=\dot{y}=b_{03}y^3+O_4^{(2)}(x,y)=b_{03}y_1^3+O_4^{(2)}\big(\frac{x_1+\alpha y_1}{a},y_1\big),\\ \dot{x}_1&=a\dot{x}-\alpha\dot{y}=a\big[x_1-x^2+a_{03}y^3+O_4^{(1)}(x,y)\big]-\alpha\big[b_{03}y^3+O_4^{(2)}(x,y)\big]\\ &=ax_1-ax^2+\tilde{a}y^3+P_4^{(0)}=ax_1-\frac{1}{a}(x_1+\alpha y_1)^2+\tilde{a}y_1^3+P_4^{(1)}(x_1,y_1),\end{aligned}$$

where $\tilde{a}=aa_{03}-\alpha b_{03},\ P_4^{(0)}=aO_4^{(1)}(x,y)-\alpha O_4^{(2)}(x,y)$ and

$$P_4^{(1)}(x_1,y_1)=P_4^{(0)}((x_1+\alpha y_1)/a,y_1).$$

By interchanging x_1 and y_1, (2.18) is changed into the form (2.13):

$$\begin{cases}\dot{x}_1=b_{03}x_1^3+O_4^{(2)}(\frac{y_1+\alpha x_1}{a},x_1),\\ \dot{y}_1=ay_1-\frac{y_1^2}{a}-\frac{2x_1y_1}{a}-\frac{x_1^2}{a}+(aa_{03}-\alpha b_{03})x_1^3+P_4^{(1)}(x_1,y_1),\end{cases}$$

where $a=\sqrt{1-4\delta}>0$ and $b_{03}=-4\beta/(1+\sqrt{1-4\delta})^2<0$. It follows from Lemma 2.2 that $(u_1,0)$ is a topological saddle of (2.5).

If $(\bar{u},\bar{v}) = (u_2, 0)$, then $a = -\sqrt{1-4\delta}$, $b = -\alpha$, $a_{20} = -1$,

$$c = d = a_{11} = a_{02} = a_{30} = a_{21} = a_{12} = b_{20} = b_{11} = b_{20} = b_{30} = b_{21} = b_{12} = 0,$$

$a_{03} = \frac{4\alpha}{(1+\sqrt{1-4\delta})^2}$ and $b_{03} = -\frac{4\beta}{(1+\sqrt{1-4\delta})^2}$. By Lemma 2.4, system (2.15) with $(\bar{u},\bar{v}) = (u_2, 0)$ becomes

$$\begin{cases} \dot{x} = ax - \alpha y - x^2 + a_{03}y^3 + O_4^{(1)}(x,y), \\ \dot{y} = b_{03}y^3 + O_4^{(2)}(x,y). \end{cases} \tag{2.19}$$

Let $x_1 = ax - \alpha y$ and $y_1 = y$. Then (2.19) becomes

$$\begin{aligned} \dot{y}_1 &= \dot{y} = b_{03}y^3 + O_4^{(2)}(x,y) = b_{03}y_1^3 + O_4^{(2)}[(x_1+\alpha y_1)/a, y_1], \\ \dot{x}_1 &= a\dot{x} - \alpha\dot{y} = a[x_1 - x^2 + a_{03}y^3 + O_4^{(1)}(x,y)] - \alpha[b_{03}y^3 + O_4^{(2)}(x,y)] \\ &= ax_1 - ax^2 + (aa_{03} - \alpha b_{03})y^3 + aO_4^{(1)}(x,y) - \alpha O_4^{(2)}(x,y) \\ &= ax_1 - \frac{1}{a}(x_1+\alpha y_1)^2 + (aa_{03} - \alpha b_{03})y_1^3 + P_4^{(1)}(x_1,y_1), \end{aligned}$$

where $P_4^1(x_1,y_1) = aO_4^{(1)}[(x_1+\alpha y_1)/a, y_1] - \alpha O_4^{(2)}[(x_1+\alpha y_1)/a, y_1]$. Let $\tilde{t} = -t$. This above system is changed into the form:

$$\begin{cases} \dot{x}_1 = -ax_1 + \frac{x_1^2}{a} + \frac{2x_1y_1}{a} + \frac{y_1^2}{a} - (aa_{03} - \alpha b_{03})y_1^3 - P_4^{(1)}(x_1,y_1), \\ \dot{y}_1 = -b_{03}y_1^3 - O_4^{(2)}(\frac{x_1+\alpha y_1}{a}, y_1), \end{cases}$$

where $-a = \sqrt{1-4\delta} > 0$ and $-b_{03} = 4\beta/(1+\sqrt{1-4\delta})^2 > 0$. It follows from Lemma 2.2 that $(u_2, 0)$ is an unstable node of (2.5).

Now we consider the phase portraits of (2.5) near the equilibrium $(1/2, 0)$.

Theorem 2.5. *Let $\alpha > 0$ and $\delta = 1/4$. Then $(1/2,0)$ is a saddle-node if $\beta \neq \gamma$ and is a topological saddle if $\beta = \gamma$.*

Proof. Since $(\bar{u},\bar{v}) = (1/2, 0)$, by (2.17), we have

$$A(1/2,0) = \begin{pmatrix} 0 & -\alpha \\ 0 & \beta - \gamma \end{pmatrix}.$$

Then $|A(1/2,0)| = 0$ and $\mathrm{tr}(1/2,0) \neq 0$ if $\beta \neq \gamma$ and $\mathrm{tr}(1/2,0) = 0$ if $\beta = \gamma$. When $(\bar{u},\bar{v}) = (1/2,0)$, (2.15) becomes

$$\begin{cases} \dot{x} = \frac{1}{2} + x - (\frac{1}{2}+x)^2 - \frac{\alpha(1/2+x)^2 y}{(1/2+x)^2+y^2} - \frac{1}{4} := f_2(x,y), \\ \dot{y} = -\gamma y + \frac{\beta(1/2+x)^2 y}{(1/2+x)^2+y^2} := g_2(x,y) \end{cases} \tag{2.20}$$

and

$$\begin{aligned} R(x,y) &= \frac{1}{(1/2+x)^2+y^2} = \frac{1}{1/4+x+x^2+y^2} = 4\big[\sum_{j=0}^{\infty}(-4)^j(x+x^2+y^2)^j\big] \\ &= 4\big[1-4(x+x^2+y^2)+16(x+x^2+y^2)^2-64(x+x^2+y^2)^3 \\ &+256(x+x^2+y^2)^4+\sum_{j=5}^{\infty}(-4)^j(x+x^2+y^2)^j\big] \\ &= 4[1-4(x+x^2+y^2)+16(x^2+2x^3+2xy^2+x^4+2x^2y^2+y^4) \\ &-64(x^3+3x^2y^2+3x^4)+256x^4+R_5(x,y)] \\ &= 4(1-4x+12x^2-4y^2-32x^3+32xy^2+80x^4-160x^2y^2+16y^4) \\ &+R_5(x,y), \end{aligned}$$

where $R_5(x,y) = 4\{-64[(x+x^2+y^2)^3-x^3-3x^2y^2-3x^4]+256[(x+x^2+y^2)^4-x^4]$
$+\sum_{j=5}^{\infty}(-4)^j(x+x^2+y^2)^j\}$.

We need the following power series of f_2 and g_2 which are different from those given in Lemma 2.4.

$$\begin{aligned} f_2(x,y) &= \frac{1}{2}+x-(\frac{1}{2}+x)^2-\alpha(\frac{1}{2}+x)^2 yR(x,y)-\frac{1}{4} \\ &= \frac{1}{4}+x-(\frac{1}{4}+x+x^2)-\alpha(y+4xy+4x^2y)[1-4x+12x^2-4y^2 \\ &-32x^3+32xy^2+80x^4-160x^2y^2+16y^4+R_5(x,y)] \\ &= -\alpha y-x^2+4\alpha y^3-16\alpha xy^3+48\alpha x^2y^3-16\alpha y^5+O_6^{(1)}(x,y), \end{aligned}$$

where $O_6^{(1)}(x,y) = -\alpha(4xy+4x^2y)[80x^4+160x^2y^2+16y^4+R_5(x,y)]$.

$$\begin{aligned} g_2(x,y) &= -\gamma y+\beta(\frac{1}{2}+x)^2yR(x,y) = -\gamma y+\beta(y+4xy+4x^2y)[1-4x \\ &\quad +12x^2-4y^2-32x^3+32xy^2+80x^4-160x^2y^2+16y^4+R_5(x,y)] \\ &= (\beta-\gamma)y-4\beta y^3+16\beta xy^3-48\beta x^2y^3+16\beta y^5+O_6^{(2)}(x,y), \end{aligned}$$

where $O_6^{(2)}(x,y) = -\beta O_6^{(1)}(x,y)/\alpha$. Hence, (2.20) becomes

$$\begin{cases} \dot{x} = -\alpha y-x^2+4\alpha y^3-16\alpha xy^3+48\alpha x^2y^3-16\alpha y^5+O_6^{(1)}(x,y), \\ \dot{y} = (\beta-\gamma)y-4\beta y^3+16\beta xy^3-48\beta x^2y^3+16\beta y^5+O_6^{(2)}(x,y). \end{cases} \tag{2.21}$$

If $\beta \neq \gamma$, we rewrite (2.21) into the following system

$$\begin{cases} \dot{x} = -\alpha y-x^2+4\alpha y^3+O_4^{(1)}(x,y), \\ \dot{y} = (\beta-\gamma)y-4\beta y^3+O_4^{(2)}(x,y), \end{cases}$$

where

$$O_4^{(1)}(x,y) = -16\alpha y^3(x-3x^2+y^2)+O_6^{(1)}(x,y) \text{ and } O_4^{(2)}(x,y) = -\frac{\beta O_4^{(1)}(x,y)}{\alpha}.$$

Let $x_1 = (\beta-\gamma)x+\alpha y$ and $y_1 = y$. Then

$$\begin{aligned} \dot{x}_1 &= (\beta-\gamma)[-\alpha y-x^2+4\alpha y^3+O_4^{(1)}(x,y)]+\alpha[(\beta-\gamma)y-4\beta y^3+O_4^{(2)}(x,y)] \\ &= -(\beta-\gamma)x^2-4\alpha\gamma y^3+(\beta-\gamma)O_4^{(1)}(x,y)+\alpha O_4^{(2)}(x,y) \\ &= -\frac{1}{\beta-\gamma}(x_1-\alpha y_1)^2-4\alpha\gamma y_1^3-\gamma O_4^{(1)}(x,y) \\ &= -\frac{1}{\beta-\gamma}x_1^2+\frac{2\alpha}{\beta-\gamma}x_1y_1-\frac{\alpha^2}{\beta-\gamma}y_1^2-4\alpha\gamma y_1^3+P_4^{(1)}(x_1,y_1) := p(x_1,y_1), \end{aligned}$$

where $P_4^{(1)}(x_1,y_1) = -\gamma O_4^{(1)}((x_1-\alpha y_1)/(\beta-\gamma),\ y_1)$.

$$\dot{y}_1 = \dot{y} = (\beta-\gamma)y_1-4\beta y_1^3+P_4^{(2)}(x_1,y_1) := \rho y_1+q(x_1,y_1),$$

where $\rho = \beta-\gamma \neq 0$ and $P_4^{(2)}(x_1,y_1) = O_4^{(2)}((x_1-\alpha y_1)/(\beta-\gamma),\ y_1)$. From $p(x_1,y_1)$, $a_{20} = -\frac{1}{\beta-\gamma} \neq 0$. By Lemma 2.2, $(1/2,0)$ is a saddle-node of (2.5).

If $\beta = \gamma$, then (2.21) becomes

$$\begin{cases} \dot{x} = -\alpha y + Q_1, \\ \dot{y} = -4\beta y^3 + 16\beta x y^3 - 48\beta x^2 y^3 + 16\beta y^5 + O_6^{(2)}(x,y) := Q_2, \end{cases} \tag{2.22}$$

where $Q_1 := -x^2 + 4\alpha y^3 - 16\alpha x y^3 + 48\alpha x^2 y^3 - 16\alpha y^5 + U$ and $U = O_6^{(1)}(x,y)$. Let $x_1 = x$ and $y_1 = -\alpha y + Q_1$. Then

$$y = \frac{1}{\alpha}(-y_1 + Q_1) := \frac{1}{\alpha} Q_3 \quad \text{and } \dot{x}_1 = \dot{x} = -\alpha y + Q_1 = y_1.$$

Hence,

$$\begin{aligned} \dot{y}_1 &= (-2x - 16\alpha y^3 + 96\alpha x y^2)\dot{x} - \alpha(1 - 12y(y - 4xy + 8x^2) + 80y^4)\dot{y} + \dot{U} \\ &= 2\alpha x y + 2x^3 + 4\alpha\beta y^3 - 2\alpha(1+8\beta) x y^3 + 16\alpha^2 y^4 + 16\alpha(1+3\beta) x^2 y^3 \\ &- 192\alpha x^3 y^2 - 64\alpha\beta y^5 + P_6^{(1)}(x,y) = 2x_1 Q_3 + 2x_1^3 - \frac{8(1+12\alpha+2\beta)}{\alpha^2} x_1 Q_3^3 \\ &+ \frac{4\beta}{\alpha^2} Q_3^3 + \frac{16}{\alpha^2} Q_3^4 + \frac{16(1+3\beta)}{\alpha^2} x_1^2 Q_3^3 - \frac{192}{\alpha} x_1^3 Q_3^2 - \frac{64\beta}{\alpha^4} Q_3^5 + P_6^{(1)}(x,y) \\ &= -2x_1 y_1 - \frac{\beta}{\alpha^2} y_1^3 + \frac{16(6\alpha+\beta)}{\alpha^2} x_1 y_1^3 - \frac{3\beta}{\alpha^2} x_1^2 y_1^2 - \frac{3\beta}{\alpha^2} x_1^3 y_1 + \frac{16}{\alpha^2} y_1^4 + 32\beta x_1^5 \\ &+ \frac{16(1-3\beta)}{\alpha^2} x_1^2 y_1^3 + \frac{8(-21+12\alpha+2\beta)}{\alpha^2} x_1^3 y_1^2 + \frac{64\beta}{\alpha^2} y_1^5 + P_6^{(2)}(x_1,y_1), \end{aligned}$$

where $P_6^{(2)}(x_1,y_1) = 64\alpha x_1^4(Q_3 + y_1) + 2\alpha x_1(Q_3^3 + y_1^3 - 3x_1^4 y_1 - 3x_1^2 y_1^2) + (32x_1^2 + \frac{3\beta}{\alpha} y_1^2)(Q_1^3 + y_1^3) + P_6^{(1)}(x_1, y_1, y), \dot{U} = U - 144\alpha x^4 y - 224\alpha x^2 y^3 - 16\alpha y^5$,

$$\begin{aligned} P_6^{(1)}(x_1,y_1,y) &= 32\alpha x_1 y^3 (3x_1^2 - y^2) - \frac{8(1+12\alpha+2\beta)}{\alpha^2} x_1 (Q_3^3 + y_1^3 + 3x_1^2 y_1^2) \\ &+ U + \frac{\beta}{\alpha^2}(Q_3 + y_1 + x_1^2)^3 + \frac{16}{\alpha^2}(Q_3^4 - y_1^4) + \frac{16(1+3\beta)}{\alpha^2} x_1^2 (Q_3^3 - y_1^3) \\ &- \frac{192}{\alpha} x_1^3 (Q_3^2 - y_1^2) - \frac{64\beta}{\alpha^4}(Q_3^5 - y_1^5) + \tilde{P}_6^{(1)}(x,y), \\ \tilde{P}_6^{(1)}(x,y) &= -2x\tilde{U} + (-96\alpha x^2 y + 16\alpha y^3)(Q_1 + \alpha x + x^2) + (-576\alpha x^3 y \\ &- 448\alpha x y^3)(Q_1 + \alpha x) - \alpha O_6^{(2)}(x,y) + 3\alpha y^2 (Q_2 + \beta y^3) + Q_2(-32\alpha x^3 \\ &+ 48\alpha x y^2 - 144\alpha x^4 - 672\alpha x^2 y^2 - 80\alpha y^4) + \dot{U}. \end{aligned}$$

Note that $P_6^{(2)}(x_1,y_1)$ is a power series in (x_1,y_1) with orders greater than or equal to 6. We rewrite $P_6^{(2)}(x_1,y_1)$ into the following form.

$$P_6^{(2)}(x_1,y_1)=h_1(x_1)+y_1g_1(x_1)+\sum_{i=2}^{\infty}y_1^iR_i(x_1),$$

where $h_1(x_1)$, $g_1(x_1)$ and $R_i(x_1)$ are polynomials and independent of y_1, and $h_1(x_1)=\sum_{i=6}^{\infty}c_ix_1^i$ and $g_1(x_1)=\sum_{i=5}^{\infty}d_ix_1^i$, where c_i and d_i are real numbers. Then

$$\begin{aligned}\dot{y}_1=&-2x_1y_1-\frac{\beta}{\alpha^2}y_1^3+\frac{16(6\alpha+\beta)}{\alpha^2}x_1y_1^3-\frac{3\beta}{\alpha^2}x_1^2y_1^2-\frac{3\beta}{\alpha^2}x_1^3y_1+\frac{16}{\alpha^2}y_1^4+32\beta x_1^5\\&+\frac{16(1-3\beta)}{\alpha^2})x_1^2y_1^3+\frac{8(-21+12\alpha+2\beta)}{\alpha^2}x_1^3y_1^2+\frac{64\beta}{\alpha^2}y_1^5+\sum_{i=6}^{\infty}c_ix_1^i\\&+y_1\sum_{i=5}^{\infty}d_ix_1^i+\sum_{i=2}^{\infty}y_1^iR_i(x_1)\\=&\,32\beta x_1^5(1+h(x_1))-2x_1y_1(1+g(x_1))+y_1^2R(x_1,y_1),\end{aligned}$$

where $h(x_1)=\sum_{i=6}^{\infty}\frac{c_i}{32\beta}x_1^{i-5}$, $g(x_1)=-\frac{3\beta}{2\alpha^2}x_1^2-\sum_{i=5}^{\infty}\frac{d_i}{2}x_1^{i-1}$ and

$$\begin{aligned}R(x_1,y_1)=&-\frac{\beta}{\alpha^2}y_1+\frac{16(6\alpha+\beta)}{\alpha^2}x_1y_1-\frac{3\beta}{\alpha^2}x_1^2+\frac{16}{\alpha^2}y_1^2+\frac{16(1-3\beta)}{\alpha^2}x_1^2y_1\\&+\frac{8(-21+12\alpha+2\beta)}{\alpha^2}x_1^3+\frac{64\beta}{\alpha^2}y_1^3+\sum_{i=2}^{\infty}y_1^{i-2}R_i(x_1).\end{aligned}$$

Hence, (2.22) becomes

$$\begin{cases}\dot{x}_1=y_1,\\ \dot{y}_1=32\beta x_1^5(1+h(x_1))-2x_1y_1(1+g(x_1))+y_1^2R(x_1,y_1).\end{cases}$$

It follows from Lemma 2.3 (1) that $(1/2,0)$ is a topological saddle of (2.5).

Let

$$\rho_1=\beta-\gamma,\quad \rho_2=2\gamma-\beta,\text{ and }\rho_3=4\gamma-\beta.$$

Lemma 2.5. *Assume that (u,v) satisfies $v=\rho_0u$ and* (2.16). *Then*

$$|A(u,v)|=\frac{2(\beta-\gamma)u_3^2}{\gamma(1+\rho_0^2)^2}\big[\alpha\gamma\rho_0-\beta(1-2u)\big].$$

Proof. Let $\xi_1 = u^2 + v^2$ and $\xi_2 = u^2 - v^2$. By (2.16), we have

$$A(u,v) = \frac{1}{\xi_1^2} \begin{pmatrix} (1-2u)\xi_1^2 - 2\alpha uv^3 & -\alpha u^2 \xi_2 \\ 2\beta uv^3 & -\gamma\xi_1^2 + \beta u^2 \xi_2 \end{pmatrix}.$$

Since $v = \rho_0 u$, $\xi_1 = (1+\rho_0^2)u^2$ and $\xi_2 = (1-\rho_0^2)u^2$,

$$\begin{aligned} A(u,v) &= \frac{u^4}{\xi_1^2} \begin{pmatrix} (1-2u)(1+\rho_0^2)^2 - 2\alpha\rho_0^3 & -\alpha(1-\rho_0^2) \\ 2\beta\rho_0^3 & -\gamma(1+\rho_0^2)^2 + \beta(1-\rho_0^2) \end{pmatrix} \\ &:= \frac{u^4}{\xi_1^2} B(u,v). \end{aligned}$$

By computation, we have

$$|B(u,v)| = (1+\rho_0^2)^2 \left\{ 2\alpha\gamma\rho_0^3 + (1-2u)[\beta(1-\rho_0^2) - \gamma(1+\rho_0^2)^2] \right\}.$$

Noting that $\rho_0^2 = \dfrac{\beta-\gamma}{\gamma}$, $1+\rho_0^2 = \dfrac{\beta}{\gamma}$ and $1-\rho_0^2 = 2 - \dfrac{\beta}{\gamma}$, we have

$$|B(u,v)| = \frac{2(1+\rho_0^2)^2(\beta-\gamma)}{\gamma} [\alpha\gamma\rho_0 - \beta(1-2u)]. \tag{2.23}$$

The result follows from $|A(u,v)| = \dfrac{u^8}{\xi_1^4}|B(u,v)|$.

Theorem 2.6. *Let* $\beta > \gamma > 0$ *and* $(\alpha,\delta) \in D^2$. *Then* (u_3, v_3) *is a saddle of* (2.5).

Proof. By Theorem 2.2, $v_3 = \rho_0 u_3$. It follows from (2.23) that

$$|B(u_3,v_3)| = \frac{2(1+\rho_0^2)^2(\beta-\gamma)}{\gamma} [\alpha\gamma\rho_0 - \beta(1-2u_3)].$$

Since $\beta = \alpha\rho_0\gamma$ and $2\beta u_3 = \gamma\rho_0(\alpha_1 - \alpha) - 2\beta\sqrt{\delta_1(\alpha) - \delta}$, we simplify $|B(u_3,v_3)|$ and obtain

$$|B(u_3,v_3)| = -\frac{4\beta(1+\rho_0^2)^2(\beta-\gamma)}{\gamma}\sqrt{\delta_1(\alpha)-\delta} < 0.$$

By Lemma 2.5, we find that

$$|A(u_3,v_3)| = \frac{u_3^8}{\xi_1^4}|B(u_3,v_3)| < 0.$$

The result follows from Lemma 2.1 (i).

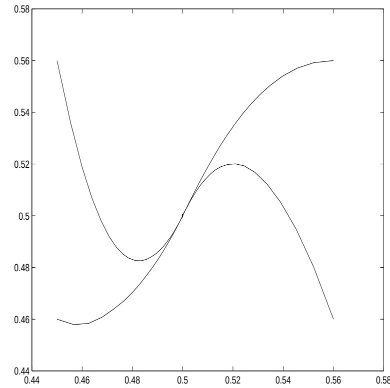

FIGURE 3. is the phase portraits of (u_3, v_3) by taking the following values: $\beta = 1,\ \gamma = 1/2$, $\alpha = 1/2$ and $\delta = 1/8$. (In this case, $\alpha_1 = 4,\ \delta_1 = 0.1406$).

By (2.16), the Jacobian matrix at (x_4, y_4) is

$$A(u_4, v_4) = \begin{pmatrix} 1 - 2u_4 - \dfrac{2\alpha u_4 v_4^3}{(u_4^2 + v_4^2)^2} & -\dfrac{\alpha u_4^2 (u_4^2 - v_4^2)}{(u_4^2 + v_4^2)^2} \\ \dfrac{2\beta u_4 v_4^3}{(u_4^2 + v_4^2)^2} & -\gamma + \dfrac{\beta u_4^2 (u_4^2 - v_4^2)}{(u_4^2 + v_4^2)^2} \end{pmatrix}. \tag{2.24}$$

Notations: Let $\beta > \gamma > 0$. We define the following notations.

$$\alpha_2 = \frac{2\beta\gamma\rho_0}{2\gamma - \beta},\ \alpha_3 = \frac{\beta^2 + 2\beta\gamma^2\rho_0^2}{2\gamma^2\rho_0},\ \alpha_4 = \frac{\beta^2 - 2\beta\gamma^2\rho_0^2}{2(\beta - \gamma)\gamma\rho_0},\ \beta_1 = \frac{2\gamma(1+\gamma)}{1+2\gamma},$$
$$\delta_2 = \frac{1}{4\beta^4}[\beta^2\big(\beta - \alpha\gamma\rho_0\big)^2 - \big(\alpha\gamma\rho_0(2\gamma - \beta) - 2\beta\gamma\rho_0^2\big)^2]. \tag{2.25}$$

By (2.8), we have $\delta_2 < \delta_1$ since

$$\delta_2 = \delta_1 - \frac{\big(\alpha\gamma\rho_0(2\gamma - \beta) - 2\beta\gamma\rho_0^2\big)^2}{4\beta^4}.$$

Lemma 2.6. *If $0 < \gamma < \beta$, then the following assertions hold.*

(1) *$\alpha_2 > 0$ if and only if $\gamma < \beta < 2\gamma$.*

(2) *$\gamma < \beta_1 < 2\gamma$.*

(3) *If $\gamma < \beta < \beta_1$, then $\alpha_2 < \alpha_3 < \alpha_1 < \alpha_4$.*

(4) *If $\gamma < \beta < \beta_1$, then $\delta_2 > 0$ if and only if $0 < \alpha < \alpha_3$ or $\alpha_4 < \alpha < \infty$.*

(5) *For $\beta < 2\gamma$, $\alpha_1 < \alpha_2$ if and only if $\beta > \beta_1$;*

Proof. (1) is obvious.

(2) By (2.25), we have

$$\beta_1 - \gamma = \frac{\gamma}{1+2\gamma} > 0 \text{ and } \beta_1 - 2\gamma = -\frac{2\gamma^2}{1+2\gamma} < 0,$$

which implies (2).

(3) Let $w = \gamma(\beta - \gamma)$. By (2.25), we have

$$\begin{aligned}
\alpha_2 - \alpha_3 &= \frac{2\beta\sqrt{w}}{2\gamma-\beta} - \frac{\beta^2+2\beta w}{2\gamma\sqrt{w}} = \frac{\beta[4\gamma w - (2\gamma-\beta)(\beta+2w)]}{2\gamma(2\gamma-\beta)\sqrt{w}} \\
&= \frac{\beta[4\beta\gamma^2 - 4\gamma^3 - 2\beta\gamma + \beta^2 - 4\beta\gamma^2 + 4\gamma^3 + 2\beta^2\gamma - 2\beta\gamma^2]}{2\gamma(2\gamma-\beta)\sqrt{w}} \\
&= \frac{\beta^2[\beta(1+2\gamma) - 2\gamma(1+\gamma)]}{2\gamma(2\gamma-\beta)\sqrt{w}} = \frac{\beta^2(1+2\gamma)}{2\gamma(2\gamma-\beta)\sqrt{w}}(\beta-\beta_1). \\
\alpha_3 - \alpha_1 &= \frac{\beta^2+2\beta w}{2\gamma\sqrt{w}} - \frac{\beta}{\sqrt{w}} = \frac{\beta[\beta+2w-2\gamma]}{2\gamma\sqrt{w}} \\
&= \frac{\beta[\beta(1+2\gamma) - 2\gamma(1+\gamma)]}{2\gamma\sqrt{w}} = \frac{\beta(1+2\gamma)}{2\gamma\sqrt{w}}(\beta-\beta_1)
\end{aligned}$$

and

$$\begin{aligned}
\alpha_1 - \alpha_4 &= \frac{\beta}{\sqrt{w}} - \frac{\beta^2-2\beta w}{2(\beta-\gamma)\sqrt{w}} = \frac{\beta[2(\beta-\gamma)(1+\gamma)-\beta]}{2(\beta-\gamma)\sqrt{w}} \\
&= \frac{\beta[\beta(1+2\gamma) - 2\gamma(1+\gamma)]}{2(\beta-\gamma)\sqrt{w}} = \frac{\beta(1+2\gamma)}{2(\beta-\gamma)\sqrt{w}}(\beta-\beta_1).
\end{aligned}$$

The result follows from the above three equations.

(4) By (2.25), we have

$$\begin{aligned}
\delta_2 &= \frac{1}{4\beta^4}\left[\beta^2 + 2\beta w - 2\alpha\gamma\sqrt{w}\right]\left[\beta^2 - 2\beta w - 2\alpha(\beta-\gamma)\sqrt{w}\right] \\
&= \frac{1}{4\beta^4} 2\gamma\sqrt{w}(\alpha_3 - \alpha) 2(\beta-\gamma)\sqrt{w}(\alpha_4 - \alpha) = \frac{w^2}{\beta^4}(\alpha_3-\alpha)(\alpha_4-\alpha).
\end{aligned}$$

By the assertion (3), wee see that $\alpha_3 < \alpha_4$ when $\gamma < \beta < \beta_1$. Hence $\delta_2 > 0$ if and only if $0 < \alpha < \alpha_3$ or $\alpha_4 < \alpha < \infty$.

(5) For $\beta < 2\gamma$, we have

$$\begin{aligned}
\alpha_1 - \alpha_2 &= \frac{\beta}{\sqrt{w}} - \frac{2\beta\sqrt{w}}{2\gamma-\beta} = \frac{\beta[2\gamma-\beta-2\gamma(\beta-\gamma)]}{(2\gamma-\beta)\sqrt{w}} = \frac{\beta[2\gamma(1+\gamma)-\beta(1+2\gamma)]}{(2\gamma-\beta)\sqrt{w}} \\
&= \frac{\beta(1+2\gamma)[\beta_1-\beta]}{(2\gamma-\beta)\sqrt{w}},
\end{aligned}$$

which implies (5).

Theorem 2.7. (1) *Assume that one of the following holds.*

(*i*) *If* $\beta_1 \le \beta < \infty$, $0 < \alpha < \alpha_1$ *and* $0 < \delta < \delta_1$.

(*ii*) *If* $\gamma < \beta < \beta_1$, $0 < \alpha \le \alpha_2$ *and* $0 < \delta < \delta_1$.

(*iii*) *If* $\gamma < \beta < \beta_1$, $\alpha_2 < \alpha < \alpha_3$ *and* $0 < \delta < \delta_2$.

Then (x_4, y_4) *is a stable focus or node.*

(2) *If* $\gamma < \beta < \beta_1$, $\alpha_2 < \alpha < \alpha_3$ *and* $\delta = \delta_2$, *then* (x_4, y_4) *is a weak focus.*

(3) *Assume that one of the following assertions holds.*

(*i*) *If* $\gamma < \beta < \beta_1$, $\alpha_2 < \alpha \le \alpha_3$ *and* $\delta_2 < \delta < \delta_1$.

(*ii*) *If* $\gamma < \beta < \beta_1$, $\alpha_3 < \alpha < \alpha_1$ *and* $0 < \delta < \delta_1$.

Then (x_4, y_4) *is an unstable focus or node.*

Proof. By Theorem 2.2, $v_4 = \rho_0 u_4$. Hence we obtain

$$\begin{aligned}|B(u_4,v_4)| &= \frac{2(1+\rho_0^2)^2(\beta-\gamma)}{\gamma}[\alpha\gamma\rho_0 + \beta(1-2u_4)]\\ &= \frac{4\beta(1+\rho_0^2)^2(\beta-\gamma)}{\gamma}\sqrt{\delta_1(\alpha)-\delta} > 0\end{aligned}$$

and by Lemma 2.5, $|A(u_4,v_4)| = \dfrac{u_4^8}{\xi_1^4}|B(u_4,v_4)| > 0$. We prove that

$$\operatorname{tr}A(u_4,v_4) = \frac{\xi(u_4)}{\beta^2}. \tag{2.26}$$

where $\xi(u_4) = \beta^2 - 2\beta\gamma^2\rho_0^2 - 2\alpha(\beta-\gamma)\gamma\rho_0 - 2\beta^2 u_4$.

Indeed,

$$\begin{aligned}\operatorname{tr}A(u_4,v_4) &= 1-\gamma-2u_4 - \frac{2\alpha u_4 v_4^3}{(u_4^2+v_4^2)^2} + \frac{\beta u_4^2(u_4^2-v_4^2)}{(u_4^2+v_4^2)^2} = 1-\gamma-2u_4\\ &\quad - 2\alpha u_4 v_4^3\frac{\gamma^2}{\beta^2 u_4^4} + \beta u_4^2(u_4^2-v_4^2)\frac{\gamma^2}{\beta^2 u_4^4} = 1-\gamma-2u_4 - \frac{2\alpha\gamma^2 v_4^3}{\beta^2 u_4^3}\\ &\quad + \frac{\gamma^2}{\beta u_4^2}(u_4^2-v_4^2) = 1-\gamma-2u_4 - \frac{2\alpha\gamma^2}{\beta^2 u_4^3}\rho_0^3 u_4^3 + \frac{\gamma^2}{\beta u_4^2}(u_4^2-\rho_0^2 u_4^2)\\ &= 1-\gamma-2u_4 - \frac{2\alpha(\beta-\gamma)\gamma\rho_0}{\beta^2} + \frac{\gamma(2\gamma-\beta)}{\beta}\\ &= \frac{1}{\beta^2}[\beta^2 - 2\beta\gamma(\beta-\gamma) - 2\alpha(\beta-\gamma)\gamma\rho_0 - 2\beta^2 u_4] = \frac{\xi(u_4)}{\beta^2}.\end{aligned}$$

Hence, $\mathrm{tr}A(x_4,y_4)$ and $\xi(u_4)$ have the same signs. Let

$$\Delta = (\beta-\alpha\gamma\rho_0)^2 - 4\beta^2\delta.$$

Then

$$u_4 = \frac{\gamma\rho_0}{2\beta}(\alpha_1-\alpha) + \sqrt{\delta_1-\delta} = \frac{\beta-\alpha\gamma\rho_0+\sqrt{\Delta}}{2\beta}$$

and

$$\xi(u_4) = \alpha(2\gamma-\beta)\sqrt{\gamma(\beta-\gamma)} - 2\beta\gamma(\beta-\gamma) - \beta\sqrt{\Delta} := T - \beta\sqrt{\Delta}.$$

(1) (i_1) If $\beta \geq 2\gamma$, then $T<0$ and $\xi(u_4)<0$. Hence, $\mathrm{tr}A(u_4,v_4)<0$ and (u_4,v_4) is a stable node or focus.

(i_2) If $\beta_1<\beta<2\gamma$, by Lemma 2.6 (1) and (3), $0<\alpha_1<\alpha_2$. This, together with $0<\alpha<\alpha_1$, implies $\alpha-\alpha_2<0$ and

$$\xi(u_4) = (2\gamma-\beta)\sqrt{\gamma(\beta-\gamma)}(\alpha-\alpha_2) - \beta\sqrt{\Delta} = T - \beta\sqrt{\Delta} < 0. \tag{2.27}$$

Hence, (u_4,v_4) is a stable node or focus. Combining (i_1) and (i_2), the result (i) of (1) follows.

(ii) If $\gamma<\beta<\beta_1$ and $0<\alpha<\alpha_2$, then by Lemma 2.6 (1) and (3), $0<\alpha_2<\alpha_1$. It follows that $\alpha<\alpha_2<\alpha_1$, $\xi(u_4)<0$ and $\mathrm{tr}A(u_4,v_4)<0$. Hence, (u_4,v_4) is a stable node or focus.

(iii) When $\alpha_2<\alpha<\alpha_1$, by (2.27), $T>0$. Hence $\xi(u_4)<0$ is equivalent to

$$0<\delta<\delta_2. \tag{2.28}$$

In fact, $\xi(u_4)<0$ if and only if

$$T = \alpha(2\gamma-\beta)\sqrt{\gamma(\beta-\gamma)} - 2\beta\gamma(\beta-\gamma) < \beta\sqrt{\Delta}$$

if and only if

$$(\alpha(2\gamma-\beta)\sqrt{\gamma(\beta-\gamma)} - 2\beta\gamma(\beta-\gamma))^2 < \beta^2[(\beta-\alpha\sqrt{\gamma(\beta-\gamma)})^2 - 4\beta^2\delta]$$

if and only if (2.28) holds. If $\gamma<\beta<\beta_1$ and $\alpha_2<\alpha<\alpha_3$, then by Lemma 2.6 (4), $\delta_2>0$. If $0<\delta<\delta_2$, by (2.28), $\xi(u_4)<0$ and $\mathrm{tr}A(u_4,v_4)<0$. (iii) follows.

(2) If $\gamma<\beta<\beta_1$ and $\alpha_2<\alpha<\alpha_3$, then by (iii) of (1), $\delta_2>0$. By (2.28), we see $\xi(u_4)=0$ if $\delta=\delta_2$. Hence $\mathrm{tr}A(u_4,v_4)=0$. Then (u_4,v_4) is a weak focus.

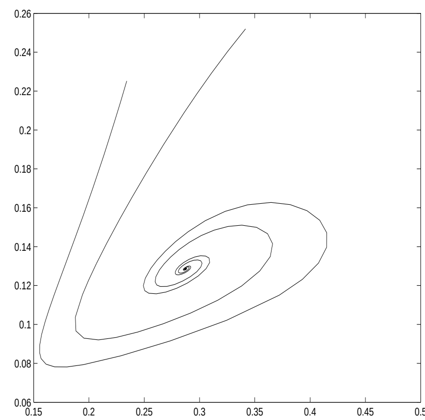

FIGURE 4. is the phase portraits of (u_4, v_4) by taking the following values: $\beta = 6/5$, $\gamma = 1$, $\alpha = 1.8$ and $\delta = 0.012$. (In this case, $\alpha_2 = 1.3416, \alpha_3 = 2.1467, \beta_1 = 4/3, \delta_1 = 0.1225, \delta_2 = 0.1193$).

(3) (i) From (iii) of the assertion (1), we see $\delta_2 > 0$. By (2.28), $\xi(u_4) > 0$ if $\delta_2 < \delta < \delta_1$. Hence $\mathrm{tr}A(u_4, v_4) > 0$. It follows that (u_4, v_4) is an unstable node or focus.

(ii) If $\gamma < \beta < \beta_1$ and $\alpha_3 < \alpha < \alpha_2$, then by Lemma 2.6 (4), we see that $\delta_2 < 0$ and (2.28) implies $\xi(u_4) > 0$. Hence, (u_4, v_4) is an unstable node or focus.

Remark 2.2. By Theorem 2.7 (1), the predator and prey can coexist. When $\gamma < \beta < \beta_1$ and $\alpha_2 < \alpha < \alpha_3$, the stability of (u_4, v_4) depends heavily on the harvesting rate δ. For a harvesting rate $\delta < \delta_2$, the predator and prey can coexist since (x_4, y_4) is stable. For a harvesting rate, $\delta_2 < \delta < \delta_1$, the equilibrium (x_4, y_4) becomes unstable which implies that over-harvesting occurs.

Finally, we analyze the stability of (u^*, v^*). The Jacobian matrix at (u^*, v^*) is

$$A(u^*, v^*) = \begin{pmatrix} 1 - 2u^* - \dfrac{2\alpha u^* v^{*3}}{(u^{*2} + v^{*2})^2} & -\dfrac{\alpha u^{*2}(u^{*2} - v^{*2})}{(u^{*2} + v^{*2})^2} \\ \dfrac{2\beta u^* v^{*3}}{(u^{*2} + v^{*2})^2} & -\gamma + \dfrac{\beta u^{*2}(u^{*2} - v^{*2})}{(u^{*2} + v^{*2})^2} \end{pmatrix}. \tag{2.29}$$

Theorem 2.8. *Assume that $\alpha, \beta, \gamma, \delta > 0$. Then the following assertions hold.*

(1) *If $\beta_1 < \beta < 2\gamma$, $\alpha < \alpha_1$ and $\delta = \delta_1$ or $\gamma < \beta < \beta_1$, $\alpha < \alpha_1$, $\alpha \neq \alpha_2$ and $\delta = \delta_1$, then (u^*, v^*) is a saddle-node of* (2.5).

(2) *If $\gamma < \beta < \beta_1$, $\alpha = \alpha_2$ and $\delta = \delta_1$, then (u^*, v^*) is a cusp of* (2.5).

Proof. By (2.29) and Theorem 2.6,

$$
\begin{aligned}
|A(u^*,v^*)| &= -\frac{2\gamma\rho_0^2}{\beta}+\frac{2\alpha\gamma^2\rho_0^3}{\beta^2}+\frac{4\gamma\rho_0^2}{\beta}u^* = -\frac{2\gamma\rho_0^2}{\beta}+\frac{2\alpha\gamma^2\rho_0^3}{\beta^2}\\
&+\frac{2\gamma^2\rho_0^3}{\beta^2}(\alpha_1-\alpha) = -\frac{2\gamma\rho_0^2}{\beta}+\frac{2\gamma^2\rho_0^3}{\beta^2}\alpha_1 = -\frac{2\gamma\rho_0^2}{\beta}+\frac{2\gamma^2\rho_0^3}{\beta^2}\alpha_1 = 0.
\end{aligned}
$$

Note that $\Delta = 0$ when $\delta = \delta_1$. Hence,

$$
\xi(u^*) = \alpha(2\gamma-\beta)\sqrt{\gamma(\beta-\gamma)}-2\beta\gamma(\beta-\gamma) = \sqrt{\gamma(\beta-\gamma)}(\alpha-\alpha_2)
$$

and $T := \mathrm{tr}A(x^*,y^*) = \frac{\xi(u^*)}{\beta^2}$.

(1) By Lemma 2.6 (5), we have $\alpha_1 < \alpha_2$. Hence $T \neq 0$ when $\alpha < \alpha_1$ or $\alpha > \alpha_1$ with $\alpha \neq \alpha_2$. Let $x = u-u^*$ and $y = v-v^*$. Then (2.5) becomes

$$
\begin{cases}
\dot{x} = (u^*+x)(1-u^*-x)-\dfrac{\alpha(u^*+x)^2(v^*+y)}{(u^*+x)^2+(v^*+y)^2}-\delta_1,\\
\dot{y} = (v^*+y)\big(-\gamma+\dfrac{\beta(u^*+x)^2}{(u^*+x)^2+(v^*+y)^2}\big).
\end{cases}
$$

By Lemma 2.4, the above system can be written as

$$
\begin{cases}
\dot{x} = ax+by+a_{20}x^2+a_{11}xy+\alpha\gamma\eta y^2+O_2^{(1)}(x,y),\\
\dot{y} = cx+dy+b_{20}x^2+b_{11}xy-\beta\gamma\eta y^2+O_2^{(2)}(x,y),
\end{cases} \tag{2.30}
$$

where $a = \frac{\alpha\rho_2\gamma\rho_0}{\beta^2}$, $b = -\frac{\alpha\gamma\rho_2}{\beta^2}$, $c = \frac{2w\rho_0}{\beta}$, $d = -\frac{2w}{\beta}$, $\eta = \frac{\rho_3\gamma\rho_0}{\beta^3 u^*}$, $a_{20} = -(1-\alpha(\beta-\gamma)\eta)$, $a_{11} = -2\alpha\gamma\rho_0\eta$, $b_{20} = -\beta(\beta-\gamma)\eta$, $b_{11} = 2\beta\gamma\rho_0\eta$, $O_2^{(1)}(x,y) = a_{30}x^3+a_{21}x^2y$ $+a_{12}xy^2+a_{03}y^3+O_4^{(1)}(x,y)$, and

$$
O_2^{(2)}(x,y) = b_{30}x^3+b_{21}x^2y+b_{12}xy^2+b_{03}y^3+O_4^{(2)}(x,y).
$$

Note that

$$
|A(u^*,v^*)| = ad-bc = -\frac{\alpha\rho_2\gamma\rho_0}{\beta^2}\frac{2w}{\beta}+\frac{\alpha\gamma\rho_2}{\beta^2}\frac{2w\rho_0}{\beta} = 0
$$

and

$$
T = \mathrm{tr}A(u^*,v^*) = a+d = \frac{\alpha\rho_2\gamma\rho_0}{\beta^2}-\frac{2w}{\beta} = \frac{\gamma\rho_0(2\gamma-\beta)}{\beta^2}(\alpha-\alpha_2) \neq 0. \tag{2.31}
$$

Let $x_1 = dx - by$ and $y_1 = y$. Then $x = \frac{1}{d}(x_1 + by_1)$, $y = y_1$ and

$$\begin{aligned}\dot{x}_1 &= d\dot{x} - b\dot{y} = d(ax + by + a_{20}x^2 + a_{11}xy + \alpha\gamma\eta y^2 + O_2^{(1)}(x,y)) - b(cx + dy \\ &+ b_{20}x^2 + b_{11}xy - \beta\gamma\eta y^2 + O_2^{(2)}(x,y)) = \tilde{\alpha}_1 x^2 + \tilde{\alpha}_2 xy + \gamma\eta(\alpha d + \beta b)y^2 \\ &+ dO_2^{(1)}(x,y) - bO_2^{(2)}(x,y) = \frac{\tilde{\alpha}_1}{d^2}(x_1 + by_1)^2 + \frac{\tilde{\alpha}_2}{d}(x_1 + by_1)y_1 \\ &+ \gamma\eta(\alpha d + \beta b)y_1^2 + P_2^{(1)}(x_1,y_1) = c_1x_1^2 + c_2x_1y_1 + c_3y_1^2 + P_2^{(1)}(x_1,y_1),\end{aligned}$$

where $\tilde{\alpha}_1 = a_{20}d - bb_{20}, \tilde{\alpha}_2 = a_{11}d - bb_{11}, c_1 = \frac{\tilde{\alpha}_1}{d^2}, c_2 = \frac{2b}{d^2}\tilde{\alpha}_1 + \frac{\tilde{\alpha}_2}{d}$,

$$c_3 = \frac{b^2}{d^2}\tilde{\alpha}_1 + \frac{b}{d}\tilde{\alpha}_2 + \gamma\eta(\alpha d + \beta b),$$

$$P_2^{(1)}(x_1,y_1) = dO_2^{(1)}(\frac{x_1 + by_1}{d}, y_1) - bO_2^{(2)}(\frac{x_1 + by_1}{d}, y_1).$$

$$\begin{aligned}\dot{y}_1 &= \dot{y} = cx + dy + b_{20}x^2 + b_{11}xy - \beta\gamma\eta y^2 + O_2^{(2)}(x,y) \\ &= \frac{c}{d}x_1 + Ty_1 + d_1x_1^2 + d_2x_1y_1 + d_3y_1^2 + P_2^{(2)}(x_1,y_1),\end{aligned}$$

where $d_1 = \frac{b_{20}}{d^2},\ d_2 = \frac{2bb_{20}}{d^2} + \frac{b_{11}}{d},\ d_3 = \frac{bb_{20}}{d^2} + \frac{bb_{11}}{d} - \beta\gamma\eta$ and $P_2^{(2)}(x_1,y_1) = O_2^{(2)}(\frac{x_1+by_1}{d}, y_1)$. Hence, (2.30) becomes

$$\begin{cases}\dot{x}_1 = c_1x_1^2 + c_2x_1y_1 + c_3y_1^2 + P_2^{(1)}(x_1,y_1), \\ \dot{y}_1 = \frac{c}{d}x_1 + Ty_1 + d_1x_1^2 + d_2x_1y_1 + d_3y_1^2 + P_2^{(2)}(x_1,y_1).\end{cases} \tag{2.32}$$

Let $x_2 = x_1$ and $y_2 = \frac{c}{d}x_1 + Ty_1$. Then $x_1 = x_2$, $y_1 = \frac{1}{T}(-\frac{c}{d}x_2 + y_2)$ and

$$\begin{aligned}\dot{x}_2 &= \dot{x}_1 = c_1x_1^2 + c_2x_1y_1 + c_3y_1^2 + P_2^{(1)}(x_1,y_1) \\ &= c_1x_2^2 + \frac{c_2}{T}(-\frac{c}{d}x_2^2 + x_2y_2) + \frac{c_3}{T^2}(-\frac{c}{d}x_2 + y_2)^2 + R_2^{(1)}(x_2,y_2) \\ &= a_{20}x_2^2 + a_{11}x_2y_2 + \frac{c_3}{T^2}y_2^2 + R_2^{(1)}(x_2,y_2),\end{aligned}$$

where $a_{20} = c_1 - \frac{c_1 c}{Td} + \frac{c_3 c^2}{T^2 d^2}$, $a_{11} = \frac{c_1}{T} - \frac{2cc_3}{T^2 d}$, $R_2^{(1)}(x_2, y_2) = O_2^{(1)}(x_2, \frac{-c/dx_2+y_2}{T})$.

$$\begin{aligned}
\dot{y}_2 &= \frac{c}{d}\dot{x}_1 + T\dot{y}_1 = \frac{c}{d}(c_1 x_1^2 + c_2 x_1 y_1 + c_3 y_1^2 + P_2^{(1)}(x_1, y_1)) + T(\frac{c}{d}x_1 + Ty_1 \\
&\quad + d_1 x_1^2 + d_2 x_1 y_1 + d_3 y_1^2 + P_2^{(2)}(x_1, y_1)) = (\frac{cc_1}{d} + Td_1)x_1^2 + (\frac{cc_2}{d} + Td_2)x_1 y_1 \\
&\quad + Ty_2 + (\frac{cc_3}{d} + Td_3)y_1^2 + R_2^{(2)}(x_2, y_2) = Ty_2 + \frac{1}{T}(\frac{cc_2}{d} + Td_2)(-\frac{c}{d}x_2^2 + x_2 y_2) \\
&\quad + (\frac{cc_1}{d} + Td_1)x_2^2 + \frac{1}{T^2}(\frac{cc_3}{d} + Td_3)(-\frac{c}{d}x_2 + y_2)^2 + R_2^{(2)}(x_2, y_2) \\
&\quad = Ty_2 + b_{20}x_2^2 + b_{11}x_2 y_2 + \frac{1}{T^2}(\frac{cc_3}{d} + Td_3)y_2^2 + R_2^{(2)}(x_2, y_2),
\end{aligned}$$

where $b_{20} = Td_1 + \frac{cc_1}{d} - \frac{c}{Td}(Td_2 + \frac{cc_2}{d}) + \frac{c^2}{T^2 d^2}(Td_3 + \frac{cc_3}{d})$, $b_{11} = d_2 + \frac{cc_2}{Td} - \frac{2c}{T^2 d}(Td_3 + \frac{cc_3}{d})$, $R_2^{(2)}(x_2, y_2) = \frac{c}{d}P_2^{(1)}(x_2, (-\frac{c}{d}x_2 + y_2)/T) + TP_2^{(2)}(x_2, (-\frac{c}{d}x_2 + y_2)/T)$.

Hence, (2.32) becomes

$$\begin{cases}
\dot{x}_2 = a_{20}x_2^2 + a_{11}x_2 y_2 + \frac{c_3}{T^2}y_2^2 + R_2^{(1)}(x_2, y_2), \\
\dot{y}_2 = Ty_2 + b_{20}x_2^2 + b_{11}x_2 y_2 + \frac{1}{T^2}(\frac{cc_3}{d} + Td_3)y_2^2 + R_2^{(2)}(x_2, y_2).
\end{cases}$$

This implies that (u^*, v^*) is a saddle-node.

(2) If $\alpha = \alpha_2$, then by (2.31), $|A(u^*, v^*)| = \mathrm{tr}A(u^*, v^*) = 0$. Hence, (2.32) becomes

$$\begin{cases}
\dot{x}_1 = c_1 x_1^2 + c_2 x_1 y_1 + c_3 y_1^2 + P_2^{(1)}(x_1, y_1) := Q_1, \\
\dot{y}_1 = \frac{c}{d}x_1 + d_1 x_1^2 + d_2 x_1 y_1 + d_3 y_1^2 + P_2^{(2)}(x_1, y_1) := Q_2.
\end{cases}$$

Let

$$x_2 = x_1 - \frac{dc_1}{c}x_1 y_1 + \frac{dd_3}{c}y_1^2 \quad \text{and } y_2 = y_1 - \frac{dd_1}{c}x_1 y_1 - \frac{d}{2c}(c_1 + d_2)y_1^2. \tag{2.33}$$

Then we have

$$\begin{aligned}
\dot{x}_2 &= (1 - \frac{dc_1}{c}y_1)Q_1 - (\frac{dc_1}{c}x_1 - \frac{2dd_3}{c}y_1)Q_2 = (c_2 + 2d_3)[x_2 + \frac{dc_1}{c}x_1 y_1 + \frac{dd_3}{c}y_1^2] \\
&\quad [y_2 + \frac{dd_1}{c}x_1 y_1 + \frac{d}{2c}(c_1 + d_2)y_1^2] + P_2^{(1)}(x_1, y_1) + c_3[\frac{dd_1}{c}x_1 y_1]^2 + \frac{d}{2c}(c_1 + d_2)y_1^2 \\
&\quad + y_2 - \frac{dc_1}{c}y_1 Q_1 - (\frac{dc_1}{c}x_1 - \frac{2dd_3}{c})(Q_2 - \frac{c}{d}x_1) = \tilde{c}x_2 y_2 + c_3 y_2^2 + O_2^{(1)}(x_2, y_2),
\end{aligned}$$

where $\tilde{c} = c_2 + 2d_3$, $(x_1, y_1) = (x_1(x_2, y_2), y_1(x_2, y_2))$ is obtained from (2.33),

$$O_2^{(1)}(x_2, y_2) = \tilde{O}_2^{(1)}(x_1(x_2, y_2), y_1(x_2, y_2), x_2, y_2)$$

and

$$\begin{aligned}O_2^{(1)}(x_1,y_1,x_2,y_2) = \tilde{c}\{[x_2+\frac{dc_1}{c}x_1y_1+\frac{dd_3}{c}y_1^2][y_2+\frac{dd_1}{c}x_1y_1+\frac{d}{2c}\tilde{c}_1y_1^2]-x_2y_2\}\\ +P_2^{(1)}(x_1,y_1)+c_3[y_2+\frac{dd_1}{c}x_1y_1+\frac{d}{2c}\tilde{c}_1y_1^2]^2-c_3y_2^2-\frac{dc_1}{c}y_1Q_1\\ -(\frac{dc_1}{c}x_1-\frac{2dd_3}{c})(Q_2-\frac{c}{d}x_1).\end{aligned}$$

$$\begin{aligned}\dot{y}_2 &= -\frac{dd_1}{c}y_1Q_1+(1-\frac{dd_1}{c}x_1-\frac{d\tilde{c}_1}{c}y_1)Q_2=\frac{c}{d}x_1-c_1x_1y_1+d_3y_1^2+R(x_1,y_1)\\ &= \frac{c}{d}[x_2+\frac{dc_1}{c}x_1y_1-\frac{dd_3}{c}y_1^2]-c_1x_1y_1+d_3y_1^2+R(x_1,y_1)\\ &= \frac{c}{d}x_2+O_2^{(2)}(x_2,y_2),\end{aligned}$$

where $\tilde{c}_1=c_1+d_2,\ O_2^{(2)}(x_2,y_2)=R(x_1(x_2,y_2),y_1(x_2,y_2))$ and

$$R(x_1,y_1)=P_2^{(2)}(x_1,y_1)-\frac{d}{c}(d_1x_1+(c_1+d_2)y_1)(Q_2-\frac{c}{d}x_1)-\frac{dd_1}{c}y_1Q_1.$$

Let $U=O_2^{(2)}(x_2,y_2)$ and

(2.34) $$x_3=\frac{c}{d}x_2+U,\ y_3=y_2.$$

Then

$$\begin{aligned}\dot{x}_3 &= \dot{x}_2+\dot{U}=\tilde{c}x_2y_2+c_3y_2^2+O_2^{(1)}(x_2,y_2)+\dot{U}\\ &= \tilde{c}\frac{d}{c}(x_3-U)y_3+c_3y_3^2+O_2^{(1)}(x_2,y_2)+\dot{U}=hx_3y_3+c_3y_3^2+P_2^{(1)}(x_3,y_3),\end{aligned}$$

where $h=d\tilde{c}/c,\ P_2^{(1)}(x_3,y_3)=\tilde{P}_2^{(1)}(x_1,y_1,x_2,y_2)$ and

$$\tilde{P}_2^{(1)}(x_1,y_1,x_2,y_2)=\dot{U}-hUy_3+O_2^{(1)}(x_2,y_2).$$

$$\dot{y}_3=\dot{y}_2=\frac{c}{d}x_2+O_2^{(2)}(x_2,y_2)=x_3.$$

Hence, we have

$$\begin{cases}\dot{x}_3=hx_3y_3+c_3y_3^2+P_2^{(1)}(x_3,y_3),\\ \dot{y}_3=x_3.\end{cases}$$

By Lemma 2.3 (2), (u^*,v^*) is a cusp of (2.5).

3. The Hopf bifurcations

By Theorem 2.7 (2), (x_4, y_4) is a weak focus. We compute the first Lyapunov number to investigate the stability of (x_4, y_4) and the directions of Hopf bifurcations.

To state the result, we consider the following system:

$$\begin{cases} \dot{u} = au + bv + p(u,v) := f_1(u,v), \\ \dot{v} = cu + dv + q(u,v) := g_1(u,v), \end{cases} \tag{3.35}$$

where $p(u,v) = \sum_{i+j=2}^{\infty} a_{ij} u^i v^j$ and $q(u,v) = \sum_{i+j=2}^{\infty} b_{ij} u^i v^j$. The Jacobian matrix of f_1 and g_1 at the equilibrium $(0,0)$ of (3.35) is

$$A(0,0) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.36}$$

If $D = |A(0,0)| = ad - bc > 0$, then by the formula (3′) of Section 4.4 in [23] (see also [16, 28]), the Lyapunov number, denoted by σ, of (3.35) is given by

$$\sigma = \frac{-3\pi}{2bD^{3/2}} \left(\sum_{i=1}^{8} \xi_i \right), \tag{3.37}$$

where

$$\xi_1 = ac(a_{11}^2 + a_{11}b_{02} + a_{02}b_{11}), \;\; \xi_2 = ab(b_{11}^2 + a_{20}b_{11} + a_{11}b_{02}),$$

$$\xi_3 = c^2(a_{11}a_{02} + 2a_{02}b_{02}), \;\; \xi_4 = -2ac(b_{02}^2 - a_{20}a_{02}), \;\; \xi_5 = -2ab(a_{20}^2 - b_{20}b_{02}),$$

$$\xi_6 = -b^2(2a_{20}b_{20} + b_{11}b_{20}), \;\; \xi_7 = (bc - 2a^2)(b_{11}b_{02} - a_{11}a_{20}),$$

$$\xi_8 = -(a^2 + bc)[3(cb_{03} - ba_{30}) + 2a(a_{21} + b_{12}) + (ca_{12} - bb_{21})].$$

A special case of [23, Theorem 1 and Remark 1 of section 4.4] is the following result.

Lemma 3.7. *Assume that $D > 0$ and $a + d = 0$. Then the following assertions hold.*

(1) *If $\sigma < 0$ (or $\sigma > 0$), then the equilibrium $(0,0)$ is a stable (or unstable) center or a stable (or unstable) focus with multiplicity one.*

(2) *If $\sigma < 0$ (or $\sigma > 0$), then a supercritical (or subcritical) Hopf bifurcation occurs at $(0,0)$ of (3.35) at the bifurcation value $\mu = a + d = 0$.*

(3) *If* $\sigma < 0$ *(or* $\sigma > 0$*), then a unique stable (or unstable) limit cycle bifurcates from* $(0,0)$ *of* (3.35) *as the bifurcation value* $\mu = a+d$ *increases from zero.*

For $\gamma < \beta < \beta_1$ and $\alpha_2 < \alpha < \alpha_3$, let $\Gamma = \Gamma(\alpha,\beta,\gamma)$. Then

$$\begin{aligned}\Gamma := {}& 16\alpha\beta\gamma\rho_1^3\rho_3^2[\gamma+\alpha\gamma\rho_0-2\beta^2\gamma\rho_0^2\rho_2(\alpha\gamma\rho_0-\beta\gamma\rho_3)]\\ &-\gamma\rho_1^2\rho_3^2[8\beta\gamma\rho_0-\alpha\rho_2(\beta^2\gamma-\alpha\rho_1)]\\ &+\beta\rho_3(\beta^2-2\rho_1(\alpha\gamma\rho_0+\beta\gamma))[4\gamma\rho_0^2\rho_2(1+\gamma+\alpha\gamma\rho_0)-\alpha\big(4\beta\gamma\rho_1^2+\alpha\gamma\rho_0\rho_2^2\big)]\\ &+2\alpha\rho_1\rho_3[2\alpha\gamma\rho_0\rho_2+8\beta\gamma\rho_0^2+\alpha^2\gamma\rho_2^2\big(\beta\gamma+\alpha\rho_1\rho_3\big)]\\ &-\alpha\beta^2\rho_2\big(\beta^2-2\rho_1(\alpha\gamma\rho_0+\beta\gamma)\big)^2-2\beta\gamma\rho_0^2(2\beta\gamma\rho_0^2-\alpha\gamma\rho_0\rho_2)[12\alpha\gamma\rho_0\rho_2^2\\ &-6\beta\gamma\rho_0\lambda_1+4(2\beta\gamma\rho_0-\alpha\rho_1)\lambda_2+\alpha(5\beta-6\gamma)\lambda_3],\end{aligned}$$

where $\lambda_1 = \beta^2-8\beta\gamma+8\gamma^2$, $\lambda_2 = \beta^2-10\beta\gamma+12\gamma^2$ and $\lambda_3 = \beta^2-18\beta\gamma+24\gamma^2$.

The following example indicates that Γ can be positive or negative.

Example 3.1. (1) For $\gamma > 0$ and $\alpha_2 < \alpha < \alpha_3$, there exists $\beta_0 > 0$ such that $\gamma+\beta_0 < \beta_1$ and $\Gamma < 0$.

(2) For $\alpha = 4\sqrt{2}\gamma$ and $\beta = 3\gamma/2$, there exists $0 < \gamma_0 < 3/8$ such that $\Gamma > 0$ for $0 < \gamma < \gamma_0$.

Proof. (1) Since $\lim_{\beta\to\gamma^+}\gamma\rho_0 = 0$, $\lim_{\beta\to\gamma^+}\Gamma(\alpha,\beta,\gamma) = -\alpha\gamma^7 < 0$. By the continuity of Γ in all its variables, for any fixed α and γ, there exists $\beta_0 > 0$ such that $\Gamma < 0$ for $\gamma < \beta < \gamma+\beta_0$.

(2) Let $\alpha = 4\sqrt{2}\gamma$ and $\beta = 3\gamma/2$. Then $\gamma < \beta < \beta_1$ and $\alpha_2 < \alpha < \alpha_3$ if $0 < \gamma < 3/8$. By the formula of Γ, we have

$$\begin{aligned}\Gamma(4\sqrt{2}\gamma,\frac{3}{2}\gamma,\gamma) = {}& 75\sqrt{2}\gamma^9(1+4\gamma-\frac{9}{2}\gamma^6+\frac{135}{32}\gamma^7)-\frac{25\gamma^7}{8\sqrt{2}}(6+\sqrt{2}\gamma^2-\frac{9}{2}\gamma^3)\\ &+\frac{15}{4}\gamma^4(\frac{9}{4}\gamma^2-\frac{35}{4}\gamma^3+\frac{7}{2}\gamma^4-\frac{50+9\sqrt{2}}{2}\gamma^5+11\sqrt{2}(3+\sqrt{2})\gamma^6)\\ &+10\sqrt{2}\gamma^6(10+12\gamma^7+40\sqrt{2}\gamma^8)-\frac{729\sqrt{2}}{4}\gamma^8-\frac{891\sqrt{2}}{2}\gamma^9+\frac{1089\sqrt{2}}{2}\gamma^{10}\\ &-\frac{9(32+43\sqrt{2})}{16}\gamma^6(\frac{3}{2}\gamma^2-2\gamma^3) := \gamma^6 G(\gamma),\end{aligned}$$

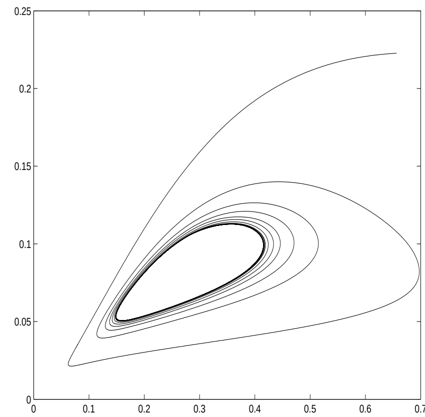

FIGURE 5. is the phase portraits by taking the following values: $\beta = 2.2,\ \gamma = 2,$ $\alpha = 2.5,\ \delta = 0.0179$. A stable limit cycle appears. (In this case, $\alpha_2 = 1.546, \alpha_2 = 2.6089,\ \beta_1 = 2.4$).

where

$$G(\gamma) = \frac{135+1600\sqrt{2}}{16} - \frac{525+150\sqrt{2}}{8}\gamma - \frac{444+7013\sqrt{2}}{32}\gamma^2 - \frac{487+2709\sqrt{2}}{8}\gamma^3 + \frac{330+18144\sqrt{2}}{32}\gamma^4 + 120\sqrt{2}\gamma^7 + 800\gamma^8 - \frac{675\sqrt{2}}{2}\gamma^9 + \frac{10125\sqrt{2}}{32}\gamma^{10}.$$

Clearly, the sign of $\Gamma(4\sqrt{2}\gamma, \frac{3}{2}\gamma, \gamma)$ is determined by the sign of $G(\gamma)$. Since

$$\lim_{\gamma\to 0^+} G(\gamma) = (135+1600\sqrt{2})/16 > 0,$$

there exists $\gamma_0 > 0$ such that

$$\Gamma|_{\alpha=4\sqrt{2}\gamma,\ \beta=3\gamma/2} > 0 \text{ for } 0 < \gamma < \gamma_0.$$

The result follows.

Theorem 3.9. *Assume that $\gamma < \beta < \beta_1$ and $\alpha_2 < \alpha < \alpha_3$ such that $\Gamma(\alpha,\beta,\gamma) \neq 0$. Then* (2.5) *has a unique stable limit cycle as δ increases from δ_2.*

Proof. Let $x = u - u_4,\ y = v - v_4$. Then (2.5) becomes

$$\begin{cases} \dot{x} = (u_4+x)(1-u_4-x) - \dfrac{\alpha(u_4+x)^2(v_4+y)}{(u_4+x)^2+(v_4+y)^2} - \delta_2 := f_4(x,y), \\ \dot{y} = (v_4+y)(-\gamma + \dfrac{\beta(u_4+x)^2}{(u_4+x)^2+(v_4+y)^2}) := g_4(x,y). \end{cases}$$

Using the Taylor's expansions to f_4 and g_4, by Lemma 2.4, we have

$$\begin{cases} \dot{x} = ax + by + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y \\ \ +a_{12}xy^2 + a_{03}y^3 + O_4^{(1)}(x,y), \\ \dot{y} = cx + dy + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y \\ \ +b_{12}xy^2 + b_{03}y^3 + O_4^{(2)}(x,y), \end{cases}$$

where $a = \frac{2\gamma\rho_0^2}{\beta}$, $b = -\frac{\alpha\gamma\rho_2}{\beta^2}$, $c = \frac{2\gamma\rho_0\rho_1}{\beta}$, $d = -\frac{2\gamma\rho_0^2}{\beta}$, $a_{20} = -1 + \frac{\alpha\gamma\rho_0\rho_1\rho_3}{\beta^3 u_4}$,

$$a_{11} = -\frac{2\alpha\gamma\rho_0^2\rho_3}{\beta^3 u_4},\ a_{02} = \frac{\alpha\gamma^2\rho_0\rho_3}{\beta^3 u_4},\ a_{30} = -\frac{4\alpha\gamma\rho_0^3\rho_2}{\beta^4 u_4^2},\ a_{21} = \frac{\alpha\gamma\rho_0^2\lambda_3}{\beta^4 u_4^2},$$
$$a_{12} = -\frac{2\alpha\gamma^2\rho_0\lambda_2}{\beta^4 u_4^2},\ a_{03} = \frac{\alpha\gamma^2\lambda_1}{\beta^4 u_4^2},\ b_{20} = -\frac{\gamma\rho_0\rho_1\rho_3}{\beta^2 u_4},\ b_{11} = \frac{2\gamma\rho_0^2\rho_3}{\beta^2 u_4},$$
$$b_{02} = -\frac{\gamma^2\rho_0\rho_3}{\beta^2 u_4},\ b_{30} = \frac{4\gamma\rho_0^3\rho_2}{\beta^3 u_4^2},\ b_{21} = -\frac{\gamma\rho_0^2\lambda_3}{\beta^3 u_4^2},\ b_{12} = \frac{2\gamma^2\rho_0\lambda_2}{\beta^3 u_4^2},\ b_{03} = -\frac{\gamma^2\lambda_1}{\beta^3 u_4^2}.$$

We calculate the first Lyapunov number σ in (3.37), where

$$\xi_1 = ac(a_{11}^2 + a_{11}b_{02} + a_{02}b_{11}) = \frac{4\gamma^2\rho_0\rho_1^2}{\beta^2}\Big[\frac{4\alpha^2\gamma\rho_0^4\rho_3^2}{\beta^6 u_4^2} + \frac{2\alpha\gamma^3\rho_0\rho_1\rho_3^2}{\beta^5 u_4^2}$$
$$+ \frac{2\alpha\gamma^3\rho_0(\rho_1\rho_3^2}{\beta^5 u_4^2}\Big] = \frac{16\alpha\gamma^4\rho_0\rho_1^3\rho_3^2}{\beta^8 u_4^2}(\gamma\rho_0 + \alpha\rho_1),$$

$$\xi_2 = ab(b_{11}^2 + a_{20}b_{11} + a_{11}b_{02}) = -\frac{2\alpha\gamma^2\rho_1\rho_2}{\beta^3}\Big[\frac{4\gamma\rho_0^4\rho_3^2}{\beta^4 u_4^2} - \frac{2\rho_1\rho_3}{\beta^2 u_4} + \frac{2\alpha\gamma^2\rho_0\rho_1^2\rho_3^2}{\beta^5 u_4^2}$$
$$+ \frac{2\alpha\gamma^3\rho_0\rho_1\rho_3^2}{\beta^5 u_4^2}\Big] = \frac{8\gamma^3\rho_1^2\rho_2\rho_3}{\beta^7 u_4^2}[\beta^2 u_4 - \gamma(\alpha + 2\gamma\rho_0)\rho_3],$$

$$\xi_3 = c^2(a_{11}a_{02}+2a_{02}b_{02}) = \frac{4\gamma\rho_1^3}{\beta^2}\big[-\frac{2\alpha^2\gamma^3\rho_0\rho_1\rho_3}{\beta^6u_4^2} - \frac{2\alpha\gamma^3\rho_1\rho_3^2}{\beta^5u_4}\big]$$
$$= -\frac{8\alpha\gamma^3\rho_1^4\rho_3}{\beta^8u_4^2}(\alpha\gamma\rho_0 - \beta\gamma\rho_3),$$
$$\xi_4 = -2ac(b_{02}^2 - a_{20}a_{02}) = -\frac{8\alpha\gamma\rho_0\rho_1^2}{\beta^2}[\frac{\gamma^3\rho_1\rho_3^2}{\beta^4u_4^2} + \frac{\alpha\gamma^2\rho_0\rho_3}{\beta^3u_4} - \frac{\alpha^2\gamma\rho_0^4\rho_3^2}{\beta^6u_4^2}]$$
$$= -\frac{8\gamma^3\rho_0\rho_1^2\rho_3}{\beta^8u_4^2}\big[\alpha\beta^3\gamma\rho_0u_4 + \gamma\rho_0^2\rho_3(\beta^2\gamma - \alpha^2\rho_1)\big],$$
$$\xi_5 = -2ab(a_{20}^2 - b_{20}b_{02}) = \frac{4\alpha\gamma^2\rho_1\rho_2}{\beta^3}\big[1 - \frac{2\alpha\gamma\rho_0\rho_1\rho_3}{\beta^3u_4} + \frac{\alpha\gamma\rho_1^3\rho_3^2}{\beta^6u_4^2} - \frac{\gamma\rho_0^4\rho_3^2}{\beta^4u_4^2}]$$
$$= \frac{4\alpha\gamma^2\rho_1\rho_2}{\beta^9u_4^2}\big[\beta^6u_4^2 - 2\alpha\beta^3\gamma\rho_0^2u_4\rho_3 - \gamma\rho_1^2\rho_3^2(\beta^2\gamma - \alpha\rho_1)\big],$$
$$\xi_6 = -b^2(2a_{20}b_{20} + b_{11}b_{20}) = -\frac{\alpha^2\gamma^2\rho_2^2}{\beta^4}\big[\frac{2\gamma\rho_0\rho_1\rho_3}{\beta^2u_4} - \frac{2\alpha\gamma\rho_1^3\rho_3^2}{\beta^5u_4^2} - \frac{2\gamma^2\rho_0\rho_1^2\rho_3}{\beta^4u_4^2}]$$
$$= -\frac{2\alpha^2\gamma^3\rho_0\rho_1\rho_2^2\rho_3}{\beta^9u_4^2}\big[\beta^3u_4 - \rho_0(\beta\gamma\rho_0 + \alpha\rho_1\rho_3)\big],$$
$$\xi_7 = (bc - 2a^2)(b_{11}b_{02} - a_{11}a_{20}) = -4\gamma^2\rho_0^4\rho_3\big(\frac{\alpha\rho_2}{\beta^3} + \frac{4\rho_0}{\beta^2}\big)\big[-\frac{\gamma^2\rho_1\rho_3}{\beta^4u_4^2} - \frac{\alpha}{\beta^3u_4}$$
$$+ \frac{\gamma^3\rho_1^2\rho_3}{\beta^6u_4^2}\big] = \frac{4\gamma^2\rho_0^4\rho_1^2\rho_3}{\beta^9u_4^2}\big(\alpha\rho_2 + 4\beta\gamma\rho_0\big)\big[\alpha\beta^3u_4 + \gamma\rho_3^2(\gamma - \alpha^2\rho_1)\big],$$
$$\xi_8 = -(a^2+bc)[3(cb_{03} - ba_{30}) + 2a(a_{21}+b_{12}) + (ca_{12} - bb_{21})] = -\big(\frac{4\gamma\rho_0^4}{\beta^2}$$
$$- \frac{2\alpha\gamma\rho_0^3\rho_2}{\beta^3}\big)\big\{3\big[-\frac{2\gamma^3\rho_0\rho_1\lambda_1}{\beta^4u_4^2} + \frac{4\alpha\gamma^3\rho_0\rho_1\rho_2^2}{\beta^5u_4^2} + \frac{4\gamma\rho_0^2}{\beta}(\frac{\alpha\gamma\rho_0^2\lambda_3}{\beta^4u_4^2} + \frac{2\gamma^2\rho_0\lambda_2}{\beta^3u_4^2})$$
$$- \frac{4\alpha\gamma\rho_0^4\lambda_2}{\beta^5u_4^2} - \frac{\alpha\gamma^2\rho_1\rho_2\lambda_3}{\beta^5u_4^2}\big]\big\} = -\frac{2\gamma^4\rho_0\rho_1^2}{\beta^8u_4^2}(2\beta\gamma\rho_0 - \alpha\rho_2)[12\alpha\gamma\rho_0\rho_2^2$$
$$- 6\beta\gamma\rho_0\lambda_1 + 4(2\beta\gamma\rho_0 - \alpha\rho_1)\lambda_2 + \alpha(5\beta - 6\gamma)\lambda_3].$$

Noting that

$$\Delta_{\delta=\delta_2} = (\beta - \alpha\gamma\rho_0)^2 - 4\beta^2\delta_2 = \frac{\gamma\rho_0^2}{\beta^2}(\alpha\rho_2 - 2\beta\gamma\rho_0)^2,$$

we have

$$u_4 = \frac{\beta - \alpha\gamma\rho_0 + \Delta_{\delta_2}}{2\beta} = \frac{1}{2\beta^3}\big[\beta^3 - \alpha\beta^2\gamma\rho_0 + \gamma\rho_0^2(\alpha\rho_2 - 2\beta\gamma\rho_0)^2\big].$$

Let $\tilde{\alpha} = \beta^2 - 2\rho_1(\alpha\gamma\rho_0 + \beta\gamma)$. By (3.37), we have

$$\sigma = \frac{-3\pi\gamma^2\rho_1}{2b\beta^9u_4^2D^{3/2}}\{16\alpha\beta\gamma\rho_1^3\rho_3^2(\gamma+\alpha\gamma\rho_0) + 4\beta^2\gamma\rho_0^2\rho_2\rho_3[\tilde{\alpha} - 8\alpha\beta\gamma\rho_1^3\rho_3(\alpha\gamma\rho_0$$
$$-\beta\gamma\rho_3)] - 4\beta\gamma\rho_0\rho_1\rho_3(\alpha\beta\gamma\rho_0\tilde{\alpha} + 2\gamma\rho_0^2\rho_3) - \alpha\rho_2[-4\alpha\beta\gamma\rho_0^2\rho_3(\gamma+\alpha\gamma\rho_0)\tilde{\alpha}$$
$$+\beta^2\tilde{\alpha}^2 - \gamma\rho_1^2\rho_3^2(\beta^2\gamma - \alpha\rho_1)] - \gamma\rho_0\alpha^2\rho_2^2\rho_3[\beta(\beta^2-\tilde{\alpha}) - 2\gamma\rho_0(\beta\gamma\rho_0 + \alpha\rho_1\rho_3)]$$
$$+4\rho_1\rho_3(\alpha\gamma\rho_0\rho_2 + 4\beta\gamma\rho_0^2) - 2\beta\gamma\rho_0^3(2\beta\gamma\rho_0 - \alpha\rho_2)[12\alpha\gamma\rho_0\rho_2^2 - 6\beta\gamma\rho_0\lambda_1$$
$$+4(2\beta\gamma\rho_0 - \alpha\rho_1)\lambda_2 + \alpha(5\beta-6\gamma)\lambda_3]\} = \frac{-3\pi\gamma^2\rho_1}{2b\beta^9u_4^2D^{3/2}}\Gamma(\alpha,\beta,\gamma).$$

Since $\gamma < \beta,\ b < 0$ and $D > 0$, the sign of σ is determined by the sign of $\Gamma(\alpha,\beta,\gamma)$. The results follow from Lemma 3.7.

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