J. Nonlinear Funct. Anal. 2015 (2015), Article ID 17

MATHRES
Journal of Nonlinear Functional Analysis
Available online at http://jnfa.mathres.org

IMPLICIT HYBRID PROJECTION ALGORITHMS FOR COMMON FIXED POINTS OF A DEMICONTINUOUS SEMIGROUP OF PSEUDOCONTRACTIONS IN HILBERT SPACES

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Abstract. In this paper, the implicit hybrid projection algorithm is considered for dealing with a demicontinuous semigroup of pseudocontractions. A strong convergence theorem of fixed points is established in the framework of real Hilbert spaces.

Keywords. Implicit hybrid projection algorithm; Demicontinuous semigroup of pseudocontractions; Fixed point; Banach space.

1. Introduction and preliminaries

Fixed point theory as an important branch of nonlinear analysis theory has been applied to solve real world problems. The study of fixed points of nonlinear mappings and their approximation algorithms constitute a topic of intensive research efforts. Many well known problems arising in various branches of science can be studied by using iteration algorithms. Since Mann iteration algorithm was introduced, the Mann iteration algorithm for nonlinear operator's fixed points has been studied by many authors. Notice that no matter implicit Mann algorithms or explicit Mann algorithms has only weak convergence, even for nonexpansive operator, see [1]. In order to obtain the strong convergence results of nonlinear operators, many authors started

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Received June 24, 2015

to modify Mann iteration algorithm; for examples, viscosity approximation method and hybrid projection method; see [2-8].

More recently, implicit viscosity approximation method for nonlinear operators has been frequently studied; see [4]. In this paper, by another way, a implicit projection algorithm is considered for treating common fixed points of a demicontinuous semigroup of pseudocontractions. Strong convergence theorems are established in the framework of real Hilbert spaces.

At first, we recall some necessary knowledge which is useful for our main result.

Let $\mathscr{H}$ be a real Hibert space with inner product $\langle \cdot,\cdot\rangle$ and induced norm $\|\cdot\|$ and let $T : C \to C$ be a nonlinear mapping, where C is a closed convex subset of $\mathscr{H}$. The set of fixed points of T is denoted by $\mathscr{F}(T)$; that is, $\mathscr{F}(T) := \{x \in C : Tx = x\}$. We remark that $\to$ and $\rightharpoonup$ denote strong and weak convergences, respectively. Let us introduce some types of operators involved in our study.

Definition 1.1. T is said to be demicontinuous on C, if $\{x_n\} \subset C$ and $x_n \to x \in C$ together imply $Tx_n \rightharpoonup Tx$.

Definition 1.2. A mapping T is said to be

(a_1) Nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall\, x,\ y \in C;$$

(a_2) Strictly pseudocontractive if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall\, x,\ y \in C;$$

(a_3) Pseudocontractive if

$$\langle Tx - Ty, x - y\rangle \le \|x - y\|^2, \quad \forall\, x,\ y \in C; \tag{1.1}$$

(a_4) Strongly pseudocontractive if there exists a constant $\alpha \in (0,1)$ such that

$$\langle Tx - Ty, x - y\rangle \le \alpha\|x - y\|^2, \quad \forall\, x,\ y \in C;$$

(a_5) $\alpha-$dissipative with $\alpha \in \mathbb{R}$ (see [9]) if

$$\langle Tx - Ty, x - y\rangle \le \alpha\|x - y\|^2, \quad \forall\, x,\ y \in C.$$

Remark 1.3. (1) It is very clear that, in a real Hilbert space $\mathscr{H}$, (1.1) is equivalent to

$$\|Tx-Ty\|^2 \leq \|x-y\|^2 + \|(I-T)x-(I-T)y\|^2, \quad \forall\, x,\ y \in C.$$

(2) the class of strict pseudocontractive mappings falls into the one between classes of nonexpansive mappings and pseudocontractive mappings; and the class of strongly pseudocontractive mappings is independent of the class of strict pseudocontractive ones.

(3) The class of strongly pseudocontractive mapping is a subset of the ones of $\alpha-$dissipative mapping with $\alpha \in \mathbb{R}$.

In the following, we give some examples for strict pseudocontractive mapping, strongly pseudocontractive mapping and pseudocontractive mapping, which were cited in [10].

(i) *Take* $C=(0,\infty)$ *and define* $T: C \to C$ *by*

$$Tx = \frac{x^2}{1+x}$$

Then T *is a strict pseudocontractive mapping but not a strongly pseudocontractive one.*

(ii) *Take* $C=\mathbb{R}^1$ *and define* $T: C \to C$ *by*

$$T(x) = \begin{cases} 1, & \text{if } x \in (-\infty,-1), \\ \sqrt{1-(1+x)^2}, & \text{if } x \in [-1,0), \\ -\sqrt{1-(x-1)^2}, & \text{if } x \in [0,1], \\ -1, & \text{if } x \in (1,+\infty). \end{cases}$$

Then T *is a strongly pseudocontractive mapping but not a strict pseudocontractive one.*

(iii) *Take* $\mathscr{H}=\mathbb{R}^2$ *and* $B=\{x\in\mathbb{R}^2 : \|x\| \leq 1\}$, $B_2=\{x\in B : \|x\|\leq \frac{1}{2}\}$, $B_2=\{x\in B : \frac{1}{2} \leq \|x\| \leq 1\}$. *If* $x \in (a,b) \in \mathscr{H}$, *we define* $x^{\perp}$ *to be* $(b,-a)\in\mathscr{H}$. *Define* $T: B \to B$ *by*

$$T(x) = \begin{cases} x+x^{\perp}, & \text{if } x \in B_1, \\ \frac{x}{\|x\|}-x+x^{\perp}, & \text{if } x \in B_2. \end{cases}$$

Then T *is a Lipschitz pseudocontractive mapping but not a strict pseudocontractive one.*

Definition 1.4. A pseudocontraction semigroup is a family $\mathfrak{F} := \{T(t) : t \geq 0\}$ of self-mappings of C such that

(b_1) $T(0)x = x$ for all $x \in C$;

(b_2) $T(s+t) = T(s)T(t)$ for all $s,\ t \geq 0$;

(b_3) $\lim_{t\to 0^+} T(t)x = x$ for all $x \in C$;

(b_4) for each $t > 0$, $T(t)$ is pseudocontractive; that is,

$$\langle T(t)x - T(t)y, j(x-y)\rangle \leq \|x-y\|^2, \quad \forall\, x,\ y \in C.$$

Obviously, the class of pseudocontractive semigroups includes the class of nonexpansive semigroup as a special case. In this paper, we use $\mathscr{F}$ to denote the set of common fixed points of $\mathfrak{F}$; that is,

$$\mathscr{F} := \{x \in C : T(t)x = x, \quad t > 0\} = \bigcap_{t>0} F(T(t)).$$

In order to prove our main result, we also need the following lemmas.

Lemma 1.5. *Let $\mathscr{H}$ be a real Hilbert space. Then the following equations hold:*

(c_1) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$ *for all* $x,\ y \in \mathscr{H}$;

(c_2) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$ *for all* $t \in [0,1]$;

(c_3) *If* $\{x_n\}$ *is a sequence in H weakly convergent to z, then*

$$\limsup_{n\to\infty} \|x_n - y\| = \limsup_{n\to\infty} \|x_n - z\| + \|y - z\| \ for\ all\ y \in \mathscr{H}.$$

Lemma 1.6. *Let C be a closed convex subset of a real Hilbert space $\mathscr{H}$. and let P_C be the metric projection from $\mathscr{H}$ onto C (i.e., for $x \in \mathscr{H}$, P_C is the only point in C such that $\|x - P_C x\| =$ $\inf\{\|x-z\| : z \in C\}$). Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if there holds the relations:*

$$\langle x - z, y - z\rangle \leq 0 \ for\ all\ y \in C.$$

Recall that a mapping $T : D \subset K \to \mathscr{H}$ is said to be weakly inward (relative to K) if $Tx \in \bar{I}_K(x)$ for $x \in D$, where $\bar{I}_K(x)$ is the closure of the inward set $\bar{I}_K(x) := \{x + c(z-x) : z \in K \ and\ c \geq 1\}$, see [11-13] for more details.

Lemma 1.7. (see [9]) *Let C be a closed convex subset of $\mathscr{H}$. Assume that $T : C \to \mathscr{H}$ is a demicontinuous weakly inward α-dissipative mapping with $\alpha < 1$. Then T has a unique fixed point in C.*

We remark here that if the demicontinuous weakly inward $\alpha-$dissipative mapping T is replaced by a demicontinuous and strongly psedocontractive operator (or demicontinuous psedocontractive operator), the result of Lemma 1.7 is also held from Definition 1.2, see [9, 14].

Lemma 1.8. (see [10]) *Let C be a nonempty closed convex subset of a real Hilbert space $\mathscr{H}$ and $T : C \to C$ be a demicontinuous pseudocontractive mapping. Then $\mathscr{F}(T)$ is closed convex subset of C and $I-T$ is demiclosed at 0.*

2. Main results

In this section, we shall introduce our implicit hybrid projection algorithm for treating common fixed points of a demicontinuous semigroup of pseudocontractions.

Theorem 2.1. Let C be a nonempty closed and convex subset of a real Hilbert space $\mathscr{H}$. Let $\mathfrak{T} := \{T(t) : t \geq 0\}$ be a demicontinuous semigroup of pseudocontractions on C with $\mathscr{F} := \bigcap_{t\geq 0} \mathscr{F}(T(t)) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following iterative process:

$$\begin{cases} x_0 \in H \text{ chosen arbitrarily}, \\ C_1(t) = C,\ C_1 = \bigcap_{t\geq 0} C_1(t), x_1 = P_{C_1} x_0, \\ y_n(t) = \alpha_n(t)x_n + (1-\alpha_n(t))T(t)y_n(t) + \alpha_n(t)e_n(t), \cdots (*) \\ C_{n+1}(t) = \{z \in C_n(t) : \|y_n(t) - z\| \leq \|x_n - z\| + \|e_n(t)\|\}, \\ C_{n+1} = \bigcap_{t\geq 0} C_{n+1}(t), \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases} \tag{2.1}$$

where, for $t \geq 0$, $\{e_n(t)\}$ is an error sequence satisfying the condition $\lim_{n\to\infty} \|e_n(t)\| = 0$. Assume that the control sequence $\{\alpha_n(t)\}$ in [0,1] satisfies the condition $\limsup_{n\to\infty} \alpha_n(t) < 1$ for all $t \geq 0$. Then $\{x_n\}$ converges strongly to $P_{\mathscr{F}} x_0$.

Proof. We will prove the result in several steps.

Step 1. we show that the equation $(*)$ in (2.1) is well defined.

For each $n \geq 0$, defined a mapping $T_n : C \to C$ as follows:

$$T_n x = \alpha_n(t)x_n + (1-\alpha_n(t))T(t)x + \alpha_n(t)e_n(t).$$

We see that T_n is a demi-continuous and strong psedocontraction. Indeed, for every $x,\ y \in C$, we have

$$\begin{aligned}\langle T_n x - T_n y, j(x-y)\rangle &= (1-\alpha_n(t))\langle T(t)x - T(t)y, j(x-y)\rangle \\ &\leq (1-\alpha_n(t))\|x-y\|^2.\end{aligned}$$

From Lemma 1.7, we see that T_n has a unique fixed point. This show that the equation $(*)$ in (2.1) is well defined.

Step 2. we show that C_n is closed and convex for each $n \geq 1$.

Obviously, C_n is closed for each $n \geq 1$. We observe that C_n is convex. Indeed, let $z_1,\ z_2 \in C_{n+1}(t)$ for each $n \geq 0$ and $t \geq 0$. Take $z = \alpha z_1 + (1-\alpha)z_2$ for $\alpha \in (0,1)$. Notice that

$$\|y_n(t) - z_1\| - \|x_n - z_1\| \leq \|e_n(t)\| \tag{2.2}$$

and

$$\|y_n(t) - z_2\| - \|x_n - z_2\| \leq \|e_n(t)\|. \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$\begin{aligned}\|y_n(t) - z\| - \|x_n - z\| &= \|y_n(t) - \alpha z_1 - (1-\alpha)z_2\| \\ &\quad - \|x_n - \alpha z_1 - (1-\alpha)z_2\| \\ &\leq \alpha\|y_n(t) - z_1\| + (1-\alpha)\|y_n(t) - z_2\| \\ &\quad - \alpha\|x_n - z_1\| - (1-\alpha)\|x_n - z_2\| \\ &\leq \|e_n(t)\|.\end{aligned}$$

This shows that $C_{n+1}(t)$ is convex for each $n \geq 0$ and $t \geq 0$, therefore, $C_{n+1} = \bigcap_{t\geq 0} C_{n+1}(t)$ is also convex for each $n \geq 0$ and $t \geq 0$. That is, C_n is convex for all $n \geq 1$.

Step 3. we show that $\mathscr{F} \subset C_n$ for all $n \geq 1$.

Indeed, from Lemma 1.5 and Definition 1.4, $\forall\, p \in \mathscr{F}$, one has

$$\begin{aligned}
\|y_n(t)-p\|^2 &= \langle \alpha_n(t)x_n + (1-\alpha_n(t))T(t)y_n(t) + \alpha_n(t)e_n(t) - p,\\
&\quad y_n(t)-p\rangle\\
&= \alpha_n(t)\langle x_n - p, y_n(t)-p\rangle + (1-\alpha_n(t))\langle T(t)y_n(t)-p,\\
&\quad y_n(t)-p\rangle + \alpha_n(t)\langle e_n(t), y_n(t)-p\rangle\\
&\le \alpha_n(t)\|x_n-p\|\|y_n(t)-p\| + (1-\alpha_n(t))\|y_n(t)-p\|^2\\
&\quad + \alpha_n(t)\|e_n(t)\|\|y_n(t)-p\|.
\end{aligned}$$

It follows that

$$\|y_n(t)-p\| \le \|x_n-p\| + \|e_n(t)\|$$

This shows that $\mathscr{F} \subset C_{n+1}(t)$ for all $n \ge 0$ and $t \ge 0$. Therefore, $\mathscr{F} \subset C_{n+1}$ for all $n \ge 0$, that is, $\mathscr{F} \subset C_n$ for all $n \ge 1$.

Step 4. we prove that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$.

Since $x_n = P_{C_n}x_0$, we see that

$$\langle x_0 - x_n, x_n - x\rangle \ge 0, \quad \forall\, x \in C_n. \tag{2.4}$$

and since $\mathscr{F} \subset C_n$, we also have that

$$\langle x_0 - x_n, x_n - w\rangle \ge 0, \quad \forall\, w \in \mathscr{F}. \tag{2.5}$$

It follows that

$$\begin{aligned}
0 &\le \langle x_0 - x_n, x_n - P_{\mathscr{F}}x_0\rangle\\
&= \langle x_0 - x_n, x_n - x_0 + x_0 - P_{\mathscr{F}}x_0\rangle\\
&\le -\|x_0-x_n\|^2 + \|x_0-x_n\|\|x_0-x_n\|\|x_0-P_{\mathscr{F}}x_0\|.
\end{aligned}$$

This shows that

$$\|x_0 - x_n\| \le \|x_0 - P_{\mathscr{F}}x_0\|. \tag{2.6}$$

This implies that $\{x_n\}$ is bounded. In view of the construction of C_n, we see that $C_{n+1} \subset C_n$ and $x_{n+1} \in C_n$. It follows from (2.6) that

$$\|x_0 - x_n\| \le \|x_0 - x_{n+1}\|.$$

From the boundedness property of $\{x_n\}$, we find that $\lim_{n\to\infty}\|x_0-x_n\|$ exists. It follows from (2.4) that

$$\langle x_0-x_n, x_n-x_{n+1}\rangle \geq 0.$$

It follows that

$$\begin{aligned}\|x_n-x_{n+1}\|^2 &= \|x_n-x_0\|^2+\|x_0-x_{n+1}\|^2-2\langle x_0-x_n, x_0-x_{n+1}\rangle\\ &= \|x_0-x_{n+1}\|^2-\|x_n-x_0\|^2-\langle x_0-x_n, x_n-x_{n+1}\rangle\\ &\leq \|x_0-x_{n+1}\|^2-\|x_n-x_0\|^2.\end{aligned}$$

Letting $n\to\infty$, we see that

$$\lim_{n\to\infty}\|x_n-x_{n+1}\|=0.$$

This completes the proof of Step 4.

Step 5. we prove that $\lim_{n\to\infty}\|y_n(t)-T(t)y_n(t)\|=0$.

In view of $x_{n+1}\in C_{n+1}$, one has $x_{n+1}\in C_{n+1}(t)$ for all $t\geq 0$. From the construction of $C_{n+1}(t)$, one has

$$\|y_n(t)-x_{n+1}\|\leq \|x_n-x_{n+1}\|+\|e_n(t)\|.$$

Since $\lim_{n\to\infty} e_n(t)=0$ for all $t\geq 0$ and Step 4, one obtains that

$$\lim_{n\to\infty}\|y_n(t)-x_{n+1}\|=0.$$

It follows that

$$\begin{aligned}\lim_{n\to\infty}\|y_n(t)-x_n\| &= \lim_{n\to\infty}\|y_n(t)-x_{n+1}\|\\ &\quad+\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.\end{aligned} \tag{2.7}$$

On the other hand, from $(*)$ in (2.1), one has

$$y_n(t)-x_n=(1-\alpha_n(t))(T(t)y_n(t)-x_n)+\alpha_n(t)e_n(t).$$

from $\lim_{n\to\infty}\|e_n(t)\|=0$ and $\limsup_{n\to\infty}\alpha_n(t)<1$ for all $t\geq 0$ and (2.7), one obtains that

$$\|x_n-T(t)y_n(t)\|\to 0, \quad as\ n\to\infty.$$

It follows that $\lim_{n\to\infty}\|y_n(t)-T(t)y_n(t)\|=0$ for all $t\geq 0$.

Step 6. we prove that the iterative sequence $\{x_n\}$ converges weakly to $P_{\mathscr{F}}x_0$.

Notice that $\{x_n\}$ and $\{y_n(t)\}$ are all bounded. In view of Step 5 and (2.7), we obtain from Lemma 1.8 that $\emptyset \neq \omega_\omega(x_n) \subset \mathscr{F}$, where $\omega_\omega(x)$ denotes the weak $\omega-$limit set of the sequence $\{x_n\}$. In view of the weak lower semicontinuity of the norm, we obtain from (2.6) that

$$\|x_0 - p\| \leq \|x_0 - P_{\mathscr{F}} x_0\|, \quad \forall\, p \in \omega_\omega(x_n).$$

Since $\omega_\omega(x_n) \subset \mathscr{F}$, we arrive at $p = P_{\mathscr{F}} x_0$, which in turn implies that $\omega_\omega(x_n) = \{P_{\mathscr{F}} x_0\}$. It follows that $\{x_n\}$ converges weakly to $p = P_{\mathscr{F}} x_0$.

Step 7. we show that the iterative sequence $\{x_n\}$ converges strongly to $p = P_{\mathscr{F}} x_0$.

In view of the weak lower semicontinuity of the norm, we obtain from (2.6) that

$$\|x_0 - P_{\mathscr{F}} x_0\| \leq \liminf_{n\to\infty} \|x_0 - x_n\| \leq \limsup_{n\to\infty} \|x_0 - x_n\| \leq \|x_0 - P_{\mathscr{F}} x_0\|,$$

which yields that $\lim_{n\to\infty} \|x_0 - x_n\| = \|x_0 - P_{\mathscr{F}} x_0\|$. It follows that $\{x_n\}$ converges strongly to $P_{\mathscr{F}} x_0$. This completes the Proof.

If we consider a single demicontinuous pseudocontractive operator T, we can obtain from Theorem 2.1 directly.

Corollary 2.2. *Let C be a nonempty closed and convex subset of a real Hilbert space $\mathscr{H}$. Let $T : C \to C$ be a demicontinuous pseudocontraction with $\mathscr{F}(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following iterative process:*

$$\begin{cases} x_0 \in H \text{ chosen arbitrarily}, \\ C_1 = C,\ x_1 = P_{C_1} x_0, \\ y_n = \alpha_n x_n + (1-\alpha_n) T y_n + \alpha_n e_n, \\ C_{n+1} = \{z \in C_n(t) : \|y_n - z\| \leq \|x_n - z\| + \|e_n\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases}$$

where $\{e_n\}$ is an error sequence satisfying the condition $\lim_{n\to\infty} \|e_n\| = 0$. Assume that the control sequences $\{\alpha_n\}$ in [0,1] satisfies the condition $\limsup_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ converges strongly to $P_{\mathscr{F}} x_0$.

Considering $e_n(t) \equiv 0$ in (2.1) for all $t \geq 0$, the following corollary can be implied using Theorem 2.1.

Corollary 2.3. *Let C be a nonempty closed and convex subset of a real Hilbert space $\mathscr{H}$. Let $\mathfrak{T} := \{T(t) : t \geq 0\}$ be a demicontinuous semigroup of pseudocontractions on C with $\mathscr{F} := \bigcap_{t\geq 0} \mathscr{F}(T(t)) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following iterative process:*

$$\begin{cases} x_0 \in H \text{ chosen arbitrarily}, \\ C_1(t) = C, \\ C_1 = \bigcap_{t\geq 0} C_1(t) \\ x_1 = P_{C_1} x_0, \\ y_n(t) = \alpha_n(t) x_n + (1-\alpha_n(t)) T(t) y_n(t), \\ C_{n+1}(t) = \{z \in C_n(t) : \|y_n(t) - z\| \leq \|x_n - z\|\}, \\ C_{n+1} = \bigcap_{t\geq 0} C_{n+1}(t), \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases}$$

assume that the control sequence $\{\alpha_n(t)\}$ in [0,1] satisfies the condition $\limsup_{n\to\infty} \alpha_n(t) < 1$ for all $t \geq 0$. Then $\{x_n\}$ converges strongly to $P_{\mathscr{F}} x_0$.

Remark 2.4. Referring to [15], one can easily see that

(1) Theorem 2.1 which relates to error sequence $\{e_n(t)\}$, is more general than Theorem 3.1 in [15];

(2) Corollary 2.3 is similar to Theorem 3.1 in [15] for treating strongly continuous semigroups of demicontinuous pseudocontractions without error term.

If we consider a single demicontinuous pseudocontractive operator T in Corollary 2.3, one can obtain the following corollary by applying Corollary 2.3.

Corollary 2.5. *Let C be a nonempty closed and convex subset of a real Hilbert space $\mathscr{H}$. Let $T : C \to C$ be a demicontinuous pseudocontraction with $\mathscr{F}(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence* *generated in the following iterative process:*

$$\begin{cases} x_0 \in H \text{ chosen arbitrarily}, \\ C_1 = C,\ x_1 = P_{C_1}x_0, \\ y_n = \alpha_n x_n + (1-\alpha_n)Ty_n, \\ C_{n+1} = \{z \in C_n(t) : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad \forall n \ge 0, \end{cases}$$

assume that the control sequences $\{\alpha_n\}$ *in [0,1] satisfies the condition* $\limsup_{n\to\infty} \alpha_n < 1$. *Then* $\{x_n\}$ *converges strongly to* $P_{\mathscr{F}}x_0$.

Remark 2.6. In this paper, the condition $\limsup_{n\to\infty} \alpha_n < 1$ for $\{\alpha_n\}$ is not strict. For example, obviously, $\{\alpha_n\} = \{\frac{1}{n}\}$ satisfies the condition.

Acknowledgements

The author was supported by Shandong Provincial Natural Science Foundation, China (Grant No. ZR2015AL001) and by the Project of Shandong Province Higher Educational Science and Technology Program (Grant No. J15LI51, No. J14LI51).

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