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ON ITERATIVE SOLUTIONS OF A COMMON ELEMENT PROBLEM

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Abstract. In this article, we investigate the problem of finding a common element of the set of solutions of a variational problem and the set of common fixed points of an infinite family nonexpansive mappings based on a viscosity approximation algorithm in a Hilbert space.

Keywords. Common element; Variational inequality; Nonexpansive mapping; Fixed point; Hilbert space.

1. Introduction

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a closed convex subset of H and let $A : C \to H$ be a nonlinear map. Let P_C be the projection of H onto the convex subset C. The classical variational inequality which denoted by $VI(C,A)$ is to find $u \in C$ such that

$$\langle Au, v-u \rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

It is know that P_C satisfies

$$\langle x-y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2 \tag{1.2}$$

for $x, y \in H$. Moreover, $P_C x$ is characterized by the properties: $P_C x \in C$ and $\langle x - P_C x, P_C x - y \rangle \geq 0$ for all $y \in C$. One can see that the variational inequality (1.1) is equivalent to a fixed point

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problem. The function $u \in C$ is a solution of the variational inequality (1.1) if and only if $u \in C$ satisfies the relation $u = P_C(u - \lambda Au)$, where $\lambda > 0$ is a constant.

In the real world, many important problems have reformulations which require finding solutions of the variational inequality, for instance, evolution equations, complementarity problems, mini-max problems, and optimization problems; see [1-23] and the references therein.

Recall that the following definitions. A mapping A is said to be γ-cocoercive, if for each $x, y \in C$, we have

$$\langle Ax - Ay, x - y\rangle \geq \gamma \|Ax - Ay\|^2, \quad \text{for a constant } \mu > 0.$$

Clearly, every μ-cocoercive map A is $1/\mu$-Lipschitz continuous. A is said to be relaxed γ-cocoerceive, if there exists a constant $u > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-\gamma)\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

A is said to be relaxed (γ, r)-cocoercive, if there exist two constants $u, v > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq (-\gamma)\|Ax - Ay\|^2 + r\|x - y\|^2, \quad \forall x, y \in C.$$

A mapping $S : C \to C$ is said to be nonexpansive if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$. Next, we denote by $F(S)$ the set of fixed points of S. A mapping $f : C \to C$ is said to be a contraction if there exists a coefficient α $(0 < \alpha < 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|,$$

for $\forall x, y \in C$. A linear bounded operator B is strongly positive if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Bx, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H.$$

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every

$(y,g) \in G(T)$ implies $f \in Tx$. Let A be a monotone map of C into H and let $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v-u, w\rangle \geq 0,\ \forall u \in C\}$ and define

$$Tv = \begin{cases} Av + N_C v, & v \in C \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C,A)$; see [1].

2. Preliminaries

Recently iterative methods for nonexpansive mappings have been applied to solve convex minimization problems. A typical problem is to minimize a quadratic function over the set of the fixed points a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in C} \frac{1}{2}\langle Bx, x\rangle - \langle x, b\rangle, \tag{2.1}$$

where B is a linear bounded operator, C is the fixed point set of a nonexpansive mapping S and b is a given point in H. In [11], it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n B)Sx_n + \alpha_n b, \quad n \geq 0,$$

converges strongly to the unique solution of the minimization problem (2.1) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. More recently, Marino and Xu [12] introduced a new iterative scheme by the viscosity approximation method:

$$x_{n+1} = (I - \alpha_n B)Sx_n + \alpha_n \gamma f(x_n), \quad n \geq 0.$$

They proved the sequence $\{x_n\}$ generated by above iterative scheme converges strongly to the unique solution of the variational inequality

$$\langle (B - \gamma f)x^*, x - x^*\rangle \geq 0, \quad x \in C,$$

which is the optimality condition for the minimization problem $\min_{x\in C} \frac{1}{2}\langle Bx, x\rangle - h(x)$, where C is the fixed point set of a nonexpansive mapping S, h is a potential function for δf (i.e., $h'(x) = \delta f(x)$ for $x \in H$.)

Concerning a family of nonexpansive mappings has been considered by many authors. The well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings. The problem of finding an optimal point that minimizes a given cost function over common set of fixed points of a family of nonexpansive mappings is of wide interdisciplinary interest and practical importance. A simple algorithmic solution to the problem of minimizing a quadratic function over common set of fixed points of a family of nonexpansive mappings is of extreme value in many applications including set theoretic signal estimation.

Recently, Yao *et al.* [13] considered a general iterative algorithm for an infinite family of nonexpansive mapping in the framework of Hilbert spaces. To be more precisely, they introduced the following general iterative algorithm.

$$x_{n+1} = \lambda_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \lambda_n A)W_n x_,$$

where f is a contraction on H, A is a stronglyy positive bounded linear operator, W_n are nonexpansive mappings which are generated by an finite family of nonexpansive mapping $T_1, T_2, \ldots$. To be more precisely,

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \gamma_n T_n U_{n,n+1} + (1-\gamma_n)I,\\
&\vdots\\
&U_{n,k} = \gamma_k T_k U_{n,k+1} + (1-\gamma_k)I,\\
&u_{n,k-1} = \gamma_{k-1} T_{k-1} U_{n,k} + (1-\gamma_{k-1})I,\\
&\vdots\\
&U_{n,2} = \gamma_2 T_2 U_{u,3} + (1-\gamma_2)I,\\
&W_n = U_{n,1} = \gamma_1 T_1 U_{n,2} + (1-\gamma_1)I,
\end{aligned} \tag{2.2}$$

where $\{\gamma_1\}, \{\gamma_2\}, \ldots$ are real numbers such that $0 \le \gamma \le 1$, $T_1, T_2, \ldots$ be an infinite family of mappings of C into itself. Nonexpansivity of each T_i ensures the nonexpansivity of W_n.

Concerning W_n we have the following lemmas which are important to prove our main results.

Lemma 2.1 [14] *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\gamma_1, \gamma_2, \ldots$ be real numbers such that $0 < \gamma_n \leq \eta < 1$ for any $n \geq 1$. Then, for every $x \in C$ and $k \in N$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 2.1, one can define the mapping W of C into itself as follows.

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad x \in C.$$

Such a W is called the W-mapping generated by $T_1, T_2, \ldots$ and $\gamma_1, \gamma_2, \ldots$. Throughout this paper, we will assume that $0 < \gamma_n \leq \eta < 1$ for all $n \geq 1$.

Lemma 2.2 [14] *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\gamma_1, \gamma_2, \ldots$ be real numbers such that $0 < \gamma_n \leq \eta < 1$ for any $n \geq 1$. Then, $F(W) = \cap_{n=1}^{\infty} F(T_n)$.*

In this paper, we introduce a composite iterative process as following:

$$\begin{cases} x_1 \in C \\ y_n = P_C\big(\beta_n \gamma f(x_n) + (I - \beta_n B)W_n P_C(I - r_n A)x_n\big), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)y_n, \quad n \geq 1, \end{cases} \tag{2.3}$$

where A is a relaxed cocoercive mapping, B is a strongly positive linear bounded operator, f is a contraction on C and W_n is a mapping generated by (2.2). We prove the sequence $\{x_n\}$ generated by the above iterative scheme converges strongly to a common element of the set of common fixed points of an infinite nonexpansive mappings and the set of solutions of the variational inequalities for relaxed (γ, r)-cocoercive maps, which solves another variation inequality $\langle \gamma f(q) - Bq, q - p\rangle \leq 0, p \in \cap_{i=1}^{\infty} F(T_i) \cap VI(C,A)$ and is also the optimality condition for the minimization problem $\min_{x\in C} \frac{1}{2}\langle Bx, x\rangle - h(x)$, where C is the intersection of the common fixed points set of a nonexpansive mappings and the set of solutions of the variational inequalities for relaxed (γ, r)-cocoercive maps, h is a potential function for δf (i.e., $h'(x) = \delta f(x)$ for $x \in H$.) The results are obtained in this paper improve and extend the recent ones announced by many authors; see the literatures.

In order to prove our main results, we need the following lemmas.

Lemma 2.3 [12] *Assume B is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|B\|^{-1}$. Then $\|I - \rho B\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.4 [11] *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n,$$

where γ_n is a sequence in (0,1) and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.5. *In a real Hilbert space H, there holds the the following inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle,$$

for all $x, y \in H$.

Lemma 2.6 [15] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

3. Main results

Theorem 3.1. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H and $A : C \to H$ be relaxed (γ, r)-cocoercive and μ-Lipschitz continuous. Let $f : C \to C$ be a contraction with the coefficient α $(0 < \alpha < 1)$ and $\{T_i\}_{i=1}^{\infty}$ be an infinite nonexpansive mappings from C into itself generated by (2.2) such that $F = \bigcap_{i=1}^{\infty} F(T_i) \cap VI(C,A) \neq \emptyset$. Let B be a strongly positive linear bounded self-adjoint operator of C into itself with coefficient $\bar{\gamma} > 0$ such that*

$\|B\| \leq 1$. *Assume that* $0 < \gamma < \bar{\gamma}/\alpha$. *Assume that* $\{x_n\}$ *is generated by*

$$\begin{cases} x_1 \in C, \\ y_n = P_C\big(\beta_n \gamma f(x_n) + (I - \beta_n B) W_n P_C (I - r_n A) x_n\big), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n, \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $(0,1)$. *If* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{r_n\}$ *are chosen such that*

(i) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(iii) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;

(iv) $\{r_n\} \subset [a,b]$ *for some* a,b *with* $0 < a < b < \frac{2(r-\gamma\mu^2)}{\mu^2}$, $r > \gamma\mu^2$.

Then $\{x_n\}$ *converges strongly to* $q \in F$, *where* $q = P_F(\gamma f + (I - B))(q)$, *which solves the variation inequality* $\langle \gamma f(q) - Bq, p - q\rangle \leq 0, \forall p \in F$.

Proof. First, we show the mapping $I - r_n A$ is nonexpansive. Indeed, from the relaxed (γ, r)-cocoercive and μ-Lipschitzian definition on A and the condition (iv), we have

$$\begin{aligned} &\|(I - r_n A)x - (I - r_n A)y\|^2 \\ &= \|(x-y) - r_n(Ax - Ay)\|^2 \\ &\leq \|x-y\|^2 - 2r_n[-\gamma\|Ax - Ay\|^2 + r\|x-y\|^2] + r_n^2\|Ax - Ay\|^2 \\ &\leq \|x-y\|^2 + 2r_n\mu^2\gamma\|x-y\|^2 - 2r_n r\|x-y\|^2 + \mu^2 r_n^2\|x-y\|^2 \\ &= (1 + 2r_n\mu^2\gamma - 2r_n r + \mu^2 r_n^2)\|x-y\|^2 \\ &\leq \|x-y\|^2, \end{aligned}$$

which implies the mapping $I - r_n A$ is nonexpansive. Since the condition (i), we may assume, with no loss of generality, that $\beta_n < \|B\|^{-1}$ for all n. From Lemma 2.3, we know that if $0 < \rho \leq \|B\|^{-1}$, then $\|I - \rho B\| \leq 1 - \rho\bar{\gamma}$. Letting $p \in F$, we have

$$\begin{aligned} \|y_n - p\| &\leq \beta_n\|\gamma f(x_n) - Bp\| + (1 - \beta_n\bar{\gamma})\|W_n P_C(I - r_n A)x_n - p\| \\ &\leq \beta_n\gamma\|f(x_n) - f(p)\| + \beta_n\|\gamma f(p) - Bp\| + (1-\beta_n\bar{\gamma})\|x_n - p\| \\ &= [1 - \beta_n(\bar{\gamma} - \gamma\alpha)]\|x_n - p\| + \beta_n\|\gamma f(p) - Bp\|. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}\|x_{n+1}-p\| &\le \alpha_n\|x_n-p\|+(1-\alpha_n)\|y_n-p\|\\ &\le \alpha_n\|x_n-p\|+(1-\alpha_n)[(1-\beta_n(\bar{\gamma}-\gamma\alpha))\|x_n-p\|+\beta_n\|\gamma f(p)-Bp\|]\end{aligned}$$

By simple inductions, we have $\|x_n-p\|\le\max\{\|x_0-p\|,\frac{\|Bp-\gamma f(p)\|}{\bar{\gamma}-\gamma\alpha}\}$, which gives that the sequence $\{x_n\}$ is bounded. Set $\rho_n=P_C(I-r_nA)x_n$. Notice that

$$\begin{aligned}\|\rho_n-\rho_{n+1}\| \le&\|(I-r_nA)x_n-(I-r_{n+1}A)x_{n+1}\|\\ =&\|(x_n-r_nAx_n)-(x_{n+1}-r_nAx_{n+1})+(r_{n+1}-r_n)Ax_{n+1}\| \qquad (3.1)\\ \le&\|x_n-x_{n+1}\|+|r_{n+1}-r_n|M_1,\end{aligned}$$

where M_1 is an appropriate constant such that $M_1\ge\sup_{n\ge1}\{\|Ax_n\|\}$. It follows that

$$\begin{aligned}\|y_n-y_{n+1}\| \le (1-\beta_{n+1}\bar{\gamma})(\|\rho_{n+1}-\rho_n\|+\|W_{n+1}\rho_n-W_n\rho_n\|)\\ +|\beta_{n+1}-\beta_n|M_2+\gamma\beta_{n+1}\alpha\|x_{n+1}-x_n\|,\end{aligned} \qquad (3.2)$$

where M_2 is an appropriate constant such that $M_2\ge\max\{\sup_{n\ge1}\{\|BW_n\rho_n\|\},\gamma\sup_{n\ge1}\{\|f(x_n)\|\}\}$. Since T_i and $U_{n,i}$ are nonexpansive, we find from (2.2)

$$\begin{aligned}\|W_{n+1}\rho_n-W_n\rho_n\| &= \|\gamma_1T_1U_{n+1,2}\rho_n-\gamma_1T_1U_{n,2}\rho_n\|\\ &\le \gamma_1\|U_{n+1,2}\rho-U_{n,2}\rho_n\|\\ &= \gamma_1\|\gamma_2T_2U_{u+1,3}\rho_n-\gamma_2T_2U_{n,3}\rho_n\|\\ &\le \gamma_1\gamma_2\|U_{u+1,3}\rho_n-U_{n,3}\rho_n\| \qquad (3.3)\\ &\le\cdots\\ &\le \gamma_1\gamma_2\cdots\gamma_n\|U_{n+1,n+1}\rho_n-U_{n,n+1}\rho_n\|\\ &\le M_3\prod_{i=1}^n\gamma_i,\end{aligned}$$

where $M_3\ge0$ is an appropriate constant such that $\|U_{n+1,n+1}\rho_n-U_{n,n+1}\rho_n\|\le M_3$, for all $n\ge0$. Substitute (3.1) and (3.3) into (3.2) yields that

$$\begin{aligned}\|y_n-y_{n+1}\| \le& [1-\beta_{n+1}(\bar{\gamma}-\alpha\gamma)]\|x_{n+1}-x_n\|\\ &+M_4(|r_{n+1}-r_n|+|\beta_{n+1}-\beta_n|+\prod_{i=1}^n\gamma_i),\end{aligned}$$

where M_4 is an appropriate appropriate constant such that $M_4 \geq \max\{M_1, M_2, M_3\}$. From the conditions (i) and (iii), we have

$$\limsup_{n\to\infty}\{\|y_{n+1}-y_n\| - |x_{n+1}-x_n\|\} \leq 0. \tag{3.4}$$

By virtue of Lemma 2.6, we obtain that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.5}$$

It follows that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.6}$$

Hence, we have

$$\lim_{n\to\infty}\|W_n\rho_n - y_n\| = 0. \tag{3.7}$$

For $p \in F$, we have

$$\begin{aligned}
&\|\rho_n - p\|^2\\
&= \|P_C(I - r_nA)x_n - P_C(I - r_nA)p\|^2\\
&\leq \|(x_n - p) - r_n(Ax_n - Ap)\|^2\\
&= \|x_n - p\|^2 - 2r_n\langle x_n - p, Ax_n - Ap\rangle + r_n^2\|Ax_n - Ap\|^2\\
&\leq \|x_n - p\|^2 - 2r_n[-\gamma\|Ax_n - Ap\|^2 + r\|x_n - p\|^2] + r_n^2\|Ax_n - Ap\|^2\\
&\leq \|x_n - p\|^2 + 2r_n\gamma\|Ax_n - Ap\|^2 - 2r_nr\|x_n - p\|^2 + r_n^2\|Ax_n - Ap\|^2\\
&\leq \|x_n - p\|^2 + (2r_n\gamma + r_n^2 - \frac{2r_nr}{\mu^2})\|Ax_n - Ap\|^2.
\end{aligned} \tag{3.8}$$

Observe that

$$\begin{aligned}
\|y_n - p\|^2 &= \|\beta_n(\gamma f(x_n) - Bp) + (I - \beta_nB)(W_n\rho_n - p)\|^2\\
&\leq (\beta_n\|\gamma f(x_n) - Bp\| + (1 - \beta_n\bar{\gamma})\|\rho_n - p\|)^2\\
&\leq \beta_n\|\gamma f(x_n) - Bp\|^2 + \|\rho_n - p\|^2 + 2\beta_n\|\gamma f(x_n) - Bp\|\|\rho_n - p\|.
\end{aligned} \tag{3.9}$$

We find that

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n(x_n-p)+(1-\alpha_n)(y_n-p)\|^2 \\
&\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|y_n-p\|^2 \\
&\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)[\beta_n\|\gamma f(x_n)-Bp\|^2+\|\rho_n-p\|^2 \\
&\quad +2\beta_n\|\gamma f(x_n)-Bp\|\|\rho_n-p\|].
\end{aligned} \tag{3.10}
$$

Substituting (3.8) into (3.10), we arrive at

$$
\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\le \|x_n-p\|^2+\beta_n\|\gamma f(x_n)-Bp\|^2+(2r_n\gamma+r_n^2-\frac{2r_nr}{\mu^2})\|Ax_n-Ap\|^2 \\
&\quad +2\beta_n\|\gamma f(x_n)-Bp\|\|\rho_n-p\|.
\end{aligned} \tag{3.11}
$$

It follows from the condition (iv) that

$$
\lim_{n\to\infty}\|Ax_n-Ap\|=0. \tag{3.12}
$$

Observe that

$$
\begin{aligned}
\|\rho_n-p\|^2 =&\|P_C(I-r_nA)x_n-P_C(I-r_nA)p\|^2 \\
\le&\langle (I-r_nA)x_n-(I-r_nA)p,\rho_n-p\rangle \\
=&\frac{1}{2}\{\|(I-r_nA)x_n-(I-r_nA)p\|^2+\|\rho_n-p\|^2 \\
&-\|(I-r_nA)x_n-(I-r_nA)p-(\rho_n-p)\|^2\} \\
\le&\frac{1}{2}\{\|x_n-p\|^2+\|\rho_n-p\|^2-\|(x_n-\rho_n)-r_n(Ax_n-Ap)\|^2\} \\
=&\frac{1}{2}\{\|x_n-p\|^2+\|\rho_n-p\|^2-\|x_n-\rho_n\|^2-r_n^2\|Ax_n-Ap\|^2 \\
&+2r_n\langle x_n-\rho_n,Ax_n-Ap\rangle\},
\end{aligned}
$$

which yields that

$$
\|\rho_n-p\|^2 \le\|x_n-p\|^2-\|\rho_n-x_n\|^2+2r_n\|\rho_n-x_n\|\|Ax_n-Ap\|. \tag{3.13}
$$

Substituting (3.13) into (3.10), we have

$$\begin{aligned}
&(1-\alpha_n)\|\rho_n - x_n\|^2 \\
&\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \beta_n\|\gamma f(x_n) - Bp\|^2 + 2r_n\|\rho_n - x_n\|\|Ax_n - Ap\| \\
&\quad + 2\beta_n\|\gamma f(x_n) - Bp\|\|\rho_n - p\| \\
&\le (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + \beta_n\|\gamma f(x_n) - Bp\|^2 \\
&\quad + 2r_n\|\rho_n - x_n\|\|Ax_n - Ap\| + 2\beta_n\|\gamma f(x_n) - Bp\|\|\rho_n - p\|.
\end{aligned}$$

From the conditions (i), (ii), (3.6) and (3.12), we have

$$\lim_{n\to\infty}\|\rho_n - x_n\| = 0. \tag{3.14}$$

On the other hand, we have $\|\rho_n - W_n\rho_n\| \le \|x_n - \rho_n\| + \|x_n - y_n\| + \|y_n - W_n\rho_n\|$. It follows from (3.5), (3.7) and (3.14) that $\lim_{n\to\infty}\|W_n\rho_n - \rho_n\| = 0$. We have for any $\varepsilon > 0$, there is N such that $\|W\rho - W_n\rho\| \le \varepsilon$ for all $\rho \in \{\rho_n\}$ and for all $n \ge N$. Therefore, we have $\|W\rho_n - W_n\rho_n\| \to 0$ as $n \to \infty$. Notice that

$$\|W\rho_n - \rho_n\| \le \|W_n\rho_n - \rho_n\| + \|W_n\rho_n - W\rho_n\|,$$

from which it follows that

$$\lim_{n\to\infty}\|W\rho_n - \rho_n\| = 0. \tag{3.15}$$

Since $P_F(\gamma f + (I-B))$ is a contraction, we find that $P_F(\gamma f + (I-B))$ has a unique fixed point, say $q \in H$. That is, $q = P_F(\gamma f + (I-B))(q)$. To show it, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(q) - Bq, x_n - q\rangle = \lim_{i\to\infty}\langle \gamma f(q) - Bq, x_{n_i} - q\rangle.$$

As $\{x_{n_i}\}$ is bounded, we have that there is a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converges weakly to p. We may assume that without loss of generality that $x_{n_i} \rightharpoonup p$. Hence we have $p \in F$. Indeed, let us first show that $p \in VI(C,A)$. Put

$$Tw_1 = \begin{cases} Aw_1 + N_C w_1, & w_1 \in C \\ \emptyset, & w_1 \notin C. \end{cases}$$

Since A is relaxed (γ, r)-cocoercve and the condition (iii), we have

$$\langle Ax-Ay, x-y\rangle \geq (-\gamma)\|Ax-Ay\|^2 + r\|x-y\|^2 \geq (r-\gamma\mu^2)\|x-y\|^2 \geq 0,$$

which yields that A is monotone. Thus T is maximal monotone. Let $(w_1, w_2) \in G(T)$. Since $w_2 - Aw_1 \in N_C w_1$ and $\rho_n \in C$, we have

$$\langle w_1 - \rho_n, w_2 - Aw_1\rangle \geq 0.$$

On the other hand, from $\rho_n = P_C(I - r_n A)x_n$, we have $\langle w_1 - \rho_n, \rho_n - (I - r_n A)x_n\rangle \geq 0$ and hence $\langle w_1 - \rho_{n_i}, w_2\rangle \geq \langle w_1 - \rho_{n_i}, A\rho_{n_i} - Ax_{n_i}\rangle - \langle w_1 - \rho_{n_i}, \frac{\rho_{n_i} - x_{n_i}}{r_{n_i}}\rangle$, which implies that $\langle w_1 - p, w_2\rangle \geq 0$. We have $p \in T^{-1}0$ and hence $p \in VI(C, A)$. Next, let us show $p \in \bigcap_{i=1}^{\infty} F(T_i)$. Since Hilbert spaces are Opial's spaces, from (3.15), we have

$$\begin{aligned}
\liminf_{i\to\infty}\|\rho_{n_i} - p\| &< \liminf_{i\to\infty}\|\rho_{n_i} - Wp\| \\
&= \liminf_{i\to\infty}\|\rho_{n_i} - W\rho_{n_i} + W_n\rho_{n_i} - Wp\| \\
&\leq \liminf_{i\to\infty}\|W\rho_{n_i} - Wp\| \\
&\leq \liminf_{i\to\infty}\|\rho_{n_i} - p\|,
\end{aligned}$$

which derives a contradiction. Thus, we have $p \in F(W) = \bigcap_{i=1}^{\infty} F(T_i)$. On the other hand, we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle\gamma f(q) - Bq, x_n - q\rangle &= \lim_{n\to\infty}\langle\gamma f(q) - Bq, x_{n_i} - q\rangle \\
&= \langle\gamma f(q) - Bq, p - q\rangle \leq 0.
\end{aligned} \tag{3.16}$$

It follows from Lemma 2.5 that

$$\begin{aligned}
\|y_n - q\|^2 \leq& \frac{(1-\beta_n\bar{\gamma})^2 + \beta_n\gamma\alpha}{1-\beta_n\gamma\alpha}\|x_n - q\|^2 + \frac{2\beta_n}{1-\alpha_n\gamma\alpha}\langle\gamma f(q) - Bq, y_n - q\rangle \\
=& \frac{(1-2\beta_n\bar{\gamma} + \beta_n\alpha\gamma)}{1-\beta_n\gamma\alpha}\|x_n - q\|^2 + \frac{\beta_n^2\bar{\gamma}^2}{1-\beta_n\gamma\alpha}\|x_n - q\|^2 \\
&+ \frac{2\beta_n}{1-\beta_n\gamma\alpha}\langle\gamma f(q) - Bq, y_n - q\rangle \\
\leq& \Big(1 - \frac{2\beta_n(\bar{\gamma} - \alpha\gamma)}{1-\beta_n\gamma\alpha}\Big)\|x_n - q\|^2 \\
&+ \frac{2\beta_n(\bar{\gamma} - \alpha\gamma)}{1-\beta_n\gamma\alpha}\Big(\frac{1}{\bar{\gamma} - \alpha\gamma}\langle\gamma f(q) - Bq, y_n - q\rangle + \frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma} - \alpha\gamma)}M_5\Big),
\end{aligned} \tag{3.17}$$

where M_5 is an appropriate constant. On the other hand, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n(x_n-p)+(1-\alpha_n)(y_n-p)\|^2\\ &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|y_n-p\|^2\end{aligned} \tag{3.18}$$

Substitute (3.17) into (3.18) yields that

$$\begin{aligned}\|x_{n+1}-p\|^2 \le \Big(1-(1-\alpha_n)\frac{2\beta_n(\bar\gamma-\alpha\gamma)}{1-\beta_n\gamma\alpha}\Big)\|x_n-q\|^2\\ +(1-\alpha_n)\frac{2\beta_n(\bar\gamma-\alpha\gamma)}{1-\beta_n\gamma\alpha}\Big(\frac{1}{\bar\gamma-\alpha\gamma}\langle\gamma f(q)-Aq,y_n-q\rangle+\frac{\beta_n\bar\gamma^2}{2(\bar\gamma-\alpha\gamma)}M_5\Big).\end{aligned} \tag{3.19}$$

Put $l_n=(1-\alpha_n)\frac{2\beta_n(\bar\gamma-\alpha_n\gamma)}{1-\beta_n\alpha\gamma}$ and $t_n=\frac{1}{\bar\gamma-\alpha\gamma}\langle\gamma f(q)-Aq,y_n-q\rangle+\frac{\beta_n\bar\gamma^2}{2(\bar\gamma-\alpha\gamma)}M_5$. That is,

$$\|x_{n+1}-q\|^2\le(1-l_n)\|x_n-q\|^2+l_nt_n. \tag{3.20}$$

Notice that

$$\begin{aligned}\langle\gamma f(q)-Aq,y_n-q\rangle &= \langle\gamma f(q)-Aq,y_n-x_n\rangle+\langle\gamma f(q)-Aq,x_n-q\rangle\\ &\le \|\gamma f(q)-Aq\|\|y_n-x_n\|+\langle\gamma f(q)-Aq,x_n-q\rangle.\end{aligned}$$

From (3.5) and (3.16) that

$$\limsup_{n\to\infty}\langle\gamma f(q)-Aq,y_n-q\rangle\le 0.$$

Apply Lemma 2.4 to (3.20) to conclude $x_n\to q$ as $n\to\infty$. This completes the proof.

If T_i, $1\le i\le\infty$ is identity, we have the following.

Corollary 3.2. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H and $A: C\to H$ be relaxed (γ,r)-cocoercive and μ-Lipschitz continuous. Let $f: C\to C$ be a contraction with the coefficient α $(0<\alpha<1)$. Assume that $VI(C,A)\neq\emptyset$. Let B be a strongly positive linear bounded self-adjoint operator of C into itself with coefficient $\bar\gamma>0$ such that $\|B\|\le 1$. Assume that $0<\gamma<\bar\gamma/\alpha$. Assume that $\{x_n\}$ is generated by*

$$\begin{cases} x_1\in C,\\ y_n=P_C\big(\beta_n\gamma f(x_n)+(I-\beta_nB)P_C(I-r_nA)x_n\big),\\ x_{n+1}=\alpha_nx_n+(1-\alpha_n)y_n,\quad n\ge 1,\end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. If $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ are chosen such that

(i) $\lim_{n\to\infty}\beta_n=0$, $\sum_{n=1}^{\infty}\beta_n=\infty$;

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(iii) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;

(iv) $\{r_n\} \subset [a,b]$ *for some* a,b *with* $0 < a < b < \frac{2(r-\gamma\mu^2)}{\mu^2}$, $r > \gamma\mu^2$.

Then $\{x_n\}$ *converges strongly to* $q \in F$, *where* $q = P_F(\gamma f + (I-B))(q)$, *which solves the variation inequality* $\langle \gamma f(q) - Bq, p-q \rangle \leq 0, \forall p \in VI(C,A)$.

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