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# BEST PROXIMITY POINT THEOREMS FOR GENERALIZED CYCLIC CONTRACTION IN METRIC SPACES

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**Abstract.** The purpose of this paper is to present existence results of the best proximity point in metric spaces for generalized cyclic contractions mappings.

**Keywords.** Generalized cyclic contraction; Best proximity point; Metric space.

## 1. Introduction

Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$. A mapping $T : A\cup B \to A\cup B$ is called a cyclic mapping if $T(A) \subset B$ and $T(B) \subset A$. A point $z \in A\cup B$ is said to be fixed point of $T$ if $Tz = z$ and a best proximity point of $T$ if $d(z,Tz) = d(A,B)$, where $d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$. All mappings do not have fixed points. For example the mapping $T : [0,\infty) \to [0,\infty)$ defined by $Tx = 1+x$, has no fixed points, since $x$ is never equal to $x+1$ for any $x \in [0,\infty)$. If the fixed-point equation $Tx = x$ does not possesses a solution, it is contemplated to resolve a problem finding an element $x$ such that $x$ is in proximity to $Tx$ in some sense. Best proximity theorems analyze the conditions under which the optimization problem, namely $min_{x\in A} d(x,Tx)$ has a solution [13].

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A mapping $T$ of $X$ into itself is called a contraction if there exists a positive real number $\alpha < 1$ with the property

$$d(Tx, Ty) \leq \alpha d(x, y) \tag{1.1}$$

for all $x, y \in X$. The well known Banach's [2] contraction mapping theorem may be stated as follows: Every contraction mapping of metric spaces X into itself has a unique fixed point. Many mathematicians worked on this principal. Banach's contraction principle has played an important role in the development of various results connected with fixed point and approximation theory. Kanan [8] proved that If $T$ is a self mapping of a metric space $X$ into itself satisfying:

$$d(Tx, Ty) \leq \eta[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$; where $\eta \in [0, 1/2]$. Then, T has a unique fixed point in $X$.
Fisher [6] proved the result with

$$d(Tx, Ty) \leq \mu[d(x, Tx) + d(y, Ty)] + \delta d(x, y)$$

for all $x, y \in X$; where $\mu, \delta \in [0, 1/2]$. Then T has unique fixed point in $X$. A similar conclusion was also obtained by Chaterjee [3] proved the result with

$$d(Tx, Ty) \leq \mu[d(x, Ty) + d(y, Tx)]$$

for all $x, y \in X$; where $\mu \in [0, 1/2]$.

The condition (1.1) entails $A \cap B$ being nonempty. Eldred and Veeramani [5] modified the condition (1.1) for the case $A \cap B = \phi$ as follows:

$$d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha) d(A, B), \tag{1.2}$$

for some $\alpha \in (0, 1)$ and for all $x \in A$ and $y \in B$ where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.

The mapping $T$ satisfying condition (1.2) is called a cyclic contraction. Eldred and Veeramani ([5], 3.10) obtained a unique best proximity point for the mapping $T$ in a uniformly convex Banach space setting. Subsequently, a number of extensions and generalizations of their results appeared in [1, 4, 14, 15] and many others. In 2011, Erdal Karapinar and Erhan [9] studied some proximity points by using different types cyclic contraction. Furthermore [7, 10, 12] examine several variants of contractions for the existence of a best proximity point. Recently, Karapinar, E. [11] have derived a best proximity point theorem for Cyclic Mappings.

## 2. Preliminaries

In this section, we first define generalized cyclic contraction mappings.

**Definition 2.1.** Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$ and $T:A\cup B\to A\cup B$ a map such that $T(A)\subseteq B$, $T(B)\subseteq A$ satisfies the following condition:

$$d(Tx,Ty)\leq \alpha d(x,y)+\beta[d(x,Tx)+d(y,Ty)]+L\min\{d(x,Ty),d(y,Tx)\}+\gamma d(A,B), \quad (2.1)$$

for all $x\in A$ and $y\in B$, where $L\geq 0$ and $\alpha,\beta,\gamma\in[0,1)$ with $\alpha+2\beta+\gamma<1$. Then a mapping $T$ said to be generalized cyclic contraction.

## 3. Main results

In this section we will consider the existence of the best proximity points, by considering some sequences which converge to that best proximity point.

**Theorem 3.1.** *Let $A$ and $B$ be two nonempty closed subsets of a metric space $(X,d)$. Suppose that the mapping $T:A\cup B\to A\cup B$ satisfy condition (2.1) with $\alpha+2\beta+\gamma<1$, then there exists a sequence $\{x_n\}$ such that* $\lim_{n\to\infty} d(x_n,x_{n+1})=d(A,B)$.

**Proof.** Suppose $x_0\in A\cup B$ be given and set a sequence $\{x_n\}$ by $x_{n+1}=Tx_n$ for all $n\in N\cup\{0\}$. From (2.1), we obtain

$$\begin{aligned}
d(x_{n+1},x_{n+2}) &= d(Tx_n,Tx_{n+1})\\
&\leq \alpha d(x_n,x_{n+1})+\beta[d(x_n,Tx_n)+d(x_{n+1},Tx_{n+1})]\\
&\quad +L\min\{d(x_n,Tx_{n+1}),d(x_{n+1},Tx_n)\}+\gamma d(A,B)\\
&\leq \alpha d(x_n,x_{n+1})+\beta[d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})]\\
&\quad +L\min\{d(x_n,x_{n+2}),d(x_{n+1},x_{n+1})\}+\gamma d(A,B)
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha d(x_n,x_{n+1})+\beta[d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})]\\
&\quad +L\min\{d(x_n,x_{n+2}),0\}+\gamma d(A,B)\\
&\leq \frac{\alpha+\beta}{1-\beta}d(x_n,x_{n+1})+\frac{\gamma}{1-\beta}d(A,B).\\
&\leq \frac{\alpha+\beta}{1-\beta}d(x_n,x_{n+1})+\frac{\gamma}{1-\beta}d(A,B),
\end{aligned}
\tag{3.1}
$$

which implies that,

$$
d(x_{n+1},x_{n+2}) \leq \lambda d(x_n,x_{n+1})+(1-\lambda)d(A,B), \tag{3.2}
$$

where $\lambda=\frac{\alpha+\beta}{1-\beta}<1$. Again, we conclude that

$$
\begin{aligned}
d(x_{n+2},x_{n+3}) &= d(Tx_{n+1},Tx_{n+2})\\
&\leq \alpha d(x_{n+1},x_{n+2})+\beta[d(x_{n+1},Tx_{n+1})+d(x_{n+2},Tx_{n+2})]\\
&\quad +L\min\{d(x_{n+1},Tx_{n+2}),d(x_{n+2},Tx_{n+1})\}+\gamma d(A,B)\\
&\leq \alpha d(x_{n+1},x_{n+2})+\beta[d(x_{n+1},x_{n+2})+d(x_{n+2},x_{n+3})]\\
&\quad +L\min\{d(x_{n+1},x_{n+3}),d(x_{n+2},x_{n+2})\}+\gamma d(A,B)\\
&\leq \alpha d(x_{n+1},x_{n+2})+\beta[d(x_{n+1},x_{n+2})+d(x_{n+2},x_{n+3})]\\
&\quad +L\min\{d(x_{n+1},x_{n+3}),0\}+\gamma d(A,B).
\end{aligned}
\tag{3.3}
$$

On the other hand, we have $\min\{d(x_{n+1},x_{n+3}),0\}=0$. So that, the inequality (3.3) implies that

$$
d(x_{n+2},x_{n+3}) \leq \lambda d(x_{n+1},x_{n+2})+(1-\lambda)d(A,B). \tag{3.4}
$$

Analogously, we concluded that from (3.2) and (3.4), we have

$$
d(x_{n+2},x_{n+3}) \leq \lambda^2 d(x_n,x_{n+1})+(1-\lambda^2)d(A,B).
$$

Hence inductively, we have

$$\begin{aligned}
d(x_n,x_{n+1}) &\leq \lambda d(x_n,x_{n-1})+(1-\lambda)d(A,B)\\
&\leq \lambda^2 d(x_{n-1},x_{n-2})+(1-\lambda^2)d(A,B)\\
&\leq \dots\\
&\leq \lambda^n d(x_0,x_1)+(1-\lambda^n)d(A,B).
\end{aligned}$$

Since $\lambda \in [0,1)$, we have $\lim_{n\to\infty}\lambda^n=0$, So the last inequality implies that

$$\lim_{n\to\infty} d(x_n,x_{n+1})=d(A,B).$$

This completes the proof.

**Theorem 3.2.** *Let $A$ and $B$ be two nonempty closed subsets of a metric space $(X,d)$. Suppose that the mapping $T: A\cup B \to A\cup B$ satisfy condition (2.1) with $\alpha+2\beta+\gamma<1$. Assume that the sequence $\{x_{2n_k}\}$ has a subsequence converging to some element $x$ in $A\cup B$. Then $x$ is a best proximity point of $T$.*

**Proof.** Suppose the sequence $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ converges to some element $x$ in $A$. Furthermore,

$$\begin{aligned}
d(A,B) &\leq d(x,x_{2n_k-1})\\
&\leq d(x,x_{2n_k})+d(x_{2n_k},x_{2n_k-1})\\
&\leq d(x,x_{2n_k})+d(A,B).
\end{aligned}$$

Therefore $d(x,x_{2n_k-1})\to d(A,B)$. Since $T$ is a generalized cyclic contraction, it follows that

$$\begin{aligned}
d(A,B) &\leq d(x_{2n_k},Tx)=d(Tx_{2n_k-1},Tx)\\
&\leq \alpha d(x_{2n_k-1},x)+\beta[d(x_{2n_k-1},Tx_{2n_k-1})+d(x,Tx)]\\
&\quad +L\min\{d(x_{2n_k-1},Tx),d(x,Tx_{2n_k-1})\}+\gamma d(A,B)\\
&\leq \alpha d(x_{2n_k-1},x)+\beta[d(x_{2n_k-1},x_{2n_k})+d(x,Tx)] \qquad (3.5)\\
&\quad +L\min\{d(x_{2n_k-1},Tx),d(x,x_{2n_k})\}+\gamma d(A,B)\\
&\leq \alpha d(x_{2n_k-1},x)+\beta[d(x_{2n_k-1},x_{2n_k})+d(x,Tx)]\\
&\quad +L\min\{d(x_{2n_k-1},Tx),0\}+\gamma d(A,B).
\end{aligned}$$

Since $d(x, x_{2n_k-1}) \to d(A,B)$ and from Theorem 3.1 that $d(x_{2n_k}, x_{2n_k-1}) \to d(A,B)$. So the last inequality implies that

$$\begin{aligned} d(x_{2n_k}, Tx) &\leq \alpha d(A,B) + \beta d(A,B) + \beta d(x,Tx) + 0 + \gamma d(A,B) \\ &\leq (\alpha+\beta+\gamma) d(A,B) + \beta d(x,Tx). \end{aligned} \tag{3.6}$$

Taking $n \to \infty$ in the inequality (3.6) and using condition $\alpha + 2\beta + \gamma < 1$, we have $d(x,Tx) = d(A,B)$. Hence $x$ is a best proximity point of $T$. This completes the proof.

Following examples illustrating our main results.

**Example 3.1.** Consider the usual metric space $d(x,y) = |x-y|$, for all $x, y \in X$. Let $X = R$. Suppose $A = [1,2]$ and $B = [-2,-1]$, then $d(A,B) = 2$. Define a mapping $T : A \cup B \to A \cup B$ as follows:

$$T(x) = \begin{cases} \frac{-1-x}{2} & \text{if } x \in A, \\ \frac{1-x}{2} & \text{if } x \in B. \end{cases}$$

It is clear that $T(A) \subset B$, $T(B) \subset A$ and 1 and $-1$ are best proximity point of $T$, where $\alpha = \gamma = 1/3$, $\beta = 1/6$, $\delta = 1/9$. So that $T$ is a rational cyclic contraction.

In this paper, we have taken values of $A = [1,2]$, $B = [-2,-1]$ and $d(A,B) = 2$ for our comparative analysis and we have derived best proximity points for the following equation which has represented as functions: $T(x) = \frac{-1-x}{2}$ if $x \in A$ and $T(y) = \frac{1-y}{2}$ if $y \in B$. Starting with $\alpha = \gamma = 1/3$, $\beta = 1/6$ and $\delta = 1/9$ we have the following table.

| Iteration | Values of (Tx - Ty) | Values of Right hand side of condition (2.1) | D(x,Tx) | d(A,B) |
|---|---|---|---|---|
| 1 | 2.000000 | 2.000000 | 2.000000 | 2.000000 |
| 2 | 2.250000 | 2.541667 | 2.000000 | 2.000000 |
| 3 | 2.500000 | 3.083333 | 2.000000 | 2.000000 |
| 4 | 2.250000 | 2.541667 | 2.750000 | 2.000000 |
| 5 | 2.500000 | 2.833333 | 2.750000 | 2.000000 |
| 6 | 2.750000 | 2.875000 | 2.750000 | 2.000000 |
| 7 | 2.500000 | 3.083333 | 3.500000 | 2.000000 |
| 8 | 2.750000 | 2.875000 | 3.500000 | 2.000000 |
| 9 | 3.000000 | 3.666667 | 3.500000 | 2.000000 |

Here we observe that the values of (Tx - Ty) are less than or equal to the values of right hand side of condition (2.1) and converges to a best proximity point before 4th iteration. This data has been tabulated by the help of MATLAB 7.0.

## 4. Applications

**Corollary 4.1.** Let $A$ and $B$ be two nonempty closed subsets of a metric space $(X,d)$. Suppose that a mapping $T : A\cup B \to A\cup B$ satisfying the following condition:

$$\int_0^{d(Tx,Ty)} \mu(t)dt \le \alpha \int_0^{d(x,y)} \mu(t)dt + \beta \int_0^{[d(x,Tx)+d(y,Ty)]} \mu(t)dt$$
$$+ L \int_0^{\min\{d(x,Ty),d(y,Tx)\}} \mu(t)dt + \gamma \int_0^{d(A,B))} \mu(t)dt,$$

for each $x \in A$ and $y \in B$ with nonnegative real numbers $\alpha, \beta, \gamma \in [0,1)$ such that with $0 < \alpha + 2\beta + \gamma < 1$ and $L \ge 0$, where $\mu : R^+ \to R^+$ is a lesbesgue- integrable mapping which is summable on each compact subset of $R^+$, non negative, and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \mu(t)dt$. Then, $\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A,B)$.

**Corollary 4.2.** Let $A$ and $B$ be two nonempty closed subsets of a metric space $(X,d)$. Suppose that a mapping $T : A \cup B \to A \cup B$ satisfying the following condition:

$$\int_0^{d(Tx,Ty)} \mu(t)dt \leq \alpha \int_0^{d(x,y)} \mu(t)dt + \beta \int_0^{[d(x,Tx)+d(y,Ty)]} \mu(t)dt$$
$$+ L \int_0^{\min\{d(x,Ty),d(y,Tx)\}} \mu(t)dt + \gamma \int_0^{d(A,B))} \mu(t)dt,$$

for each $x \in A$ and $y \in B$ with nonnegative real numbers $\alpha, \beta, \gamma \in [0,1)$ such that with $0 < \alpha + 2\beta + \gamma < 1$ and $L \geq 0$, where $\mu : R^+ \to R^+$ is a lesbesgue- integrable mapping which is summable on each compact subset of $R^+$, non negative, and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \mu(t)dt$. Suppose that a sequence $\{x_{2n}\}$ has a convergent subsequence in $A$. Then there exists $x \in A$ such that $d(x,Tx) = d(A,B)$.

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