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STRUCTURE OF POSITIVE SOLUTION SETS FOR SECOND-ORDER STURM-LIOUVILLE SEMIPOSITONE PROBLEMS ON TIME SCALES

YANBIN SANG*, QIAN MENG

School of Science, North University of China, Taiyuan, Shanxi 030051, China

Abstract. We are concerned with second-order time scale superlinear and sublinear semipositone boundary value problems with Sturm-Liouville boundary conditions. We obtain the existence of unbounded connected component of positive solutions sets for above problems. The methods to show our main results are the global bifurcation theories.

Keywords. Time scales; Positive solutions; Semipositone; Unbounded connected component; Fixed point.

1. Introduction

In this paper, we are interested in structure of positive solutions sets for the following secondorder boundary value problem with Sturm-Liouville boundary conditions on time scales

$$(py^{\Delta})^{\nabla}(t)+\lambda f(t,y(t))=0,\ \ t\in(a,b]_{\mathbb{T}}, \tag{1.1}$$

$$\alpha y(a)-\beta(py^{\Delta})(a)=0,\ \ \gamma y^{\sigma}(b)+\delta(py^{\Delta})(b)=0, \tag{1.2}$$

where $\lambda>0$,

$$p:\ [a,\sigma(b)]_{\mathbb{T}}\longrightarrow(0,+\infty),\quad p\in C[a,\sigma(b)]_{\mathbb{T}}, \tag{1.3}$$

$$\beta,\ \delta\in(0,+\infty),\ \ \alpha,\ \gamma\in[0,+\infty),\ \ \beta\gamma+\alpha\delta+\alpha\gamma\int_a^{\sigma(b)}\frac{\Delta\tau}{p(\tau)}>0. \tag{1.4}$$

*Corresponding author

E-mail address: syb6662004@163.com (Y. Sang)

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We point out that the continuous function $f:(a,b]_{\mathbb{T}}\times[0,+\infty)\longrightarrow\mathbb{R}$ is semipositone, i.e., there exists $M\geq 0$ such that $f(t,y)\geq -M$ for all $t\in(a,b]_{\mathbb{T}}$ and $y\geq 0$. Some preliminary definitions and theorems on time scales can be seen in [1-4] which are excellent references for the calculus of time scales.

The study of dynamic equations on time scales goes back to its founder Hilger [5], and is a new area of still fairly theoretical exploration in mathematics. In particular, the theory is widely applied to biology, heat transfer and economic modelling, for details, see [5-11] and references therein. In recent years, there was much attention focused on the existence of positive solutions of second-order boundary value problems on time scales. We refer the readers to [1, 12-25] for some recent results. But very little work has been done on structure of positive solutions sets for semipositone problems on time scales. We would like to mention some results of Kaufmann and Kosmatov [13], Anderson and Wong [14], Sun and Li [15], Davidson and Rynne [16], Luo [17], which motivated us to consider the BVP (1.1) and (1.2).

In 2007, Kaufmann and Kosmatov [13] considered the following nonlinear Sturm-Liouville problem

$$-x^{\Delta\Delta}(t)=\lambda f(t,x(\sigma(t))),\ \ t\in(0,1]_{\mathbb{T}}, \tag{1.5}$$

$$\alpha x(0)-\beta x^{\Delta}(0)=0,\ \ \gamma x(\sigma(1))+\delta x^{\Delta}(\sigma(1))=0, \tag{1.6}$$

where $\alpha,\ \beta,\ \gamma,\ \delta\geq 0,\ \gamma\beta+\alpha\delta+\alpha\gamma\sigma(1)>0$. For the function f, the authors imposed the following hypotheses:

(H_1) $f:\mathbb{T}\times[0,+\infty)\longrightarrow\mathbb{R}$ is continuous, and there exists $M\geq 0$ such that $f(t,y)+M\geq 0$ on $\mathbb{T}\times[0,+\infty)$;

(H_2) $\lim\limits_{y\to\infty}\frac{f(t,y)}{y}=\infty$ holds uniformly on $[\mu,\nu]$, where $\mu=\min\{\tau\in\mathbb{T}:\ \tau\geq\frac{\sigma(1)}{4}\},\ \nu=\max\{\tau\in\mathbb{T}:\ \tau\leq\frac{3\sigma(1)}{4}\}$. The authors obtained the following theorem.

Theorem 1.1. (See [13]). *Assume that* (H_1)-(H_2) *are satisfied. Then, for a sufficiently small* $\lambda>0$*, the BVP* (1.5) *and* (1.6) *has at least one positive solution.*

We note that Anderson and Wong [14] have used the same conditions with (H_1) and (H_2) to study the solvability of semipositone BVP (1.1) and (1.2), and established the similar theorem (see Theorem 3.2 in [14]). Sun and Li [15] studied the following semipositone BVP

$$-x^{\Delta\Delta}(t) = f(t, x(\sigma(t))),\ \ t \in [0,T]_{\mathbb{T}}, \tag{1.7}$$

$$x(0) = 0 = x(\sigma^2(T)), \tag{1.8}$$

where $T > 0$ is fixed and $0,\ T \in \mathbb{T},\ f : [0,T]_{\mathbb{T}} \times [-\sigma(T)\sigma^2(T)M, +\infty) \longrightarrow [-M, +\infty)$ is continuous and $M > 0$ is constant. However, different from (H_1) and (H_2) , the conditions imposed on f here are local. In other words, the solvability of the BVP (1.7) and (1.8) only depends on ″height″ of f on some bounded sets of its domain, not the growth of f outside these bounded sets.

On the other hand, recently there has been growing interest in structure of solutions sets for dynamic equations on time scales by using global bifurcation theories. In 2002, Davidson and Rynne [16] investigated the BVP on time scales

$$-u^{\Delta\Delta}(t) + q(t)u^{\sigma}(t) = \lambda u^{\sigma}(t) + f(\lambda, t, u^{\sigma}(t)),\ \ t \in \mathbb{T}, \tag{1.9}$$

$$u(0) = u(1) = 0, \tag{1.10}$$

where $\lambda \in \mathbb{R}$ and the function $q : \mathbb{T} \longrightarrow \mathbb{R}$ and $f : \mathbb{R} \times \mathbb{T} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous, with $f(\lambda, t, x) = o(|x|)$ for x near 0, uniformly for λ in bounded subsets of $\mathbb{R}$. The authors have shown that unbounded continua of nontrivial solutions (λ, u) bifurcate from the trivial solutions at the eigenvalues of the linearization of (1.9) and (1.10).

Very recently, Luo [17] considered the existence of positive solutions for the following second order dynamic equation on a time scale

$$-x^{\Delta\Delta}(t) = f(t, x^{\sigma}(t)),\ \ t \in [0,T]_{\mathbb{T}}, \tag{1.11}$$

$$x(0) = 0 = x(\sigma^2(T)). \tag{1.12}$$

The author gave a global description of the branches of positive solutions for the problem (1.11) and (1.12).

We should point out that conditions ″the nonlinear terms in the BVP (1.9)-(1.10) and (1.11)-(1.12) are nonnegative ″ are crucial in obtaining global structures of positive solutions sets for above problems. In this work, by means of two known abstract theorems obtained in [26, 27], respectively, we will show some existence results for unbounded connected components of the positive solution sets to semipositone BVP (1.1) and (1.2). Other new result on global structure of semipositone time scales BVP, please see [25].

Let $\mathbb{T}$ be a time scale which has the subspace topology inherited from the standard topology on $\mathbb{R}$. For each interval I of $\mathbb{R}$, we define $I_{\mathbb{T}} = I \cap \mathbb{T}$.

2. Several lemmas and known abstract results

Let E be a real Banach space which is ordered by a normal cone P, that is, $x \le y$ if and only if $y - x \in P$. Let θ denote the zero element of the real Banach space E, $e \in P \backslash \{\theta\}$, $\|e\| \le 1$ and $Q = \{x \in P | x \ge \|x\| e\}$. It is easy to see that Q is also a cone of E.

Firstly, we consider the following sup-linear operator equation

$$x = \lambda KFx, \quad x \in P,$$

where $\lambda > 0$ is a parameter. Let

$$S(P) = \overline{\{(\lambda, x) \in [0, +\infty) \times P |\, x \ne \theta,\ x = \lambda Ax\}},$$

$$S(Q) = \overline{\{(\lambda, x) \in [0, +\infty) \times Q |\, x \ne \theta,\ x = \lambda Ax\}}.$$

The first abstract theorem used in our proof is the following.

Theorem 2.1. (See [26]). *Assume that the following conditions are satisfied*:

(A_1) $K : E \longmapsto E$ *is a linear completely continuous operator,* $K : P \longmapsto Q$*;* $F : P \longmapsto E$ *is a bounded and continuous operator*;

(A_2) *there exist* $g_0 \in P$ *and* $\sigma_0 \ge 0$ *such that* $g_1 =: Kg_0 \le \sigma_0 e$ *and* $Fx \ge -g_0$*, for all* $x \in P$;

(A_3) *there exists a linear completely continuous operator* $B : P \longmapsto P$ *such that* $Be \ge \theta$*, and* $Ax \ge Bx$ *for each* $x \in Q$ *whenever* $g_0 = \theta$;

(A_4) $\lim\limits_{x \in D, \|x\| \to \infty} \frac{\|KF(x)\|}{\|x\|} = \infty$*, where* $D = \{x \in E | x \ge \|x\| e/2\}$;

(A_5) $\lim\limits_{x \in D, \|x\| \to 0^+} \frac{\|KF(x)\|}{\|x\|} = \infty$*, when* $g_0 = \theta$;

Then

(a) $S(P)$ *possesses an unbounded connected component* C^* *which tends to* $(0, +\infty)$ *whenever* F *is semipositone*;

(b) *$S(Q)$ possesses an unbounded connected component C^* which comes from $(0,\theta)$ and tends to $(0,\infty)$ whenever F is positone.*

Secondly, we are concerned with the following sub-linear operator equation

$$x = \lambda KFx + e_0, \qquad x \in P,$$

where $\lambda > 0$ is a parameter. Let

$$\tilde{S}(P) = \overline{\{(\lambda, x) \in [0, +\infty) \times P |\ x \neq \theta,\ x = \lambda KFx + e_0\}}.$$

The second abstract result that we will utilize in our proof is the following.

Theorem 2.2. (See [27]). *Assume that the following conditions are satisfied*:

(B_1) *$K : P \longmapsto Q$ is a linear completely continuous operator, $F : P \longmapsto P$ is a bounded and continuous operator, $e_0 \in (-P)$*;

(B_2) *there exist $e_1 \in P$ and $\sigma_1 \geq 0$ such that $-e_0 \leq e_1 \leq \sigma_1 e$ and $e_0 + e_1 \in Q$*;

(B_3) $\lim\limits_{x\in Q, \|x\|\to\infty} \frac{\|KF(x-e_1)\|}{\|x\|} = 0.$

Then the following conclusions holds:

(i) *if $e_0 = \theta$, moreover suppose $\lim\limits_{x\in Q, \|x\|\to\theta} \frac{\|KF(x)\|}{\|x\|} = +\infty$, then $\tilde{S}(P)$ possesses an unbounded connected component C^* which comes from $(0,\theta)$ and $Prj_\lambda C^* = [0, +\infty)$, where $Prj_\lambda C$ denotes the projection of C on the λ-axis*;

(ii) *if $e_0 < \theta$, moreover suppose that there exist $\eta_0,\ \zeta_0 > 0$ such that $\|KFx\| \geq \eta_0$ whenever $x \geq \zeta_0 e$, then $\tilde{S}(P)$ possesses an unbounded connected component C^* which tends to (∞, ∞) and $Prj_\lambda C^* \supset [\lambda^*, +\infty)$, where λ^* is a positive number.*

To prove the main results in this paper, we will employ several lemmas. These lemmas are based on the linear equation

$$-(py^\Delta)^\nabla(t) = u(t),\ \ t \in (a,b]_{\mathbb{T}}, \tag{2.1}$$

with boundary conditions (1.2). Define the constant d via

$$d := \beta\gamma + \alpha\delta + \alpha\gamma \int_a^{\sigma(b)} \frac{\Delta\tau}{p(\tau)}. \tag{2.2}$$

Lemma 2.3. (See [14]). *Assume* (1.3) *and* (1.4). *Then the nonhomogeneous boundary value problem* (2.1), (1.2) *has a unique solution y for which the formula*

$$y(t) = \int_a^b G(t,s)u(s)\nabla s, \ \ t \in [a,\sigma(b)]_{\mathbb{T}}$$

holds, where the Green function $G(t,s)$ is given by

$$G(t,s) = \frac{1}{d}\begin{cases} \left(\beta+\alpha\int_a^s \frac{\Delta\tau}{p(\tau)}\right)\left(\delta+\gamma\int_t^{\sigma(b)}\frac{\Delta\tau}{p(\tau)}\right): \ a \le s \le t \le \sigma(b), \\ \left(\beta+\alpha\int_a^t \frac{\Delta\tau}{p(\tau)}\right)\left(\delta+\gamma\int_s^{\sigma(b)}\frac{\Delta\tau}{p(\tau)}\right): \ a \le t \le s \le \sigma(b), \end{cases} \tag{2.3}$$

for all $t,s \in [a,\sigma(b)]_{\mathbb{T}}$, where d is given in (2.2).

Lemma 2.4. (See [14]). *Assume* (1.4) *and* (1.5). *Then the Green function $G(t,s)$ in* (2.3) *satisfies*

$$g(t)G(s,s) \le G(t,s) \le G(s,s), \ \ t,s \in [a,\sigma(b)]_{\mathbb{T}},$$

where g is given by

$$g(t) = \min_{t\in[a,\sigma(b)]_{\mathbb{T}}}\left\{\frac{\delta+\gamma\int_t^{\sigma(b)}\frac{\Delta\tau}{p(\tau)}}{\delta+\gamma\int_a^{\sigma(b)}\frac{\Delta\tau}{p(\tau)}}, \frac{\beta+\alpha\int_a^t\frac{\Delta\tau}{p(\tau)}}{\beta+\alpha\int_a^{\sigma(b)}\frac{\Delta\tau}{p(\tau)}}\right\} \in [0,1]. \tag{2.4}$$

Let the Banach space $E = C[a,\sigma(b)]_{\mathbb{T}}$ be equipped with norm $\|u\| = \sup_{t\in[a,\sigma(b)]_{\mathbb{T}}} |u(t)|$. Define a cone $P \subset E$ as

$$P = \{u \in E | u = u(t) \ge 0 \ \ for \ \ t \in [a,\sigma(b)]_{\mathbb{T}}\}.$$

Then, P is a solid of E. Furthermore, we let

$$Q = \{u \in P | u = u(t) \ge g(t)\|u\| \ \ for \ \ t \in [a,\sigma(b)]_{\mathbb{T}}\}.$$

Then Q is also a cone of E.

3. Sup-linear BVP (1.1) and (1.2)

Theorem 3.1. *Let* $L = \overline{\{(\lambda,y)|\lambda \in [0,+\infty), \ y \neq \theta, \ y \in P, \ y \text{ is a solution of } (1.1) \text{ and } (1.2)\}}$. *Assume that the following conditions are satisfied*

(C_1) *there exists* $t_1, \ t_2 \in (a,b]_{\mathbb{T}}$ *such that*

$$\lim_{y\to\infty}\frac{f(t,y)}{y} = \infty$$

uniformly on $[t_1,t_2]_{\mathbb{T}}$;

(C_2) *when* $M=0$*, for each* $[\tilde{t}_1,\tilde{t}_2]_{\mathbb{T}}\subset(a,b]_{\mathbb{T}}$,

$$\lim_{y\to 0^+}\frac{f(t,y)}{y}=\infty$$

uniformly on $[\tilde{t}_1,\tilde{t}_2]_{\mathbb{T}}$;

(C_3) *when* $M=0$*, there exists* $\omega(t)\geq 0$ *such that* $f(t,y)\geq\omega(t)y$ *for all* $(t,y)\in(a,b]_{\mathbb{T}}\times[0,+\infty)$.

Then

(i) L *possesses an unbounded connected component* C^* *which tends to* $(0,\infty)$ *when* $M>0$;

(ii) L *possesses an unbounded connected component* C^* *which comes from* $(0,\theta)$ *and tends to* $(0,+\infty)$ *when* $M=0$.

Proof. It is known that the problem (1.1) and (1.2) are equivalent to the fixed point equation

$$u(t)=\lambda KFu(t),\quad t\in[a,\sigma(b)]_{\mathbb{T}},$$

where the operator $K:E\longrightarrow E$ and $F:P\longrightarrow E$ are defined by

$$(Fu)(t)=f(t,u(t)),\qquad t\in[a,\sigma(b)]_{\mathbb{T}},\ \ u\in P,$$

and

$$(Ku)(t)=\int_a^b G(t,s)u(s)\nabla s,\qquad t\in[a,\sigma(b)]_{\mathbb{T}},\ \ u\in E.$$

Let $g_0(t)=M$ (for all $t\in[a,\sigma(b)]_{\mathbb{T}}$) and

$$g_1(t)=M\int_a^b G(t,s)\nabla s,\quad t\in[a,\sigma(b)]_{\mathbb{T}}.$$

By Lemma 2.4, we get that $K:P\longrightarrow Q$ is a linear completely operator. Furthermore, in virtue of Lemma 3.1 in [14], we have that there exists $\sigma_0>0$ such that $g_1\leq\sigma_0 g(t)$. By (C_3), we can know that (A_3) holds. By (C_1), for each $X>0$, there exists $G_0>0$ such that

$$f(t,y)\geq Xy\quad for\ any\ y\geq G_0\ \ and\ \ t\in[t_1,t_2]_{\mathbb{T}}.$$

Set $G_1=2G_0(\min_{t\in[t_1,t_2]_{\mathbb{T}}}g(t))^{-1}$. Therefore, we get for each $y\in D$ with $\|y\|\geq G_1$,

$$y(t)\geq\frac{1}{2}g(t)\|y\|\geq\frac{1}{2}G_1\min_{t\in[t_1,t_2]_{\mathbb{T}}}g(t)=G_0,\tag{3.1}$$

and so

$$f(t,y(t)) \geq Xy(t), \quad \textit{for all } t \in [t_1,t_2]_{\mathbb{T}}. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\begin{aligned}(KFy)(t) &= \textstyle\int_a^b G(t,s)[f(s,y(s))+M]\nabla s - g_1(t) \\ &\geq \textstyle\int_{t_1}^{t_2} G(t,s)[f(s,y(s))+M]\nabla s - \|g_1\| \\ &\geq X \textstyle\int_{t_1}^{t_2} G(t,s)y(s)\nabla s - \|g_1\| \\ &\geq \tfrac{1}{2}X \textstyle\int_{t_1}^{t_2} G(t,s)g(s)\nabla s\|y\| - \|g_1\|,\end{aligned}$$

for all $t \in [a,\sigma(b)]_{\mathbb{T}}$. Thus, for $y \in D$ with $\|y\| \geq G_1$, we obtain

$$\|KFy\| \geq \frac{1}{2}X \int_{t_1}^{t_2} G(t,s)g(s)\nabla s\|y\| - \|g_1\|,$$

for all $t \in [t_1,t_2]$. Consequently,

$$\frac{\|KFy\|}{\|y\|} \geq \frac{1}{2}X \min_{t\in[t_1,t_2]} \int_{t_1}^{t_2} G(t,s)g(s)\nabla s\|y\| - \frac{\|g_1\|}{\|y\|},$$

and noting $\frac{\|g_1\|}{\|y\|} \longrightarrow 0$ as $\|y\| \longrightarrow \infty$, we have

$$\lim_{y\in D,\|y\|\to\infty} \frac{\|KFy\|}{\|y\|} = \infty.$$

This implies that (A_4) holds.

Corollary 3.1. *Assume that all conditions of Theorem* 3.1 *hold. Then*

(a) *when $M > 0$, there exists $\lambda^* > 0$ such that* (1.1), (1.2) *has at least one positive solution for* $0 < \lambda < \lambda^*$;

(b) *when $M = 0$, there exists $\lambda^* > 0$ such that* (1.1), (1.2) *has at least two positive solutions $x_\lambda^{(1)}$ and $x_\lambda^{(2)}$ for $0 < \lambda < \lambda^*$, $\|x_\lambda^{(1)}\| \leq 1 \leq \|x_\lambda^{(2)}\|$ and $\|x_\lambda^{(1)}\| \to 0$, $\|x_\lambda^{(2)}\| \to \infty$ as $\lambda \to 0^+$.*

4. Sub-linear BVP (1.1) and (1.2)

To begin, we list some assumptions which we shall use later as follows:

(D_1) $\lim\limits_{y\to+\infty} \frac{f(t,y)}{y} = 0$ uniformly on $(a,b]_{\mathbb{T}}$;

(D_2) there exists $[t_1,t_2]_{\mathbb{T}} \subset (a,b]_{\mathbb{T}}$ such that

$$\lim_{y\to 0^+} \frac{f(t,y)}{y} = \infty$$

uniformly on $[t_1,t_2]_{\mathbb{T}}$;

(D_3) there exist $[\tilde{t}_1,\tilde{t}_2]_{\mathbb{T}} \subset (a,b]_{\mathbb{T}}$ and positive numbers ξ and η such that $f(t,y)+M \geq \xi$ for all $(t,y) \in [\tilde{t}_1,\tilde{t}_2]_{\mathbb{T}} \times [\eta,+\infty)$.

Theorem 4.1. *Let* $L = \overline{\{(\lambda,y)|\lambda \in [0,+\infty),\ y \neq \theta,\ y \in Q,\ y \text{ is a solution of } (1.1) \text{ and } (1.2)\}}$. *Assume that* (D_1) *is satisfied. Then the following conclusions hold*
(i) *if* $M \equiv 0$*, and* (D_2) *holds, then* L *possesses an unbounded connected component* C^* *which comes from* $(0,\theta)$ *and* $Prj_\lambda C^* = [0,+\infty)$;
(ii) *if* $M > 0$*, and* (D_3) *holds, then* L *possesses an unbounded connected component* C^* *which tends to* (∞,∞) *and* $Prj_\lambda C^* \supset [\lambda^*,+\infty)$*, where* λ^* *is a positive number.*

Proof. From the proof of Theorem 3.1, it is easy to check that (B_1) and (B_2) hold. In view of Krein-Rutman Theorem [28], there exist $\varphi \in P\{\theta\}$ and $h \in P^*\{\theta\}$ such that

$$K\varphi = r(K)\varphi, \quad K^*h = r(K)h.$$

It is easy to see that $\varphi \in Q$ and h can be taken in the form

$$h(y) = \int_a^b \varphi(t)y(t)\nabla t, \quad for\ all\ t \in E.$$

If $M \equiv 0$, then we have $e_0 = \theta$. By (D_2), for each $X > 0$, there exists $\delta > 0$ such that for each $0 < y \leq \delta$,

$$f(t,y) \geq Xy, \quad t \in [t_1,t_2]_{\mathbb{T}}.$$

Thus, we get for each $y \in Q$ with $0 < \|y\| \leq \delta$

$$\frac{h(F(y))}{h(y)} = \frac{\int_a^b \varphi(s)f(s,y(s))\nabla s}{\int_a^b \varphi(s)y(s)\nabla s} \geq \frac{X\int_{t_1}^{t_2} \varphi(s)y(s)\nabla s}{\int_a^b \varphi(s)y(s)\nabla s} \geq \frac{X\int_{t_1}^{t_2} \varphi(s)g(s)\nabla s}{\int_a^b \varphi(s)\nabla s}.$$

Since X is arbitrarily given, then we have when $e_0 = \theta$,

$$\lim_{y\in Q,\|y\|\to 0} \frac{h(F(y))}{h(y)} = +\infty.$$

Therefore, condition of conclusion (i) in Theorem 2.2 holds when $M \equiv 0$.

If $M \not\equiv 0$, we let $\eta_0 = [\min_{t\in[\tilde{t}_1,\tilde{t}_2]_{\mathbb{T}}} g(t)]^{-1}\eta$ and $\xi_0 = \xi \int_{\tilde{t}_1}^{\tilde{t}_2} \varphi(t)\nabla t$. For each $y \geq \eta_0 g(t)$, we have

$$y(t) \geq \eta_0 \min_{t\in[\tilde{t}_1,\tilde{t}_2]_{\mathbb{T}}} g(t) = \eta \quad \text{for all } t \in [\tilde{t}_1,\tilde{t}_2]_{\mathbb{T}}.$$

By (D_3), we find for each $y \geq \eta_0 g(t)$, $f(t,y(t)) + M \geq \xi$ for all $t \in [\tilde{t}_1,\tilde{t}_2]_{\mathbb{T}}$. Consequently, we have for each $y \geq \eta_0 g(t)$,

$$h(F(y)) = \int_a^b \varphi(s)[f(s,y(s)) + M]\nabla s \geq \int_{\tilde{t}_1}^{\tilde{t}_2} \varphi(s)[f(s,y(s)) + M]\nabla s \geq \xi \int_{\tilde{t}_1}^{\tilde{t}_2} \varphi(s)\nabla s = \xi_0,$$

which implies that assumption of conclusion (ii) in Theorem 2.2 holds when $M \not\equiv 0$.

In the following, we will prove that (B_3) of Theorem 2.2 holds. For each $\varepsilon > 0$, in virtue of (D_1), there exists $X_0 > 0$ such that for each $y > X_0$

$$f(t,y) + M \leq \varepsilon y \quad \text{for all } t \in (a,b]_{\mathbb{T}}. \tag{4.1}$$

Set $\overline{X}_0 = \max_{t\in(a,b]_{\mathbb{T}}, y\in[0,X_0]} f(t,y)$. By (4.1), we have

$$f(t,y) + M \leq \varepsilon y + \overline{X}_0, \quad \text{for all } t \in (a,b]_{\mathbb{T}} \text{ and } y \in [0,+\infty).$$

Thus, for each $y \in Q$ with $\|y\| \geq \max\{\sigma_0, \frac{\overline{X}_0}{\varepsilon}\} + 1$,

$$\begin{aligned}
\frac{h(F(y-g_1)+M)}{h(y)} &= \frac{\int_a^b \varphi(s)[f(s,y(s)-g_1(s))+M]\nabla s}{\int_a^b \varphi(s)y(s)\nabla s} \\
&\leq \frac{\varepsilon \int_a^b \varphi(s)[y(s)-g_1(s)]\nabla s + \overline{X}_0 \int_a^b \varphi(s)\nabla s}{\|y\| \int_a^b \varphi(s)g(s)\nabla s} \\
&\leq \frac{\varepsilon \int_a^b \varphi(s)\nabla s}{\int_a^b \varphi(s)g(s)\nabla s} + \frac{\overline{X}_0 \int_a^b \varphi(s)\nabla s}{\|y\| \int_a^b \varphi(s)g(s)\nabla s} \\
&\leq \frac{2\varepsilon \int_a^b \varphi(s)\nabla s}{\int_a^b \varphi(s)g(s)\nabla s}.
\end{aligned}$$

The result follows.

Corollary 4.1. *Assume that (D_1) is satisfied. Then the following conclusions hold*

(i) *when $M \equiv 0$, and (D_2) holds, then the BVP* (1.1) *and* (1.2) *has at least one positive solution for each $\lambda > 0$;*

(ii) *when $M > 0$, and (D_3) holds, then there exists $\lambda^* > 0$ such that the BVP* (1.1) *and* (1.2) *has at least one positive solution for each $\lambda \geq \lambda^*$.*

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