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ON NONEXPANSIVE MAPPINGS AND AN INVERSE-STRONGLY MONOTONE MAPPING IN HILBERT SPACES

QING YUAN

School of Sciences, Hangzhou Normal University, Zhejiang, China

Abstract. The aim of this paper is to study fixed point problems of an infinite family of nonexpansive mappings and solution problems of a monotone variational inequality. Strong convergence theorems of common solutions are established in the framework of Hilbert spaces.

Keywords. Nonexpansive mapping; Variational inequality; Fixed point; Inverse-strongly monotone mapping; Strong convergence.

1. Introduction

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty, closed and convex subset of H. Let $A : C \to H$ be a mapping. Recall that A is said to be α-inverse-strongly monotone iff there exists a positive real number α such that

$$\langle Ax-Ay, x-y\rangle \geq \alpha\|Ax-Ay\|^2, \quad \forall x,y \in C.$$

A mapping $S : C \to C$ is said to be nonexpansive iff

$$\|Sx-Sy\| \leq \|x-y\|, \quad \forall x,y \in C.$$

E-mail address: zjyuanq@yeah.net

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In this paper, we use $F(S)$ to denote the fixed point set of S. A mapping $f: C \to C$ is said to be a κ-contraction iff there exists a positive real number $\kappa \in (0,1)$ such that

$$\|f(x) - f(y)\| \le \kappa \|x - y\|, \quad \forall x, y \in C.$$

A linear bounded operator B on H is strongly positive iff there exists a positive real number $\bar{\gamma}$ such that

$$\langle Bx, x\rangle \ge \bar{\gamma}\|x\|^2, \quad \forall x \in H.$$

A set-valued mapping $T: H \to 2^H$ is called monotone iff for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \ge 0$. A monotone mapping $T: H \to 2^H$ is maximal if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal iff for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \ge 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone map of C into H and let $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w\rangle \ge 0, \ \forall u \in C\}$ and define

$$Tv = \begin{cases} Av + N_C v, & v \in C \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ iff $v \in VI(C, A)$; see [1] and the reference therein.

The classical variational inequality is to find $u \in C$ such that

$$\langle Au, v - u\rangle \ge 0, \quad \forall v \in C. \tag{1.1}$$

In this paper, we use $VI(C, A)$ to denote the solution set of the variational inequality. Let P_C be the metric projection from H onto the subset C. For a given point $z \in H$, $u \in C$ satisfies the inequality

$$\langle u - z, v - u\rangle \ge 0, \quad \forall v \in C,$$

if and only if $u = P_C z$. It is known that projection operator P_C is nonexpansive. It is also know that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{1.2}$$

One can see that the variational inequality (1.1) is equivalent to a fixed point problem. The point $u \in C$ is a solution of the variational inequality (1.1) if and only if $u \in C$ satisfies the relation $u = P_C(u - \lambda Au)$, where $\lambda > 0$ is a constant.

For finding a common element in the fixed point set of nonexpansive mappings and in the solution set of the variational inequality involving inverse-strongly mappings, many authors considered projection algorithms. Strong convergence theorems are established in Hilbert or Banach spaces; see [2-6] and the references therein.

Concerning a family of nonexpansive mappings has been considered by many authors; see, for example, [7-16] and the references therein. The well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings. The problem of finding an optimal point that minimizes a given cost function over common set of fixed points of a family of nonexpansive mappings is of wide interdisciplinary interest and practical importance.

Consider a sequence mappings W_n which are generated by an finite family of nonexpansive mapping $T_1, T_2, \ldots$ as follows

$$
\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \gamma_n T_n U_{n,n+1} + (1-\gamma_n)I,\\
&U_{n,n-1} = \gamma_{n-1} T_{n-1} U_{n,n} + (1-\gamma_{n-1})I,\\
&\vdots\\
&U_{n,k} = \gamma_k T_k U_{n,k+1} + (1-\gamma_k)I, \qquad (1.6)\\
&u_{n,k-1} = \gamma_{k-1} T_{k-1} U_{n,k} + (1-\gamma_{k-1})I,\\
&\vdots\\
&U_{n,2} = \gamma_2 T_2 U_{u,3} + (1-\gamma_2)I,\\
&W_n = U_{n,1} = \gamma_1 T_1 U_{n,2} + (1-\gamma_1)I,
\end{aligned}
$$

where $\{\gamma_1\}, \{\gamma_2\}, \ldots$ are real numbers such that $0 \le \gamma \le 1$, $T_1, T_2, \ldots$ be an infinite family of mappings of C into itself. Nonexpansivity of each T_i ensures the nonexpansivity of W_n.

Concerning W_n we have the following lemmas which are important to prove our main results.

Lemma 1.1 [17]. *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty,* *and let* $\gamma_1, \gamma_2, \ldots$ *be real numbers such that* $0 < \gamma_n \le \eta < 1$ *for any* $n \ge 1$*. Then, for every* $x \in C$ *and* $k \in N$*, the limit* $\lim_{n\to\infty} U_{n,k}x$ *exists.*

Using Lemma 1.1, one can define the mapping W of C into itself as follows. $Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$, for every $x \in C$. Such a W is called the W-mapping generated by $T_1, T_2, \ldots$ and $\gamma_1, \gamma_2, \ldots$. Throughout this paper, we will assume that $0 < \gamma_n \le \eta < 1$ for all $n \ge 1$.

Lemma 1.2 [17]. *Let* C *be a nonempty closed convex subset of a strictly convex Banach space* E*. Let* $T_1, T_2, \ldots$ *be nonexpansive mappings of* C *into itself such that* $\cap_{n=1}^{\infty} F(T_n)$ *is nonempty, and let* $\gamma_1, \gamma_2, \ldots$ *be real numbers such that* $0 < \gamma_n \le \eta < 1$ *for any* $n \ge 1$*. Then,* $F(W) = \cap_{n=1}^{\infty} F(T_n)$*.*

Lemma 1.3 [18] *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \delta_n,$$

where γ_n *is a sequence in (0,1) and* $\{\delta_n\}$ *is a sequence such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 1.4. [19] *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in [0,1] with* $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$*. Suppose* $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all integers* $n \ge 0$ *and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.5 [7] *Assume* B *is a strongly positive linear bounded operator on a Hilbert space* H *with coefficient* $\bar{\gamma} > 0$ *and* $0 < \rho \le \|B\|^{-1}$*. Then* $\|I - \rho B\| \le 1 - \rho\bar{\gamma}$*.*

Lemma 1.6 [14] *Let* K *be a nonempty closed convex subset of a Hilbert space* H*,* $\{T_i : C \to C\}$ *be a family of infinitely nonexpansive mappings with* $\cap_{i=1}^{\infty} F(T_i)$*,* $\{\gamma_n\}$ *be a real sequence such that* $0 < \gamma_n \le b < 1$ *for each* $n \ge 1$*. If* C *is any bounded subset of* K*, then* $\lim_{n\to\infty} \sup_{x\in C} \|Wx - W_n x\| = 0$.

Lemma 1.7 [5] *Let H be a Hilbert space. Let B be a strongly positive linear bounded self-adjoint operator with the constant $\bar{\gamma} > 0$ and f a contraction with the constant κ. Assume that $0 < \gamma < \bar{\gamma}/\kappa$. Let T be a nonexpansive mapping with a fixed point $x_t \in H$ of the contraction $x \mapsto t\gamma f(x) + (I - tB)Tx$. Then $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point $\bar{x}$ of T, which solves the variational inequality*

$$\langle (A - \gamma f)\bar{x}, z - \bar{x} \rangle \leq 0, \quad \forall z \in F(T).$$

Equivalently, we have $P_{F(T)}(I - A + \gamma f)\bar{x} = \bar{x}$.

2. Main results

Theorem 2.1. *Let H be a real Hilbert space and C a nonempty closed convex subset of H such that it is closed for linear operator. Let $A : H \to H$ be an α-inverse-strongly monotone mapping and f a κ-contraction on H. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings from C into itself such that $F := \bigcap_{i=1}^{\infty} F(T_i) \cap VI(C, A) \neq \emptyset$. Let B be a strongly positive linear bounded self-adjoint operator of C into itself with the constant $\bar{\gamma} > 0$. Let $\{x_n\}$ be a sequence generated in*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\beta_n \gamma f(x_n) + (1 - \alpha_n)(I - \beta_n B)W_n P_C(I - r_n A)x_n, \quad n \geq 1, \tag{2.1}$$

where W_n is generated in (1.6), $\{\alpha_n\}$ and $\{\beta_n\}$ are real number sequences in $(0, 1)$. Assume that the control sequence $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy the following restrictions: (i) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$; (ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$; (iii) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$, (iv) $\{r_n\} \subset [a, b]$ $for\ some\ a, b\ with\ 0 < a < b < 2\alpha$. Assume that $0 < \gamma < \bar{\gamma}/\kappa$. Then $\{x_n\}$ strongly converges to some point q, where $q \in F$, where $q = P_F(\gamma f + (I - B))(q)$, which solves the variation inequality

$$\langle \gamma f(q) - Bq, p - q \rangle \leq 0, \quad \forall p \in F.$$

Proof. It is not hard to see that $I - r_n A$ is firmly nonexpansive. Put $y_n = \beta_n \gamma f(x_n) + (1 - \alpha_n)(I - \beta_n B)W_n P_C(I - r_n A)x_n$. Since the condition (i), we may assume, with no loss of generality, that $\beta_n < \|B\|^{-1}$ for all n. From Lemma 1.3, we know that if $0 < \rho \leq \|B\|^{-1}$, then $\|I - \rho B\| \leq 1 - \rho\bar{\gamma}$.

Letting $p \in F$, we have

$$\begin{aligned}\|y_n - p\| &\le \beta_n\|\gamma f(x_n) - Bp\| + (1-\beta_n\bar{\gamma})\|W_nP_C(I-r_nA)x_n - p\| \\ &\le \beta_n\gamma\|f(x_n)-f(p)\| + \beta_n\|\gamma f(p) - Bp\| + (1-\beta_n\bar{\gamma})\|x_n - p\| \\ &= [1-\beta_n(\bar{\gamma}-\kappa\gamma)]\|x_n-p\| + \beta_n\|\gamma f(p) - Bp\|.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\|x_{n+1}-p\| &\le \alpha_n\|x_n-p\| + (1-\alpha_n)\|y_n-p\| \\ &\le \alpha_n\|x_n-p\| + (1-\alpha_n)[(1-\beta_n(\bar{\gamma}-\gamma\kappa))\|x_n-p\| + \beta_n\|\gamma f(p) - Bp\|]\end{aligned}$$

By simple induction, we have

$$\|x_n - p\| \le \max\{\|x_0-p\|, \frac{\|Bp-\gamma f(p)\|}{\bar{\gamma}-\gamma\kappa}\},$$

which gives that the sequence $\{x_n\}$ is bounded, so is $\{y_n\}$.

Next, we prove $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$. Put $\rho_n = P_C(I-r_nA)x_n$. Next, we compute

$$\begin{aligned}\|\rho_n - \rho_{n+1}\| \le& \|(I-r_nA)x_n - (I-r_{n+1}A)x_{n+1}\| \\ =& \|(x_n - r_nAx_n) - (x_{n+1} - r_nAx_{n+1}) + (r_{n+1}-r_n)Ax_{n+1}\| \\ \le& \|x_n - x_{n+1}\| + |r_{n+1}-r_n|M_1,\end{aligned} \tag{2.2}$$

where M_1 is an appropriate constant. It follows that

$$\begin{aligned}\|y_n - y_{n+1}\| \le& (1-\beta_{n+1}\bar{\gamma})(\|\rho_{n+1}-\rho_n\| + \|W_{n+1}\rho_n - W_n\rho_n\|) \\ &+ |\beta_{n+1}-\beta_n|M_2 + \gamma\beta_{n+1}\kappa\|x_{n+1}-x_n\|,\end{aligned} \tag{2.3}$$

where M_2 is an appropriate constant. Since T_i and $U_{n,i}$ are nonexpansive, we have from (1.6) that

$$\begin{aligned}\|W_{n+1}\rho_n - W_n\rho_n\| &\le \gamma_1\|U_{n+1,2}\rho - U_{n,2}\rho_n\| \\ &\le \gamma_1\gamma_2\|U_{u+1,3}\rho_n - U_{n,3}\rho_n\| \\ &\le \cdots \\ &\le \gamma_1\gamma_2\cdots\gamma_n\|U_{n+1,n+1}\rho_n - U_{n,n+1}\rho_n\| \\ &\le M_3\prod_{i=1}^{n}\gamma_i,\end{aligned} \tag{2.4}$$

where $M_3 \geq 0$ is an appropriate constant such that $\|U_{n+1,n+1}\rho_n - U_{n,n+1}\rho_n\| \leq M_3$, for all $n \geq 0$. Substitute (2.2) and (2.4) into (2.3) yields that

$$\|y_n - y_{n+1}\| \leq [1 - \beta_{n+1}(\bar{\gamma} - \kappa\gamma)]\|x_{n+1} - x_n\| + M_4(|r_{n+1} - r_n| + |\beta_{n+1} - \beta_n| + \prod_{i=1}^{n}\gamma_i),$$

where M_4 is an appropriate appropriate constant such that $M_4 \geq \max\{M_1, M_2, M_3\}$. From the conditions (i) and (iii), we have

$$\limsup_{n\to\infty}\{\|y_{n+1} - y_n\| - |x_{n+1} - x_n\|\} \leq 0. \tag{2.5}$$

By virtue of Lemma 1.5, we obtain that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{2.6}$$

This implies from (2.6) that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{2.7}$$

Next, we show $\lim_{n\to\infty}\|W\rho_n - \rho_n\| = 0$. Observing that

$$y_n - W_n\rho_n = \beta_n(\gamma f(x_n) - BW_n\rho_n)$$

and the condition (i), we can easily get

$$\lim_{n\to\infty}\|W_n\rho_n - y_n\| = 0. \tag{2.8}$$

Notice that

$$\|\rho_n - p\|^2 = \|x_n - p\|^2 - r_n(2\alpha - r_n)\|Ax_n - Ap\|^2. \tag{2.9}$$

On the other hand, we have

$$\|y_n - p\|^2 \leq \beta_n\|\gamma f(x_n) - Bp\|^2 + \|\rho_n - p\|^2 + 2\beta_n\|\gamma f(x_n) - Bp\|\|\rho_n - p\|, \tag{2.10}$$

from which it follows that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|y_n - p\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)[\beta_n\|\gamma f(x_n) - Bp\|^2 + \|\rho_n - p\|^2 \\ &\quad + 2\beta_n\|\gamma f(x_n) - Bp\|\|\rho_n - p\|].\end{aligned} \tag{2.11}$$

Substituting (2.9) into (2.11), we arrive at

$$\begin{aligned}\|x_{n+1}-p\|^2 \leq \|x_n-p\|^2+\beta_n\|\gamma f(x_n)-Bp\|^2 \\ -(1-\alpha_n)r_n(2\alpha-r_n)\|Ax_n-Ap\|^2 \\ +2\beta_n\|\gamma f(x_n)-Bp\|\|\rho_n-p\|].\end{aligned} \tag{2.12}$$

It follows that

$$\begin{aligned}&(1-\alpha_n)r_n(2\alpha-r_n)\|Ax_n-Ap\|^2 \\ &\leq \beta_n\|\gamma f(x_n)-Bp\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2+2\beta_n\|\gamma f(x_n)-Bp\|\|\rho_n-p\| \\ &\leq \beta_n\|\gamma f(x_n)-p\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\| \\ &\quad +2\beta_n\|\gamma f(x_n)-Bp\|\|\rho_n-p\|.\end{aligned}$$

In view of the restrictions (i), and (iv), we find from (2.7) that

$$\lim_{n\to\infty}\|Ax_n-Ap\|=0. \tag{2.13}$$

Since

$$\begin{aligned}\|\rho_n-p\|^2 =&\|P_C(I-r_nA)x_n-P_C(I-r_nA)p\|^2 \\ \leq&\langle (I-r_nA)x_n-(I-r_nA)p,\rho_n-p\rangle \\ =&\frac{1}{2}\{\|(I-r_nA)x_n-(I-r_nA)p\|^2+\|\rho_n-p\|^2 \\ &-\|(I-r_nA)x_n-(I-r_nA)p-(\rho_n-p)\|^2\} \\ \leq&\frac{1}{2}\{\|x_n-p\|^2+\|\rho_n-p\|^2-\|(x_n-\rho_n)-r_n(Ax_n-Ap)\|^2\} \\ =&\frac{1}{2}\{\|x_n-p\|^2+\|\rho_n-p\|^2-\|x_n-\rho_n\|^2-r_n^2\|Ax_n-Ap\|^2 \\ &+2r_n\langle x_n-\rho_n,Ax_n-Ap\rangle\},\end{aligned}$$

which yields that

$$\|\rho_n-p\|^2 \leq\|x_n-p\|^2-\|\rho_n-x_n\|^2+2r_n\|\rho_n-x_n\|\|Ax_n-Ap\|. \tag{2.14}$$

Substituting (2.14) into (2.11), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 \leq \|x_n-p\|^2+\beta_n\|\gamma f(x_n)-Bp\|^2+2r_n\|\rho_n-x_n\|\|Ax_n-Ap\| \\ -(1-\alpha_n)\|\rho_n-x_n\|^2+2\beta_n\|\gamma f(x_n)-Bp\|\|\rho_n-p\|.\end{aligned}$$

This implies that

$$\begin{aligned}
&(1-\alpha_n)\|\rho_n - x_n\|^2 \\
&\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \beta_n\|\gamma f(x_n) - Bp\|^2 + 2r_n\|\rho_n - x_n\|\|Ax_n - Ap\| \\
&\quad + 2\beta_n\|\gamma f(x_n) - Bp\|\|\rho_n - p\| \\
&\le (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + \beta_n\|\gamma f(x_n) - Bp\|^2 \\
&\quad + 2r_n\|\rho_n - x_n\|\|Ax_n - Ap\| + 2\beta_n\|\gamma f(x_n) - Bp\|\|\rho_n - p\|.
\end{aligned}$$

In view of the restrictions (i) and (ii), we find from (2.7) and (2.13) that

$$\lim_{n\to\infty}\|\rho_n - x_n\| = 0. \tag{2.15}$$

On the other hand, we have

$$\|\rho_n - W_n\rho_n\| \le \|x_n - \rho_n\| + \|x_n - y_n\| + \|y_n - W_n\rho_n\|.$$

It follows from (2.6), (2.8) and (2.15) that $\lim_{n\to\infty}\|W_n\rho_n - \rho_n\| = 0$. From Lemma 1.6, we find that $\|W\rho_n - W_n\rho_n\| \to 0$ as $n \to \infty$. Notice that

$$\|W\rho_n - \rho_n\| \le \|W_n\rho_n - \rho_n\| + \|W_n\rho_n - W\rho_n\|,$$

from which it follows that

$$\lim_{n\to\infty}\|W\rho_n - \rho_n\| = 0. \tag{2.16}$$

Next, we show $\limsup_{n\to\infty}\langle\gamma f(q) - Bq, x_n - q\rangle \le 0$, where $q = P_F(\gamma f + (I-B))(q)$. To show it, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle\gamma f(q) - Bq, x_n - q\rangle = \lim_{i\to\infty}\langle\gamma f(q) - Bq, x_{n_i} - q\rangle.$$

As $\{x_{n_i}\}$ is bounded, we have that there is a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converges weakly to p. We may assume, without loss of generality, that $x_{n_i} \rightharpoonup p$. Hence we have $p \in F$. Indeed, let us first show that $p \in VI(C,A)$. Put

$$Tw_1 = \begin{cases} Aw_1 + N_C w_1, & w_1 \in C \\ \emptyset, & w_1 \notin C. \end{cases}$$

Since A is inverse-strongly monotone, we see that T is maximal monotone. Let $(w_1,w_2)\in G(T)$. Since $w_2-Aw_1\in N_Cw_1$ and $\rho_n\in C$, we have

$$\langle w_1-\rho_n,w_2-Aw_1\rangle\geq 0.$$

It follows that

$$\langle w_1-\rho_n,\frac{\rho_n-x_n}{r_n}+Ax_n\rangle\geq 0.$$

This implies from (2.15) that $\langle w_1-p,w_2\rangle\geq 0$. We have $p\in T^{-1}0$ and hence $p\in VI(C,A)$. Next, let us show $p\in\bigcap_{i=1}^{\infty}F(T_i)$. Since Hilbert spaces are Opial's spaces, from (2.16), we have

$$\begin{aligned}\liminf_{i\to\infty}\|\rho_{n_i}-p\|&<\liminf_{i\to\infty}\|\rho_{n_i}-Wp\|\\&=\liminf_{i\to\infty}\|\rho_{n_i}-W\rho_{n_i}+W_n\rho_{n_i}-Wp\|\\&\leq\liminf_{i\to\infty}\|W\rho_{n_i}-Wp\|\\&\leq\liminf_{i\to\infty}\|\rho_{n_i}-p\|,\end{aligned}$$

which derives a contradiction. Thus, we have from Lemma 2.2 that $p\in F(W)=\bigcap_{i=1}^{\infty}F(T_i)$. On the other hand, we have

$$\limsup_{n\to\infty}\langle\gamma f(q)-Bq,x_n-q\rangle\leq 0. \tag{2.17}$$

Finally, we show $x_n\to q$ strongly as $n\to\infty$. Notice that

$$\begin{aligned}&\|y_n-q\|^2\\&\leq(1-\beta_n\bar{\gamma})^2\|W_n\rho_n-q\|^2+2\beta_n\langle\gamma f(x_n)-Bq,y_n-q\rangle\\&\leq(1-\beta_n\bar{\gamma})^2\|x_n-q\|^2+\kappa\gamma\beta_n(\|x_n-q\|^2+\|y_n-q\|^2)\\&\quad+2\beta_n\langle\gamma f(q)-Bq,y_n-q\rangle.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|y_n - q\|^2 \leq & \frac{(1-\beta_n\bar{\gamma})^2 + \beta_n\gamma\kappa}{1-\beta_n\gamma\kappa}\|x_n - q\|^2 + \frac{2\beta_n}{1-\alpha_n\gamma\kappa}\langle \gamma f(q) - Bq, y_n - q\rangle \\ = & \frac{(1-2\beta_n\bar{\gamma}+\beta_n\kappa\gamma)}{1-\beta_n\gamma\kappa}\|x_n - q\|^2 + \frac{\beta_n^2\bar{\gamma}^2}{1-\beta_n\gamma\kappa}\|x_n - q\|^2 \\ & + \frac{2\beta_n}{1-\beta_n\gamma\kappa}\langle \gamma f(q) - Bq, y_n - q\rangle \\ \leq & [1 - \frac{2\beta_n(\bar{\gamma}-\kappa\gamma)}{1-\beta_n\gamma\kappa}]\|x_n - q\|^2 \\ & + \frac{2\beta_n(\bar{\gamma}-\kappa\gamma)}{1-\beta_n\gamma\kappa}[\frac{1}{\bar{\gamma}-\kappa\gamma}\langle \gamma f(q) - Bq, y_n - q\rangle + \frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma}-\kappa\gamma)}M_5],\end{aligned} \tag{2.18}$$

where M_5 is an appropriate constant. On the other hand, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|P_C y_n - p\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2.\end{aligned} \tag{2.19}$$

Substitute (2.17) into (2.18) yields that

$$\begin{aligned}\|x_{n+1} - p\|^2 \leq & [1 - (1-\alpha_n)\frac{2\beta_n(\bar{\gamma}-\alpha\gamma)}{1-\beta_n\gamma\alpha}]\|x_n - q\|^2 \\ & + (1-\alpha_n)\frac{2\beta_n(\bar{\gamma}-\alpha\gamma)}{1-\beta_n\gamma\alpha}[\frac{1}{\bar{\gamma}-\alpha\gamma}\langle \gamma f(q) - Bq, y_n - q\rangle + \frac{\beta_n\bar{\gamma}^2}{2(\bar{\gamma}-\alpha\gamma)}M_5].\end{aligned} \tag{2.19}$$

Put $l_n = (1-\alpha_n)\frac{2\beta_n(\bar{\gamma}-\alpha_n\gamma)}{1-\beta_n\alpha\gamma}$ and

$$t_n = \frac{1}{\bar{\gamma}-\alpha\gamma}\langle \gamma f(q) - Bq, y_n - q\rangle + \frac{\beta_n\bar{\gamma}^2}{2(\bar{\gamma}-\alpha\gamma)}M_5.$$

That is,

$$\|x_{n+1} - q\|^2 \leq (1-l_n)\|x_n - q\|^2 + l_n t_n. \tag{2.21}$$

In view of (2.6) and (2.17), we see that

$$\limsup_{n\to\infty}\langle \gamma f(q) - Aq, y_n - q\rangle \leq 0. \tag{2.22}$$

It follows from the condition (i) and (2.22) that

$$\lim_{n\to\infty} l_n = 0, \sum_{n=1}^{\infty} l_n = \infty \text{ and } \limsup_{n\to\infty} t_n \leq 0.$$

Apply Lemma 1.4 to (2.21) to conclude $x_n \to q$ as $n \to \infty$. This completes the proof.

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References

[1] R.T. Rockafellar, On the maximality of sums of nonlinear monotone operators, Trans. Amer. Math. Soc. 149 (1970) 75-88.

[2] J.K. Kim, Strong convergence theorems by hybrid projection methods for equilibrium problems and fixed point problems of the asymptotically quasi-ϕ-nonexpansive mappings, Fixed Point Theory Appl. 2011 (2011), Article ID 10.

[3] L. Yang, Z. Fu, J.K. Kim, Hybrid projection method for generalized mixed equilibrium problem and fixed point problem of infinite family of asymptotically quasi-phi-nonexpansive mappings in Banach spaces, Appl. Math. Comput. 218 (2012), 6072-6082.

[4] Y. Shehu, Strong convergence theorems for infinite family of relatively quasi nonexpansive mappings and systems of equilibrium problems, Appl. Math. Comput. 218 (2012), 5146-5156.

[5] S. Thianwan, Strong convergence theorems by hybrid methods for a finite family of nonexpansive mappings and inverse-strongly monotone mappings, Nonlinear Anal. 3 (2009), 605-614.

[6] H. Zegeye, N. Shahzad, An algorithm for finding a common point of the solution set of a variational inequality and the fixed point set of a Bregman relatively nonexpansive mapping, Appl. Math. Comput. 248 (2014), 225-234.

[7] G. Marino, H.K. Xu, A general iterative method for nonexpansive mappings in Hilbert spaces, J. Math. Anal. Appl. 318 (2006), 43-52.

[8] F. Deutsch, I. Yamada, Minimizing certain convex functions over the intersection of the fixed point set of nonexpansive mappings, Numer. Funct. Anal. Optim. 19 (1998), 33-56.

[9] S.Y. Cho, S.M. Kang, Approximation of common solutions of variational inequalities via strict pseudocontractions, Acta Math. Sci. 32 (2012), 1607-1618.

[10] S.Y. Cho, X. Qin, S.M. Kang, Hybrid projection algorithms for treating common fixed points of a family of demicontinuous pseudocontractions, Appl. Math. Lett. 25 (2012), 854-857.

[11] X. Qin, S.S. Chang, Y.J. Cho, Iterative methods for generalized equilibrium problems and fixed point problems with applications, Nonlinear Anal. 11 (2010), 2963-2972.

[12] A.P. Chouhan, A.S. Ranadive, Absorbing maps and common fixed point theorem in Menger space, Adv. Fixed Point Theory 2 (2012), 108-119.

[13] J. Ye, J. Huang, Strong convergence theorems for fixed point problems and generalized equilibrium problems of three relatively quasi-nonexpansive mappings in Banach spaces, J. Math. Comput. Sci. 1 (2011), 1-18.

[14] S.S. Chang, H.W.J. Lee, C.K. Chan, A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization, Nonlinear Anal. 70 (2009), 3307-3319.

[15] F. Deutsch, H. Hundal, The rate of convergence of Dykstra's cyclic projections algorithm: the polyhedral case, Numer. Funct. Anal. Optim. 15 (1994), 537-565.

[16] H.H. Bauschke, The approximation of fixed points of compositions of nonexpansive mappings in Hilbert space, J. Math. Anal. Appl. 202 (1996), 150-159.

[17] K. Shimoji, W. Takahashi, Strong convergence to common fixed points of infinite nonexpansive mappings and applications, Taiwanese J. Math. 5 (2001), 387-404.

[18] W. Takahashi, Nonlinear Functional Analysis Yokohama-Publishers, 2000.

[19] T. Suzuki, Strong convergence of Krasnoselskii and Mann'ss type sequences for one-parameter nonexpansive semigroups without Bochner integrals, J. Math. Anal. Appl. 305 (2005), 227-239.