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# CONTINUOUS SELECTIONS OF SOLUTION SETS OF QUANTUM STOCHASTIC EVOLUTION INCLUSIONS

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**Abstract.** Given a Cauchy problem

$$
\begin{aligned}
dx(t) \in & -\big(E_1(t,x(t))d\Lambda_\pi(t) + F_1(t,x(t))dA_f(t) \\
& + G_1(t,x(t))dA_g^+(t) + H_1(t,x(t))dt\big) + \big(E_2(t,x(t))d\Lambda_\pi(t) + F_2(t,x(t))dA_f(t) \\
& + G_2(t,x(t))dA_g^+(t) + H_2(t,x(t))dt\big) \\
x(0) = & \, a,
\end{aligned}
$$

where $E_1, F_1, G_1, H_1$ are hypermaximal monotone multivalued maps and $E_2, F_2, G_2, H_2$ are Lipschitzian multifunctions. For each $a$, suppose the set of adapted weakly absolutely continuous quantum stochastic processes which are weak solutions of the Cauchy problem is $S^T(a)$. We prove the existence of a continuous selection of the multifunction $\langle \eta, a\xi \rangle \mapsto S^T(a)(\eta, \xi)$, the matrix elements of $S^T(a)$.

**Keywords.** Lower semicontinuous multifunctions; Evolution inclusions; Selections; Weak solution.

## 1. Introduction

The existence of continuous selections of multifunctions associated to the solution sets of certain differential inclusions was established in [7], [16] and the references therein. A generalization of such results to a non commutative setting was established in [3]. Some further studies on the solution sets was established in [4]. Quantum stochastic evolutions arising from quantum stochastic calculus of Hudson and Parthasarathy setting [14] was considered in [12]. The

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quantum stochastic differential inclusions in [12] was perturbed by hypermaximal monotone differential inclusions which was a generalization of [11]. In [15], mild solution of quantum stochastic evolution inclusions was established under a Filippov-type assumption. Some results on evolution inclusions were also considered in [1], [6], [9] and the references therein.

In this work, we consider the Cauchy problem

$$\begin{aligned} dx(t) &\in -\big(E_1(t,x(t))d\Lambda_\pi(t) + F_1(t,x(t))dA_f(t) \\ &\quad + G_1(t,x(t))dA_g^+(t) + H_1(t,x(t))dt\big) + \big(E_2(t,x(t))d\Lambda_\pi(t) + F_2(t,x(t))dA_f(t) \\ &\quad + G_2(t,x(t))dA_g^+(t) + H_2(t,x(t))dt\big) \\ x(0) &= a, \end{aligned}$$

where $E_1, F_1, G_1, H_1$ are hypermaximal monotone multivalued maps and $E_2, F_2, G_2, H_2$ are Lipschitzian multifunctions.

The existence of continuous selections of matrix element associated with the solution set of the cauchy problem for each $a$ shall be established. By using the matrix element of Hudson and Parthasarathy [14] Boson stochastic calculus, an equivalent form of the Cauchy problem above which will be employed in our work was established in [10] and [12]. The sesquilinear equivalent form is

$$\begin{aligned} \frac{d}{dt}\langle \eta, X(t)\xi\rangle &\in -\mathbb{P}_1(t,X(t))(\eta,\xi) + \mathbb{P}_2(t,X(t))(\eta,\xi) \\ X(0) &= a \; t \in [0,T]. \end{aligned}$$

For $\mathbb{P}_1, \mathbb{P}_2 : [0,T] \times \widetilde{\mathscr{A}} \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ with $\mathbb{P}_1$ hypermaximal monotone and $\mathbb{P}_2$ Lipschitzian.

It is noteworthy that when $P_1 \equiv 0$, then we have some of the results in [3], hence this work extends some of the results in [3]. In the rest of the work; Section 2 shall be for preliminaries results and notations while the main result shall be established in Section 3.

## 2. Preliminaries

In what follows, $\mathbb{D}$ is some pre-Hilbert space whose completion is $\mathscr{R}$, $\gamma$ is a fixed Hilbert and $L^2_\gamma(\mathbb{R}_+)$ is the space of square integrable $\gamma$-valued maps on $\mathbb{R}_+$.

The inner product of the Hilbert space $\mathscr{R}\otimes\Gamma(L^2_\gamma(\mathbb{R}_+))$ will be denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the norm induced by $\langle\cdot,\cdot\rangle$ .

Let $\mathbb{E}$ be linear space generated by the exponential vectors in Fock space $\Gamma(L^2_\gamma(\mathbb{R}_+))$. We define the locally convex space $\mathscr{A}$ of noncommutative stochastic processes whose topology $\tau_w$, is generated by the family of seminorms $\{\| x \|_{\eta\xi}=| \langle\eta,x\xi\rangle |, x\in\mathscr{A},\eta,\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}\}$. The completion of $(\mathscr{A},\tau_w)$ is denoted by $\widetilde{\mathscr{A}}$. The underlying elements of $\widetilde{\mathscr{A}}$ consist of linear maps from $\mathbb{D}\underline{\otimes}\mathbb{E}$ into $\mathscr{R}\otimes\Gamma(L^2_\gamma(\mathbb{R}_+))$ having domains of their adjoints containing $\mathbb{D}\underline{\otimes}\mathbb{E}$. For a fixed Hilbert space $\gamma$, the spaces $L^p_{loc}(\widetilde{\mathscr{A}})$, $L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $L^p_{loc}(I\times\widetilde{\mathscr{A}})$ are adopted as in [10].

For a topological space $\mathscr{N}$, let $clos(\mathscr{N})$ be the collection of all nonempty closed subsets of $\mathscr{N}$; we shall employ the Hausdorff topology on $clos(\widetilde{\mathscr{A}})$ as defined in [10]. Moreover, for $A,B\in clos(\mathbb{C})$ and $x\in\mathbb{C}$ , a complex number, we define the Hausdorff distance, $\rho(A,B)$ as :

$$\mathbf{d}(x,B)\equiv\inf_{y\in B}| x-y |,\ \delta(A,B)\equiv\sup_{x\in A}\mathbf{d}(x,B)$$

$$\text{and } \rho(A,B)\equiv\max(\delta(A,B),\delta(B,A)).$$

Then $\rho$ is a metric on $clos(\mathbb{C})$ and induces a metric topology on the space.

As explained in [10], we consider the space $\mathrm{wac}(\widetilde{\mathscr{A}})$ which is the completion of the locally convex space $(Ad(\widetilde{\mathscr{A}})_{wac},\tau^{wac})$ in what follows. The topology $\tau^{wac}$ is generated by the family of seminorms $\{|\,.\,|_{\eta\xi}:\eta,\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}\}$ defined for each $\Phi\in Ad(\widetilde{\mathscr{A}})_{wac}$ (adapted weakly absolutely continuous $\widetilde{\mathscr{A}}$-valued stochastic process) by

$$| \Phi |_{\eta\xi}=\| \Phi(0) \|_{\eta\xi}+\int_0^T | \frac{d}{ds}\langle\eta,\Phi(s)\xi\rangle | \, ds.$$

Associated with $wac(\widetilde{\mathscr{A}})$, we define for arbitrary $\eta,\xi\in\mathbb{D}\underline{\otimes}\mathbb{E}$, the space of complex valued functions

$$wac(\widetilde{\mathscr{A}})(\eta,\xi)=\{\langle\eta,\Phi(.)\xi\rangle:\Phi\in wac(\widetilde{\mathscr{A}})\}.$$

Each element $\Phi_{\eta\xi}(.)=\langle\eta,\Phi(.)\xi\rangle$ of $wac(\widetilde{\mathscr{A}})(\eta,\xi)$ is an absolutely continuous complex valued function on the interval $[0,T]$. We assume that $A$ is a subset of $\widetilde{\mathscr{A}}$. such that the set of complex numbers

$$A(\eta,\xi)=\{\langle\eta,a\xi\rangle:a\in A\}$$

is compact in $\mathbb{C}$ with diameter $D_{\eta\xi}=\sup_{x,y\in A(\eta,\xi)} | x-y |$ .

Let $L^1([0,T],\mathbb{D}\underline{\otimes}\mathbb{E})$ be the space of all Bochner integrable maps from $[0,T]$ to $\mathbb{D}\underline{\otimes}\mathbb{E}$ and $C([0,T],\mathbb{D}\underline{\otimes}\mathbb{E})$ the space of continuous maps from $[0,T]$ to $\mathbb{D}\underline{\otimes}\mathbb{E}$ . The spaces $L^1(I,\mathbb{D}\underline{\otimes}\mathbb{E})$ and $C([0,T],\mathbb{D}\underline{\otimes}\mathbb{E})$ are locally convex spaces with topologies $\tau_1$ and $\tau_{con}$ respectively, generated by the family of seminorms:

$$\tau_1 : \{\| \,.\, \|_{1,\eta\xi} : \eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\} \text{ with } \| z \|_{1,\eta\xi} = \int_I dt \mid \langle \eta, z(t)\xi \rangle \mid$$

and

$$\tau_{con} : \{\| \,.\, \|_{con,\eta\xi} : \eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\} \text{ with } \| z \|_{con,\eta\xi} = \sup_{t\in I} \mid \langle \eta, z(t)\xi \rangle \mid .$$

Let $T > 0$, $I = [0,T]$, denote by $\mathfrak{L}$ the $\sigma$-algebra of all Lebesgue measurable subsets of $I$ and $\mathfrak{B}(\widetilde{\mathscr{A}})$, the family of all Borel subsets of $\widetilde{\mathscr{A}}$. The characteristic function of a subset $E$ of $I$ is denoted by $\chi_E$.

A multifunction $(t,x) \to \Phi(t,x)(\eta,\xi)$ will be said to be measurable if it is $\mathfrak{L}\otimes\mathfrak{B}(\widetilde{\mathscr{A}})$-measurable.

A subset $K$ of $L^1(I,\mathbb{D}\underline{\otimes}\mathbb{E})$ is said to be *decomposable* if for every $u,v \in K$ and $A \in \mathfrak{L}$, we have $u\chi_A + v\chi_{I\setminus A} \in K$.

We denote by $\mathscr{D}$ the family of all decomposable closed nonempty subsets of $L^1(I,\widetilde{\mathscr{A}})$. Let $S$ be a separable metric space and let $\mathfrak{A}$ be a $\sigma$-algebra of subsets of $S$;

A multivalued map $\Phi : S \to 2^{\widetilde{\mathscr{A}}}$ is said to be *lower semicontinuous*(l.s.c.) if for every closed subset $C$ of $\widetilde{\mathscr{A}}$, the set $\{s \in S : \Phi(s) \subset C\}$ is closed in $S$. By a multivalued stochastic process indexed by $I \subseteq \mathbb{R}_+$, we mean a multifunction on $I$ with values in $clos(\widetilde{\mathscr{A}})$. If $\Phi$ is a multivalued stochastic process indexed by $I \subseteq \mathbb{R}_+$, then a selection of $\Phi$ is a stochastic process $X : I \to \widetilde{\mathscr{A}}$ with the property that $X(t) \in \Phi(t)$ for almost all $t \in I$.

A multivalued stochastic process $\Phi$ will be called

(i) adapted if $\Phi(t) \subseteq \widetilde{\mathscr{A}_t}$ for each $t \in \mathbb{R}_+$; (ii) measurable if $t \mapsto d_{\eta\xi}(x,\Phi(t))$ is measurable for arbitrary $x \in \widetilde{\mathscr{A}}$, $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ ; (iii) locally absolutely $p$-integrable if $t \mapsto \| \Phi(t) \|_{\eta\xi}$, $t \in \mathbb{R}_+$, lies in $L^p_{loc}(\widetilde{\mathscr{A}})$ for arbitrary $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

The set of all absolutely $p$-integrable multivalued stochastic processes will be denoted by $L^p_{loc}(\widetilde{\mathscr{A}})_{mvs}$ and for $p \in (0,\infty)$, $L^p_{loc}(I\times\widetilde{\mathscr{A}})_{mvs}$ is the set of maps $\Phi : I\times\widetilde{\mathscr{A}} \to clos(\widetilde{\mathscr{A}})$ such that $t \mapsto \Phi(t,X(t))$, $t \in I$ lies in $L^p_{loc}(\widetilde{\mathscr{A}})_{mvs}$ for every $X \in L^p_{loc}(\widetilde{\mathscr{A}})$.

Consider multivalued stochastic processes $E,F,G,H \in L^2_{loc}([0,T]\times\widetilde{\mathscr{A}})_{mvs}$ and $(0,a)$ be a fixed point in $[0,T]\times\widetilde{\mathscr{A}}$. Then, a relation of the form

$$X(t) \in a + \int_0^t (E(s,X(s))d\Lambda_\pi(s) + F(s,X(s))dA_f(s)$$
$$+ G(s,X(s))dA_g^+(s) + H(s,X(s))ds \; t \in [0,T])$$

will be called a stochastic integral inclusion with coefficients $E,F,Gand\ H$.

The stochastic differential inclusion corresponding to the integral inclusion above is;

$$\begin{aligned} dX(t) &\in E(t,X(t))d\Lambda_\pi(t) + F(t,X(t))dA_f(t) \\ &+ G(t,X(t))dA_g^+(t) + H(t,X(t))dt \\ X(0) &= a \text{ almost all } t \in [0,T]. \end{aligned} \tag{1}$$

Let $\mathbb{P}:[0,T]\times\widetilde{\mathscr{A}} \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ be sesquilinear form valued stochastic process defined in [10] in terms of $E,F,G,H$ by using the matrix elements in Hudson and Parthasarathy quantum stochastic calculus [14], it was established that problem $(1)$ is equivalent to

$$\begin{aligned} \frac{d}{dt}\langle \eta, X(t)\xi\rangle &\in \mathbb{P}(t,X(t))(\eta,\xi) \\ X(0) &= a. \end{aligned} \tag{2}$$

Suppose $P:\widetilde{\mathscr{A}} \to 2^{sesq.(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ is a multifunction; the domain of $P$; $D(P) = \{x \in \widetilde{\mathscr{A}} : P(x)(\eta,\xi) \neq \emptyset\}$; range of $P$; $range(P) = \cup_{x\in\widetilde{\mathscr{A}}} P(x)(\eta,\xi)$; graph of $P$; $graph(P) = \{(x,y) \in \widetilde{\mathscr{A}}\times\mathbb{C} : y \in P(x)(\eta,\xi)\}$. $D(P)$ is convex and for each $x \in D(P)$, the set $P(x)(\eta,\xi)$ is closed and convex.

We shall adopt the definition of hypermaximal monotone multifunction for regular multifunction $Reg(\widetilde{\mathscr{A}_0})$ in [11].

A sesquilinear form valued map $\mathbb{P}$ is said to be

(i) Monotone if

$$Re(\langle (a-b)(\eta\otimes\xi), \Phi_{\eta,\xi}(x,y)\rangle_{(2)}) \geq 0$$

and $a \in P_{\alpha,\beta}(x)\otimes 1$, $b \in P_{\alpha,\beta}(y)\otimes 1$, $x,y \in D(\mathbb{P})$, and $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, with $\eta = u\otimes e(\alpha)$, $\xi = v\otimes e(\beta)$, $\alpha,\beta \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$, $u,v \in \mathbb{D}$.

(ii) Maximal monotone if the graph of $\mathbb{P}$ is not properly contained in the graph of any other monotone member of $Reg(\widetilde{\mathscr{A}})_0$.

(iii) Hypermaximal monotone if $\mathbb{P}$ is maximal monotone and

(a) the range of the map

$$x \mapsto id_{\widetilde{\mathscr{A}}}(x) \otimes 1 + P_{\alpha\beta}(x) \otimes 1,\ x \in D(\mathbb{P}), \alpha, \beta \in L^{\infty}_{\gamma,loc}(\mathbb{R}_+)$$

is all of $\widetilde{\mathscr{A}} \otimes 1$ and

(b) $(id_{\widetilde{\mathscr{A}}}(.) + P_{\alpha\beta} \otimes 1)^{-1}$, $\alpha, \beta \in L^{\infty}_{\gamma,loc}(\mathbb{R}_+)$ is a continuous single-valued map from $\widetilde{\mathscr{A}} \otimes 1$ to $D(\mathbb{P})$. $id_{\widetilde{\mathscr{A}}}(.)$ is the identity map on $\widetilde{\mathscr{A}}$.

The existence of solution was established in [10] for problem (2) ( or equivalently (1)) for the case of $\mathbb{P}$ (or the coefficients $E, F, G, H$) Lipschitzian. Furthermore, the existence of solution of quantum stochastic evolution arising from hypermaximal monotone $-\mathbb{P}$ (or $-(E, F, G, H)$) was established in [12]. We shall be concerned with the existence of continuous selection of the solution set of the differential inclusions of [10] and [12]. In what follows we shall be considering the nonclassical equivalent form stated in (2) as:

$$\begin{aligned} &\frac{d}{dt}\langle \eta, X(t)\xi \rangle \in -\mathbb{P}_1(t, X(t))(\eta, \xi) + \mathbb{P}_2(t, X(t))(\eta, \xi) \\ &X(0) = a\ t \in [0, T]. \end{aligned} \tag{3}$$

For $\mathbb{P}_1, \mathbb{P}_2 : [0, T] \times \widetilde{\mathscr{A}} \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ with $\mathbb{P}_1$ hypermaximal monotone and $\mathbb{P}_2$ Lipschitzian.

For each $a \in A \subset \widetilde{\mathscr{A}}$, we denote by $S^{(T)}(a)$, the set of adapted weakly absolutely continuous quantum stochastic processes which are weak solutions of (3), equipped with topology wac$(\widetilde{\mathscr{A}})$ defined above. $S^T(a)(\eta, \xi) = \{\langle \eta, \Phi(.)\xi \rangle : \Phi \in S^T(a)\}$.

For each $a \in A$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, we shall be concerned with the existence of continuous selections of the map $\langle \eta, a\xi \rangle \mapsto S^{(T)}(a)(\eta, \xi)$. The case of $\mathbb{P}_1 \equiv 0$ was considered in [3]

For an arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, a sesquilinear-form valued map $\Psi : S \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ will be said to be lower semicontinuous if for every closed subset $C$ of $\mathbb{C}$ the set $\{s \in S : \Psi(s)(\eta, \xi) \subset C\}$ is closed in $S$.

Let $\mathbb{P}_1$ be hypermaximal monotone, we shall assume the following hypotheses $(S)$ on $\mathbb{P}_2$:

$S_{(i)}$ : $(t, x) \mapsto \mathbb{P}_2(t, x)(\eta, \xi)$ is measurable.

$S_{(ii)}$ : There exists a map $K_{\eta\xi} : [0, T] \to \mathbb{R}_+$ lying in $L^1_{loc}([0, T])$ such that

$$\rho(\mathbb{P}_2(t, x)(\eta, \xi), \mathbb{P}_2(t, y)(\eta, \xi)) \leq K_{\eta\xi}(t) \parallel x - y \parallel_{\eta\xi} \text{ a.e. in } [0, T].$$

$S_{(iii)}$ : there exists $\beta \in L^1_{loc}([0,T])$ such that

$$d(0,\mathbb{P}_2(t,0)(\eta,\xi)) \leq \beta(t), a.e. t \in I.$$

Consider a multifunction $P : I \times \widetilde{\mathscr{A}} \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$; for $a \in A \subset \widetilde{\mathscr{A}}$ and $A(\eta,\xi) \in \mathbb{C}$, we define the map

$\langle \eta, a\xi \rangle \to \Psi_P(a_{\eta\xi})$ as:

$$\Psi_P(a_{\eta\xi}) = \{v_{\eta\xi} \in L^1(I,\mathbb{D}\underline{\otimes}\mathbb{E}) : v_{\eta\xi}(t) \in P(t,a)(\eta,\xi) \text{ a.e. } I\}. \tag{4}$$

The following Lemmas from [7] shall be employed in the our main result.

**Lemma 2.1.** *Consider the multivalued stochastic process* $P : I \times \widetilde{\mathscr{A}} \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ *, assume*
*(i)* $(t,x) \to P(t,x)(\eta,\xi)$ *is measurable (ii)* $(t,.) \to P(t,.)(\eta,\xi)$ *is lower semicontinuous.*
*Then the map* $\Psi_P$ *given by* (4) *is lower semicontinuous from* $A(\eta,\xi)$ *into* $\mathscr{D}$ *if and only if there exists a continuous* $\beta : A(\eta,\xi) \to L^1([0,T],\mathbb{R})$ *such that for every* $a \in A$, $a_{\eta\xi} \equiv \langle \eta, a\xi \rangle \in A(\eta,\xi)$,

$$\beta(a_{\eta\xi})(t) \geq d(0,P(t,a)(\eta,\xi)) \text{ a.e. } t \in [0,T]. \tag{5}$$

*Proof*: Suppose $\Psi_P(.)$ is lower semicontinuous and $\psi(.)$ is its continuous selection then $\beta(a_{\eta\xi})(t) =\mid \psi(a_{\eta\xi})(t) \mid$ satisfies (5). To prove the converse, let $C \subseteq L^1([0,T],\mathbb{D}\underline{\otimes}\mathbb{E})$ be an arbitrary closed set and let $a_{\eta\xi,n} \to a_{\eta\xi,0}$ be such that $\Psi_P(a_{\eta\xi}) \subseteq C$. Take any $v_{\eta\xi,0} \in \Psi_P(a_{\eta\xi,0})$ and measurable selections $v_{\eta\xi,n}(t)$ of $t \to P(t,a_n)(\eta,\xi)$ such that

$$\mid v_{\eta\xi,n}(t) - v_{\eta\xi,0} \mid < d(v_{\eta\xi,0}(t),P(t,a_n)(\eta,\xi)) + \frac{1}{n} \text{ a.e. } \in [0,T]. \tag{6}$$

The existence of such $v_{\eta\xi,n}$ follows from [5]. Moreover, since for every $t$ the multifunction $(t,.) \to P(t,.)(\eta,\xi)$ is lower semicontinuous then for every $x \in \widetilde{\mathscr{A}}$

$$(t,.) \to d(\langle \eta, x\xi \rangle, P(t,.)(\eta,\xi)) \text{ is upper semicontinuous.} \tag{7}$$

Therefore from (6) we obtain that

$$v_{\eta\xi,n}(t) \to v_{\eta\xi,0}(t) \text{ a.e. } in\ [0,T]. \tag{8}$$

We show that $v_{\eta\xi,n} \to v_{\eta\xi,0}$ in $L^1([0,T],\mathbb{D}\underline{\otimes}\mathbb{E})$. From (6) we have

$$| v_{\eta\xi,n}(t) - v_{\eta\xi,0} | < | v_{\eta\xi,0} | + \beta(a_{\eta\xi,n})(t) + \frac{1}{n} \text{ a.e. in } [0,T]. \tag{9}$$

Let $u_{\eta\xi,n}(t) = | v_{\eta\xi,0} | + \beta(a_{\eta\xi,n})(t) + \frac{1}{n}$ , the sequence $u_{\eta\xi,n}(.)$ is convergent in $L^1(I,\mathbb{R})$. Thus it is bounded and uniformly integrable in $L^1(I,\mathbb{R})$, so is the sequence of functions $t \to | v_{\eta\xi,n}(t) - v_{\eta\xi,0}(t) |$ . Therefore, $v_{\eta\xi,n} \to v_{\eta\xi,0}$ in $L^1(I,\mathbb{D}\underline{\otimes}\mathbb{E})$, from (8).

Since $C$ is closed and $v_{\eta\xi,n} \in C$, then $v_{\eta\xi,0} \in C$ as well. But $v_{\eta\xi,0}$ was arbitrarily chosen in $\Psi_P(a_{\eta\xi,0})$, hence $\Psi_P(a_{\eta\xi,0}) \subseteq C$. □ Consider the maps $\Psi, \Phi : A(\eta,\xi) \to \mathscr{D}$, we define the set

$$\Phi(a_{\eta\xi}) = cl\{u_{\eta\xi} \in \Psi(a_{\eta\xi}) : | u_{\eta\xi}(t) - \varphi(a_{\eta\xi})(t) | < \psi(a_{\eta\xi})(t) \text{ a.e. } I\}, \tag{10}$$

where $\varphi : A(\eta,\xi) \to L^1(I,\mathbb{D}\underline{\otimes}\mathbb{E})$ and $\psi : A(\eta,\xi) \to L^1(I,\mathbb{R})$ are continuous. Along with Lemma 2.1 above, the following lemma which is an adaptation of Prop. 2.2 in [7] shall be employed in the prove of our main result.

**Lemma 2.2.** *Let the multivalued stochastic process $\Psi : A(\eta,\xi) \to \mathscr{D}$ be lower semicontinuous. Assume that*

*(i) $\varphi : A(\eta,\xi) \to L^1(I,\mathbb{D}\underline{\otimes}\mathbb{E})$ and $\psi : A(\eta,\xi) \to L^1(I,\mathbb{R})$ are continuous.*

*(ii) For every $a_{\eta\xi} \in A(\eta,\xi)$ the set $\Phi(a_{\eta\xi})$ defined in (10) is nonempty.*

*Then the multivalued stochastic process $\Phi : A(\eta,\xi) \to \mathscr{D}$ is lower semicontinuous, therefore it admits a continuous selection. For $a \in \overline{D(\mathbb{P}_1(t,.)(\eta,\xi))}$ and $p \in L^1(I,\mathbb{D}\underline{\otimes}\mathbb{E})$, we consider the Cauchy problem*

$$(H_p)\ \frac{d}{dt}\langle \eta, X(t)\xi \rangle \in -\mathbb{P}_1(t,X(t))(\eta,\xi) + \langle \eta, p(t)\xi \rangle,\ x(0) = a.$$

**Definition 2.1.** A function $x : I \to \widetilde{\mathscr{A}}$ is called weak solution of $(H_p)$ if there exist sequences $\{\langle \eta, p_n\xi \rangle\}_{n\in\mathbb{N}} \subset L^1(I,\mathbb{D}\underline{\otimes}\mathbb{E})$ and $\{\langle \eta, x_n\xi \rangle\}_{n\in\mathbb{N}} \subset C(I,\mathbb{D}\underline{\otimes}\mathbb{E})$, $\big(p_n, p \in L^1(I,\widetilde{\mathscr{A}}), x_n \in C(I,\widetilde{\mathscr{A}})\big)$ such that $\langle \eta, x_n\xi \rangle$ is an adapted weakly absolutely continuous process on every compact subset of $(0,T]$, $x_n$ is a solution of $(H_{p_n})$; $\langle \eta, p_n\xi \rangle \to \langle \eta, p\xi \rangle$ in $L^1(I,\mathbb{D}\underline{\otimes}\mathbb{E})$ and $\langle \eta, x_n\xi \rangle \to \langle \eta, x\xi \rangle$ in $C(I,\mathbb{D}\underline{\otimes}\mathbb{E})$.

It was established in [12] that for each $a \in \overline{D(\mathbb{P}_1(t,.)(\eta,\xi))}$ and $p \in L^1(I,\widetilde{\mathscr{A}})$, there exists a unique weak solution $x_p(.,a)$ of the Cauchy problem $(H_p)$.

**Remark 2.1.** Let $A_0 = \overline{D(\mathbb{P}_1(t,.)(\eta,\xi))}$ and $A_0(\eta,\xi)$ a closed subset of $\mathbb{C}$ be defined as $A_0(\eta,\xi) = \{a_{\eta\xi,0} \equiv \langle \eta, a_0\xi\rangle : a_0 \in A_0\}$. We remark that, if $p,q \in L^1(I,\widetilde{\mathscr{A}})$ and $x_p(.,a), x_q(.,a)$ are weak solutions of the Cauchy problem $(H_p)$, $(H_q)$ then for any $0 \le t \le T$

$$\| x_p(t,a) - x_q(t,a) \|_{\eta\xi} \le \int_0^t \| p(u) - q(u) \|_{\eta\xi}\, du. \tag{11}$$

Also for all $t \in [0,T]$, we have

$$\| x_p(t,a_1) - x_p(t,a_2) \|_{\eta\xi} \le \| x_p(0,a_1) - x_p(0,a_2) \|_{\eta\xi} = \| a_1 - a_2 \|_{\eta\xi}\ .$$

Let $\mathbb{P}_2 : [0,T] \times \widetilde{\mathscr{A}} \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ satisfies $(S)$ and $a \in A_0$. Consider the Cauchy problem

$$(H_a)\ \frac{d}{dt}\langle \eta, X(t)\xi\rangle \in -\mathbb{P}_1(t,X(t))(\eta,\xi) + \mathbb{P}_2(t,X(t))(\eta,\xi)\ X(0) = a.$$

**Definition 2.2.** A function $x(.,a) : I \to \widetilde{\mathscr{A}}$ is called a weak solution of $(H_a)$ if there exists $\langle \eta, p(t)\xi\rangle \in L^1(I, sesq(\mathbb{D}\underline{\otimes}\mathbb{E}))$, a selection of $\mathbb{P}_2(.,x(.,a))(\eta,\xi)$ such that $x(.,a)$ is a weak solution of the Cauchy problem $(H_{p(.,a)})$.

We denote by $S^T(a)$ the set of all solutions of $(H_a)$ and prove the existence of a continuous selection from the map $S^T(a)(\eta,\xi)$, the matrix element of $S^T(a)$ where

$$S^T(a)(\eta,\xi) = \{\langle \eta, \Phi(.)\xi\rangle : \Phi \in S^T(a)\}.$$

## 3. Main results

**Theorem 3.1.** *Let* $\mathbb{P}_1, \mathbb{P}_2 : I \times \widetilde{\mathscr{A}} \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$ *be such that*

*(i)* $\mathbb{P}_1$ *is hypermaximal monotone.*

*(ii)* $(t,x) \to \mathbb{P}_2(t,x)(\eta,\xi)$ *is measurable.*

*(iii) There exists a map* $K_{\eta\xi} : [0,T] \to \mathbb{R}_+$ *lying in* $L^1_{loc}([0,T])$ *such that*

$$\rho(\mathbb{P}_2(t,x)(\eta,\xi), \mathbb{P}_2(t,y)(\eta,\xi)) \le K_{\eta\xi}(t) \| x - y \|_{\eta\xi}\ \ a.e.\ in\ [0,T].$$

*(iv) There exists* $\beta \in L^1_{loc}([0,T])$ *such that* $d(0,\mathbb{P}_2(t,0)(\eta,\xi)) \le \beta_{\eta\xi}(t)$ *a.e.* $t \in [0,T]$. *If* $A_0 \subset \widetilde{\mathscr{A}}$, *then there exists an adapted stochastic process*

$x : I \times A_0 \to sesq(\mathbb{D}\underline{\otimes}\mathbb{E})$ *such that* *(a)* $\langle \eta, x(.,a)\xi\rangle \in S^T(a)(\eta,\xi)$ *for every* $a \in A_0$; *and*

*(b)* $\langle \eta, a\xi\rangle \to \langle \eta, x(.,a)\xi\rangle$ *is continuous from* $A_0(\eta,\xi)$ *to* $C(I,\mathbb{D}\underline{\otimes}\mathbb{E})$.

*Proof* For $\eta,\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and $a \in A_0$, let $x_0(.,a) : [0,T] \to \widetilde{\mathscr{A}}$ be the unique weak solution of the Cauchy problem

$$\frac{d}{dt}\langle \eta, x(t)\xi\rangle \in -\mathbb{P}_1(t,x(t))(\eta,\xi),\ x(0) = a,$$

and for $K_{\eta\xi}$ and $\beta$ given by (iii) and (iv). Define $\alpha : A_0(\eta,\xi) \to L^1_{loc}([0,T])$ by

$$\alpha(a_{\eta\xi})(t) = \beta(t) + K_{\eta\xi}(t) \parallel x_0(t,a) \parallel_{\eta\xi}. \tag{12}$$

By Remark 2.1, the map $\langle \eta, a\xi\rangle \to \langle \eta, x_0(.,a)\xi\rangle$ is weakly continuous from $A_0(\eta,\xi)$ to $C(I,\mathbb{D}\underline{\otimes}\mathbb{E})$. From eq.(12), it follows that $\alpha(.)$ is continuous from $A_0(\eta,\xi)$ to $L^1_{loc}([0,T])$. Moreover, for each $a_{\eta\xi} \in A_0(\eta,\xi)$ we have

$$d\Big(0, \mathbb{P}_2(t,x_0(t,a))(\eta,\xi)\Big)\alpha(a_{\eta\xi})(t) \text{ a.e. } [0,T]. \tag{13}$$

Fix $\varepsilon > 0$ and set $\varepsilon_n = \frac{\varepsilon}{2^{n+1}}$, $n \in \mathbb{N}$. Define $\Phi_0 : A_0(\eta,\xi) \to 2^{L(I,\mathbb{D}\underline{\otimes}\mathbb{E})}$ and $\Psi_0 : A_0(\eta,\xi) \to 2^{L(I,\mathbb{D}\underline{\otimes}\mathbb{E})}$ by

$$\Phi_0(a_{\eta\xi}) = \{v_{\eta\xi} \in L(I,\mathbb{D}\underline{\otimes}\mathbb{E}) : v_{\eta\xi} \in \mathbb{P}_2(t,x_0(t,a))(\eta,\xi)\ a.e.\ t \in [0,T]\}, \tag{14}$$

$$\Psi_0(a\eta\xi) = cl\{v_{\eta\xi} \in \Phi_0 :\mid v_{\eta\xi} \mid< \alpha(a_{\eta\xi})(t) + \varepsilon_0\ a.e.\ t \in [0,T]\}. \tag{15}$$

Using (13) and Lemma 2.1, $\Phi_0(.)$ is lower semicontinuous and $\Psi_0(a_{\eta\xi}) \neq \emptyset$ for each $a_{\eta\xi} \in A_0(\eta,\xi)$. Hence by Lemma 2, there exists $\varphi_0 : A_0(\eta,\xi) \to L(I,\mathbb{D}\underline{\otimes}\mathbb{E})$ a continuous selection of $\Psi_0(.)$. Set $p_0(t,a)(\eta,\xi) = \varphi_0(a\eta\xi)(t)$. Then $P_0(.,a)(\eta,\xi)$ is continuous, $p_0(t,a)(\eta,\xi) \in \mathbb{P}_2(t,x_0(t,a))(\eta,\xi)$ and

$$\mid p_0(t,a)(\eta,\xi) \mid \leq \alpha(a_{\eta\xi})(t) + \varepsilon_0 \text{ a.e } t \in [0,T].$$

Set $m_{\eta\xi}(t) = \int_0^t K_{\eta\xi}(u)du$ and for $a \in A_0,\ a_{\eta\xi} \in A_0(\eta,\xi)$ $n \geq 1$ define

$$\begin{aligned}\beta_n(a_{\eta\xi})(t) = &\int_0^t \alpha(a_{\eta\xi})(u)\frac{[m_{\eta\xi}(t) - m_{\eta\xi}(u)]^{n-1}}{(n-1)!}du \\ &+ T\big(\sum_{i=0}^n \varepsilon_i\big)\frac{[m_{\eta\xi}(t)]^{n-1}}{(n-1)!},\ t \in I.\end{aligned} \tag{16}$$

Since $\alpha(.)$ is continuous from $A_0(\eta,\xi)$ to $L^1_{loc}(I,\mathbb{R})$ by (16) it follows $\beta_n(.)$ also is continuous from $A_0(\eta,\xi)$ to $L^1_{loc}(I,\mathbb{R})$. Let $x_1(.,a): I \to \widetilde{\mathscr{A}}$ be the unique weak solution of the Cauchy problem

$$\frac{d}{dt}\langle \eta, x(t)\xi\rangle \in -\mathbb{P}_1(t,x(t))(\eta,\xi) + \langle \eta, p_0(t)\xi\rangle$$
$$x(0) = a.$$

By equation (11), we have

$$\begin{aligned} |\langle \eta, x_1(t)\xi\rangle - \langle \eta, x_0(t)\xi\rangle| &\leq \int_0^t |p_0(u,a)(\eta,\xi)|\, du \\ &\leq \int_0^t \alpha(a_{\eta\xi})(u)du + \varepsilon_0 T \\ &< \beta_1(a_{\eta\xi})(t) \end{aligned}$$

for each $a_{\eta\xi} \in A_0(\eta,\xi)$ and $t \in I\setminus\{0\}$. Setting $\langle \eta, p_n(u,a)\xi\rangle \equiv p_n(u,a)(\eta,\xi)$, we claim there exist two sequences $\{p_n(.,a)\}_{n\in\mathbb{N}}$ and $\{x_n(.,a)\}n \in \mathbb{N}$ such that for each $n \geq 1$, the followings hold:

(a) $a_{\eta\xi} \to \langle \eta, p_n(.,a)\xi\rangle$ is continuous from $A_0(\eta,\xi)$ into $L(I, \mathbb{D}\underline{\otimes}\mathbb{E})$;
(b) $\langle \eta, p_n(t,a)\xi\rangle \in \mathbb{P}_2(t, x_n(t,a))(\eta,\xi)$ for each $a_{\eta\xi} \in A_0(\eta,\xi)$ and a.e. $t \in I$;
(c) $|\langle \eta, p_n(t,a)\xi\rangle - \langle \eta, p_{n-1}(t,a)\xi\rangle| \leq K_{\eta\xi}(t)\beta_n(a_{\eta\xi})(t)$ for a.e. $t \in [0,T]$ and
(d) $x_n(.,a)$ is the unique weak solution of the Cauchy problem $(H_{p_n(.,a)})$.

Then by (11) and (c) for $t \in I\setminus\{0\}$ we have

$$\begin{aligned} |\langle \eta, x_{n+1}(t,a)\xi\rangle - \langle \eta, x_n(t,a)\xi\rangle| &\leq \int_0^t |\langle \eta, p_n(u,a)\xi\rangle - \langle \eta, p_{n-1}(u,a)\xi\rangle|\, du \\ &\leq \int_0^t K_{\eta\xi}(u)\beta_n(a_{\eta\xi})(u)du \\ &= \int_0^t \alpha(a_{\eta\xi})(u)\frac{[m_{\eta\xi}(t) - m_{\eta\xi}(u)]^n}{n!}du \\ &+ T\left(\sum_{i=0}^n \varepsilon_i\right)\frac{[m_{\eta\xi}(t)]^n}{n!} \\ &< \beta_{n+1}(a_{\eta\xi})(t). \end{aligned} \tag{17}$$

Hence by (iii), we find that

$$d\Bigg(\langle\eta,p(t,a)\xi\rangle,\mathbb{P}_2(t,x_{n+1}(t,a))(\eta,\xi)\Bigg)\leq K_{\eta\xi}(t)\parallel x_{n+1}(t,a)-x_n(t,a)\parallel_{\eta\xi}$$
$$< K_{\eta\xi}(t)\beta_{n+1}(a_{\eta\xi})(t). \tag{18}$$

By (18) and Lemma 2.1, we have that the multivalued map $\Psi_{n+1}:A_0(\eta,\xi)\longrightarrow 2^{L^1(I,\mathbb{D}\underline{\otimes}\mathbb{E})}$ defined by

$$\Psi_{n+1}(a_{\eta\xi})=\{v_{\eta\xi}\in L^1(I,\mathbb{D}\underline{\otimes}\mathbb{E}):v_{\eta\xi}(t)\in\mathbb{P}(t,x_{n+1}(t,a))(\eta,\xi)\text{ a.e. in }I\} \tag{19}$$

is l.s.c. with decomposable closed nonempty values, and by (18)

$$\Phi_{n+1}(a_{\eta\xi})=cl\{v_{\eta\xi}\in\Psi_{n+1}(a_{\eta\xi}):\mid v_{\eta\xi}(t)-\langle\eta,p_{n_k}(t,a)\xi\rangle\mid$$
$$< K_{\eta\xi}(t)\beta_{n+1}(a_{\eta\xi})(t)\ in\ I\} \tag{20}$$

is a non-empty set. Then by Lemma 2.2, there exists $\varphi_{n+1}:A_0(\eta,\xi)\to sesq(\mathbb{D}\underline{\otimes}\mathbb{E})$ a continuous selection of $\Phi_{n+1}(.)$. Setting $\langle\eta,p_{n+1}(t,a)\xi\rangle=\varphi_{n+1}(a_{\eta\xi})(t)$ for $a_{\eta\xi}\in A_0(\eta,\xi)$, $t\in I$, we have that $p_{n+1}$ satisfies properties (a)- (c) of our claim. By virtue of (c) and (17), we have (20).

$$\begin{aligned}\mid\langle\eta,p_n(.,a)\xi\rangle-\langle\eta,p_{n-1}(.,a)\xi\rangle\mid&=\int_0^T\mid\langle\eta,p_n(u,a)\xi\rangle-\langle\eta,p_{n-1}(u,a)\xi\rangle\mid du\\&\leq\int_0^T\alpha(a_{\eta\xi})(u)\frac{[m_{\eta\xi}(T)-m_{\eta\xi}(u)]^n}{n!}du\\&+T\Bigg(\sum_{i=0}^n\varepsilon_i\frac{[m_{\eta\xi}(T)]^n}{n!}\Bigg)\\&\leq\frac{[\parallel K_{\eta\xi}\parallel_1]^n}{n!}\Bigg(\parallel\alpha(a_{\eta\xi})\parallel+T\varepsilon\Bigg).\end{aligned} \tag{21}$$

Since $a_{\eta\xi}\to\parallel\alpha(a_{\eta\xi})\parallel$ is continuous, then it is locally bounded. Therefore (21) implies that for every $a\in A_0$, $a_{\eta\xi}\in A_0(\eta,\xi)$ the sequence $(p_n(.,a'))_{n\in\mathbb{N}}$ satisfies the Cauchy condition uniformly with respect to $a'$ on some neighbourhood of $a$. Hence, if $p(.,a)$ is the limit of $(p_n(.,a))_{n\in\mathbb{N}}$ then $a_{\eta\xi}\to\langle\eta,p(.,a)\xi\rangle$ is weakly continuous from $A_0(\eta,\xi)$ into $L^1(I,\mathbb{D}\underline{\otimes}\mathbb{E})$. Moreover, using (17) and (21), we have

$$\begin{aligned}\mid\langle\eta,x_{n+1}(.,a)\xi\rangle-\langle\eta,x_n(.,a)\xi\rangle\mid&\leq\mid\langle\eta,p_n(.,a)\xi\rangle-\langle\eta,p_{n-1}(.,a)\xi\rangle\mid\\&\leq\frac{[\parallel K_{\eta\xi}\parallel_1]^n}{n!}(\parallel\alpha(a_{\eta\xi})\parallel_1+T\varepsilon).\end{aligned}$$

So, $(\langle\eta, x_n(.,a)\xi\rangle)$ is Cauchy in $C(I, \mathbb{D}\underline{\otimes}\mathbb{E})$ with respect to $a$. Let $\langle\eta, x_n(.,a)\xi\rangle \to \langle\eta, x(.,a)\xi\rangle$. Then the map $a_{\eta\xi} \to \langle\eta, x(.,a)\xi\rangle$ is weakly continuous from $A_0(\eta,\xi)$ to $C(I, \mathbb{D}\underline{\otimes}\mathbb{E})$ $\langle\eta, x_n(.,a)\xi\rangle \to \langle\eta, x(.,a)\xi\rangle$ uniformly and

$$d\Bigg(\langle\eta, p_n(t,a)\xi\rangle, \mathbb{P}(t, x(t,a))(\eta,\xi)\Bigg) \leq K_{\eta\xi}(t) \mid \langle\eta, x_n(.,a)\xi\rangle - \langle\eta, x(.,a)\xi\rangle \mid .$$

Passing to the linit along a subsequence $(p_{n_k})_{k\in\mathbb{N}}$ of $(p_n)n \in \mathbb{N}$ converging pointwise to $p$ in $L^1(I, \widetilde{\mathscr{A}})$. Hence we obtain

$$\langle\eta, p(t,a)\xi\rangle \in \mathbb{P}(t, x(t,a))(\eta,\xi) \text{ for each } a \in A_0 \text{ and } a.e.t \in I. \tag{22}$$

Let $x*(.,a)$ be the unique weak solution of the Cauchy problem

$$\frac{d}{dt}\langle\eta, x(t)\xi\rangle \in -\mathbb{P}_1(t, x(t))(\eta,\xi) + \langle\eta, p(t,a)\xi\rangle,\ x(0) = a. \tag{23}$$

By equation (11), we have

$$\mid \langle\eta, x_n(t,a)\xi\rangle - \langle\eta, x^*(t,a)\xi\rangle \mid \leq \int_0^t \mid \langle\eta, p_n(u,a)\xi\rangle - \langle\eta, p(u,a)\xi\rangle \mid du.$$

If $n \to \infty$, then $\langle\eta, x^*(.,a)\xi\rangle \equiv \langle\eta, x(.,a)\xi\rangle$. Therefore, $x(.,a)$ is the weak solution of $(23)$, and by $(22)$ it follows that

$$\langle\eta, x(.,a)\xi\rangle \in S^T(a)(\eta,\xi) \text{ for every } a \in A_0.$$

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