J. Nonlinear Funct. Anal. 2015 (2015), Article ID 18

MATHRES

Journal of Nonlinear Functional Analysis

Available online at http://jnfa.mathres.org

STRONG CONVERGENCE OF THE CQ METHOD FOR FIXED POINTS OF SEMIGROUPS OF NONEXPANSIVE MAPPINGS

HOSSEIN PIRI

Department of Mathematics, Faculty of Basic Science, University of Bonab, Bonab, Iran

Abstract. In this paepr, the Ishikawa iteration and the Halpern iteration are investigated based on the CQ method for fixed points of left amenable semigroups of nonexpansive mappings. Strong convergence theorems are established in the framework of Hilbert spaces.

Keywords. Amenable semigroup; CQ method; Fixed point; Nonexpansive mapping.

1. Introduction

Let H be a real Hilbert space with inner product $\langle , \rangle$ and induced norm $\| . \|$ and C a closed convex subset of H. A mapping T of C into itself is called nonexpansive if $\| Tx - Ty \| \leq \| x - y \|$, for all $x, y \in C$. By $ne(C)$, we denote the set of all nonexpansive mapping of C into itself and by $Fix(T)$, we denote the set of fixed points of T (i.e., $Fix(T) = \{x \in H : Tx = x\}$), it is well known that $Fix(T)$ is closed and convex.

Three iteration processes are often used to approximate a fixed point of a nonexpansive mapping T. The first one is introduced by Halpern [2] and is defined as follows: Take an initial guess $x_0 \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) T x_n, \qquad n \geq 0, \tag{1}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$.

E-mail address: hossein_piri1979@yahoo.com; h.piri@bonabu.ac.ir

Received June 30, 2015

The second iteration process is known as Manns iteration process [9] which is defined as

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \qquad n \geq 0, \tag{2}$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}$ is in the interval $[0,1]$.

The third iteration process is referred to as Ishikawas iteration process [4] which is defined recursively by

$$\begin{cases} y_n = \alpha_n x_n + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Ty_n, \qquad \forall n \in \mathbb{N}\cup\{0\}, \end{cases} \tag{3}$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $[0,1]$.

Yanes and Xu [16] proposed the following modification of the Ishikawa iteration for a single nonexpansive mapping T in a Hilbert space H.

$$\begin{cases} x_0 \in C \ \ chosen \ \ arbitrarily, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tz_n, \\ z_n = \beta_n x_n + (1-\beta_n)Tx_n, \\ C_n = \{v \in C : \| y_n - v \|^2 \leq \| x_n - v \|^2 \\ \qquad +(1-\alpha_n)(\| z_n \|^2 - \| x_n \|^2 + 2\langle x_n - z_n, v\rangle)\}, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{4}$$

where P_C denotes the metric projection from H onto a closed convex subset C of H.

They also proposed the following modification of the Halpern iteration for a single nonexpansive mapping T in a Hilbert space H.

$$\begin{cases} x_0 \in C \ \ chosen \ \ arbitrarily, \\ y_n = \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n = \{v \in C : \| y_n - v \|^2 \leq \| x_n - v \|^2 \\ \qquad +\alpha_n(\| x_0 \|^2 + 2\langle x_n - x_0, v\rangle)\}, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \tag{5}$$

It is the purpose of this paper to adapt the iterations (4) and (5) to a amenable semigroup of nonexpansive mappings on a closed convex subset of a Hilbert space. More precisely motivated and inspired by Yanes and Xu [16], Lau *et al.* [7], Katchang and Kumam [6], Piri [10, 11] and Piri and Badali [12], we introduce the following iteration processes for semigroup of nonexpansive mappings, with a closed convex subset C of a Hilbert space H.

$$(6)\qquad \begin{cases} x_0 \in C \ \ chosen \ \ arbitrarily, \\ y_n = \alpha_n x_n + (1-\alpha_n) T_{\mu_n} z_n, \\ z_n = \beta_n x_n + (1-\beta_n) T_{\mu_n} x_n, \\ C_n = \{v \in C : \| y_n - v \|^2 \leq \| x_n - v \|^2 \\ \qquad + (1-\alpha_n)(\| z_n \|^2 - \| x_n \|^2 + 2\langle x_n - z_n, v \rangle)\}, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0 \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

and

$$(7)\qquad \begin{cases} x_0 \in C \ \ chosen \ \ arbitrarily, \\ y_n = \alpha_n x_0 + (1-\alpha_n) T_{\mu_n} x_n, \\ C_n = \{v \in C : \| y_n - v \|^2 \leq \| x_n - v \|^2 \\ \qquad + \alpha_n(\| x_0 \|^2 + 2\langle x_n - x_0, v \rangle)\}, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0 \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

for finding a common element of the set of common fixed points for a left amenable semigroup $\varphi = \{T(t) : t \in S\}$ of nonexpansive mappings, with respect to a left regular sequence $\{\mu_n\}_{n=1}^{\infty}$ of means defined on an appropriate space of bounded real valued functions of the semigroup. We prove that under assumptions on parameters like that in Yanes and Xu [16], both iteration processes (6) and (7) converges strongly to $x^* \in \mathscr{F} = Fix(\varphi)$, where $x^* = P_{\mathscr{F}} x^*$.

2. Preliminaries

Let S be a semigroup and let $l^\infty(S)$ be the space of all bounded real valued functions defined on S with supremum norm. For $s \in S$ and $f \in l^\infty(S)$, we define elements $l(s)f$ and $r(s)f$ in $l^\infty(S)$ by

$$(l(s)f)(t) = f(st), \qquad (r(s)f)(t) = f(ts), \qquad \forall t \in S.$$

Let X be a subspace of $l^\infty(S)$ containing 1 and let X^* be its topological dual. An element μ of X^* is said to be a mean on X if $\| \mu \| = \mu(1) = 1$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let X be left invariant (resp. right invariant), i.e., $l(s)(X) \subset X$ (resp. $r(s)(X) \subset X$) for each $s \in S$. A mean μ on X is said to be left invariant (resp. right invariant) if $\mu((l(s)f) = \mu(f)$ (resp. $\mu(r(s)f) = \mu(f)$) for each $s \in S$ and $f \in X$. X is said to be left (resp. right) amenable if X has a left (resp. right) invariant mean. X is amenable if X is both left and right amenable. As is well known, $l^\infty(S)$ is amenable when S is a commutative semigroup, see [7]. A net $\{\mu_\alpha\}$ of means on X is said to be strongly left regular if

$$\lim_\alpha \| l(s)^*\mu_\alpha - \mu_\alpha \| = 0,$$

for each $s \in S$, where $l(s)^*$ is the adjoint operator of $l(s)$.

Let C be a closed convex subset of a Banach space E and let T be a mapping of C into itself. Let $F(T)$ denote the fixed point set of T and let $ne(C)$ be the set of all nonexpansive mappings of C into itself. Then $\varphi = \{T(t) : t \in S\}$ is called a representation of S as nonexpansive mappings on C if $T(s) \in ne(C)$ for each $s \in S$, $T(e) = I$ and $T(st) = T(s)T(t)$ for each $s,t \in S$. We denote by $Fix(\varphi)$ the set of common fixed points of φ, i.e.

$$Fix(\varphi) = \bigcap_{t \in S}\{x \in C : T(t)x = x\}.$$

We denote by $l^\infty(S,E)$ the Banach space of all bounded mappings of S into a Banach space E with supremum norm, and by $l_c^\infty(S,E)$ the subspace of elements $f \in l^\infty(S,E)$ such that $f(S) = \{f(s) : s \in S\}$ is a relatively weakly compact subset of E. Let X be a subspace of $l^\infty(S)$ containing 1 such that for each $f \in l^\infty(S,E)$ and $x^* \in E^*$, the function $s \to \langle f(s), x^*\rangle$ is contained in X. Then, for each $\mu \in X^*$ and $f \in l_c^\infty(S,E)$, let us define a continuous linear functional $\tau(\mu)f$ on E^* by

$$\tau(\mu)f : x^* \to \mu\langle f(.), x^*\rangle.$$

It follows from the bipolar theorem that $\tau(\mu)f$ is contained in E. We know that if μ is a mean on X, then $\tau(\mu)f$ is contained in the closure of convex hull of $\{f(s) : s \in S\}$. We also know that for each $\mu \in X^*$, $\tau(\mu)$ is a bounded linear mapping of $l_c^\infty(S,E)$ into E such that for each $f \in l_c^\infty(S,E)$, $\| \tau(\mu) \| \leq \| \mu \| \| f \|$; see [5]. Let $\varphi = \{T(t) : t \in S\}$ be a representation of S as nonexpansive mappings on C such that $T(.)x \in l_c^\infty(S,E)$ for some $x \in C$. If for each $x^* \in E^*$ the function $s \to \langle T(s)x, x^* \rangle$ is contained in X, then there exists a unique point x_0 of E such that$\mu\langle T(s)x, x^* \rangle = \langle x_0, x^* \rangle$ for each $x^* \in E^*$; see [3] and [14]. We denote such a point x_0 by $T(\mu)x$.

Lemma 2.1. [7] *Let S be a semigroup and C be a nonempty closed convex subset of a reflexive Banach space E. Let $\varphi = \{T(t) : t \in S\}$ be a nonexpansive semigroup on H such that $\{T(t)x : t \in S\}$ is bounded for some $x \in C$, let X be a subspace of $B(S)$ such that $1 \in X$ and the mapping $t \to \langle T(t)x, y^* \rangle$ is an element of X for each $x \in C$ and $y^* \in E^*$, and μ is a mean on X. Then the followings hold.*

(i) *$T(\mu)$ is nonexpansive mapping from C into C.*

(ii) *$T(\mu)x = x$ for each $x \in Fix(\varphi)$.*

(iii) *$T(\mu)x \in \overline{co}\{T(s)x : s \in S\}$ for each $x \in C$.*

Notation 2.2.

(a) *$\rightharpoonup$ denotes weak convergence and $\to$ denotes strong convergence.*

(b) *$\omega_\omega\{x_n\} = \{x \in H : \exists\{x_{n_j}\} \subset \{x_n\}$ and $x_{n_j} \rightharpoonup x\}\}$ denotes the weak $\omega-$limit set of $\{x_n\}$.*

Let C be a nonempty subset of a Hilbert space H and $T : C \to H$ a mapping. Then T is said to be demiclosed at $v \in H$ if, for any sequence $\{x_n\}$ in C, the following implication holds:

$$x_n \rightharpoonup u \in C \qquad and \qquad Tx_n \to v \qquad imply \qquad Tu = v.$$

Lemma 2.3. [15] *Let C be a nonempty closed convex subset of a Hilbert space H and suppose that $T : C \to H$ is nonexpansive. Then, the mapping $I - T$ is demiclosed at zero.*

Let C be a nonempty subset of a normed space E and let $x \in E$. An element $y_0 \in C$ is said to be the best approximation to x if

$$\| x - y_0 \| = d(x,C),$$

where $d(x,C)=\inf_{y\in C}\parallel x-y\parallel$. The number $d(x,C)$ is called the distance from x to C or the error in approximating x by C. The (possibly empty) set of all best approximation from x to C is denoted by

$$P_C(x)=\{y\in C:\parallel x-y\parallel=d(x,C)\}.$$

This defines a mapping P_C from X into 2^C and is called metric (nearest point) projection onto C. It is well known that P_C is a non expansive mapping of H onto C.

Lemma 2.4. [17] *Let C be a nonempty convex subset of a Hilbert space H and P_C be the metric projection mapping from H onto C. Let $x\in H$ and $y\in C$. Then, the following are equivalent.*

(i) $y=P_C(x)$,

(ii) $\langle x-y,y-z)\rangle\geq 0, \qquad \forall z\in C.$

Lemma 2.5. [16] *Let H be a real Hilbert space. Then, for all $x,y\in H$*

$$\parallel x-y\parallel^2=\parallel x\parallel^2-\parallel y\parallel^2-2\langle x-y,y\rangle.$$

Lemma 2.6. [16] *Let H be a real Hilbert space. Given a closed convex subset $C\subset H$ and points $x,y,z\in H$. Given also a real number $a\in\mathbb{R}$. The set $D=\{v\in C:\parallel y-v\parallel^2\leq\parallel x-v\parallel^2+\langle z,v\rangle+a\}$ is closed and convex.*

Notation 2.7. *The open ball of radius r centered at 0 is denoted by B_r and for a subset D of H, by $\overline{co}D$, we denote the closed convex hull of D. For $\varepsilon>0$ and a mapping $T:D\to H$, we let $F_\varepsilon(T;D)$ be the set of $\varepsilon-$ approximate fixed points of T, i.e. $F_\varepsilon(T;D)=\{x\in D:\parallel x-Tx\parallel\leq\varepsilon\}$.*

3. Main results

Now, we are in a position to prove our main results.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, S a semigroup and $\varphi=\{T(t):t\in S\}$ is a nonexpansive semigroup of C into itself such that $\mathscr{F}=Fix(\varphi)\neq\emptyset$. Let X be a left invariant subspace of $B(S)$ such that $1\in X$, $t\to\langle T_tx,y\rangle$ an element of X for each $x,y\in C$ and $\{\mu_n\}_{n=0}^\infty$ is a left regular sequence of means on X. Assume* *that* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *are sequences in* $[0,1]$ *such that* $\alpha_n \leq 1-\delta$ *for some* $\delta \in (0,1]$*and* $\lim_{n\to\infty}\beta_n = 1$. *Define a sequence* $\{x_n\}_{n=0}^{\infty}$ *in* C *by the iteration algorithm*

$$(8)\qquad \begin{cases} x_0 \in C \ \ chosen \ \ arbitrarily, \\ y_n = \alpha_n x_n + (1-\alpha_n)T(\mu_n)z_n, \\ z_n = \beta_n x_n + (1-\beta_n)T(\mu_n)x_n, \\ C_n = \{v \in C : \| y_n - v \|^2 \leq \| x_n - v \|^2 \\ \qquad +(1-\alpha_n)(\| z_n \|^2 - \| x_n \|^2 + 2\langle x_n - z_n, v\rangle)\}, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}x_0, \end{cases}$$

Then $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $P_{\mathscr{F}}x_0$.

Proof. First observe that C_n is convex by Lemma 2.6. Next, we show that $\mathscr{F} \subset C_n$ for all $n \in \mathbb{N}\cup\{0\}$. Indeed, for all $p \in \mathscr{F}$ from Lemma 2.1, we have

$$\begin{aligned} \| y_n - p \|^2 &= \| \alpha_n x_n + (1-\alpha_n)T(\mu_n)z_n - p \|^2 \\ &= \| \alpha_n(x_n - p) + (1-\alpha_n)(T(\mu_n)z_n - p) \|^2 \\ &= \alpha_n \| x_n - p \|^2 + (1-\alpha_n) \| T(\mu_n)z_n - p \|^2 - \alpha_n(1-\alpha_n) \| x_n - T(\mu_n)z_n \| \\ &\leq \alpha_n \| x_n - p \|^2 + (1-\alpha_n) \| T(\mu_n)z_n - p \|^2 \\ &\leq \alpha_n \| x_n - p \|^2 + (1-\alpha_n) \| z_n - p \|^2 \\ &= \| x_n - p \|^2 + (1-\alpha_n)\left(\| z_n - p \|^2 - \| x_n - p \|^2\right) \\ &= \| x_n - p \|^2 + (1-\alpha_n)\left(\| z_n \|^2 - \| x_n \|^2 + 2\langle x_n - z_n, p\rangle\right). \end{aligned}$$

So $p \in C_n$ for all $n \in \mathbb{N}\cup\{0\}$. Next, we show by induction that

$$(9)\qquad \mathscr{F} \subset C_n \cap Q_n, \quad \forall n \in \mathbb{N}\cup\{0\}.$$

From $Q_0 = C$, we have $\mathscr{F} \subset C_0 \cap Q_0$. Suppose that $\mathscr{F} \subset C_n \cap Q_n$ for some $n \in \mathbb{N}\cup\{0\}$. Since $x_{n+1} = P_{C_n\cap Q_n}x_0$, by Lemma 2.4, we have

$$\langle x_{n+1} - z, x_0 - x_{n+1}\rangle \geq 0, \quad \forall z \in C_n \cap Q_n.$$

As $\mathscr{F} \subset C_n \cap Q_n$ by the induction assumption, the last inequality holds, in particular, for all $z \in \mathscr{F}$. This together with the definition of Q_{n+1} implies that $\mathscr{F} \subset Q_{n+1}$. Hence (9) holds. Since $x_{n+1} = P_{C_n \cap Q_n} x_0$. Therefore $x_{n+1} \in Q_n$. This asserts that

$$\langle x_n - x_{n+1}, x_n - x_0 \rangle \leq 0, \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Consequently,

$$\begin{aligned} 0 &\leq \langle x_n - x_{n+1}, x_0 - x_n \rangle \\ &= \langle x_n - x_0, x_0 - x_n \rangle + \langle x_0 - x_{n+1}, x_0 - x_n \rangle \\ &= -\langle x_0 - x_n, x_0 - x_n \rangle + \langle x_0 - x_{n+1}, x_0 - x_n \rangle \\ &\leq - \| x_0 - x_n \|^2 + \| x_0 - x_{n+1} \| \| x_0 - x_n \| . \end{aligned}$$

This implies that

$$\| x_0 - x_n \| \leq \| x_0 - x_{n+1} \|, \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{10}$$

By the definition of Q_n, we have $x_n = P_{Q_n} x_0$ and since $\mathscr{F} \subset Q_n$, we get

$$\| x_0 - x_n \| \leq \| x_0 - p \|, \quad \forall p \in \mathscr{F}. \tag{11}$$

In particular,

$$\| x_0 - x_n \| \leq \| x_0 - q \|, \quad q = P_{\mathscr{F}} x_0. \tag{12}$$

It follows from (10) and (12) that $\{\| x_n - x_0 \|\}_{n=0}^{\infty}$ is nondecreasing sequence of real numbers which is bounded from above, so it is convergence. we assume that

$$\lim_{n \to \infty} \| x_n - x_0 \| = r \in \mathbb{R}. \tag{13}$$

Next, we show that

$$\lim_{n \to \infty} \| x_0 - x_n \| = 0.$$

The fact that $x_{n+1} \in Q_n$ implies that $\langle x_{n+1} - x_n, x_n - x_0 \rangle \geq 0$. This together with Lemma 2.5 implies that

$$
\begin{aligned}
\| x_{n+1} - x_n \|^2 &= \| (x_{n+1} - x_0) - (x_0 - x_n) \|^2 \\
&= \| x_{n+1} - x_0 \|^2 - \| x_0 - x_n \|^2 - 2\langle x_{n+1} - x_0, x - n - x_0 \rangle \\
&\leq \| x_{n+1} - x_0 \|^2 - \| x_0 - x_n \|^2 . \qquad (14)
\end{aligned}
$$

It follows from (13) and (14) that

$$
\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0. \qquad (15)
$$

The fact that $x_{n+1} \in C_n$ implies that

$$
\begin{aligned}
&\| y_n - x_{n+1} \|^2 \\
&\leq \| x_n - x_{n+1} \|^2 + (1-\alpha_n)[\| z_n \|^2 - \| x_n \|^2 + 2\langle x_n - z_n, x_{n+1} \rangle]. \qquad (16)
\end{aligned}
$$

On the other hand, for $p \in \mathscr{F}$, from (11) and Lemma 2.1, we have

$$
\begin{aligned}
&\| z_n \|^2 - \| x_n \|^2 + 2\langle x_n - z_n, x_{n+1} \rangle \\
&= \| z_n - x_n \|^2 + 2\langle x_n - z_n, x_{n+1} - x_n \rangle \\
&\leq \| z_n - x_n \|^2 + 2 \| x_n - z_n \| \| x_{n+1} - x_n \| \\
&= (1-\beta_n) \| x_n - T(\mu_n)x_n \|^2 \\
&\quad + 2(1-\beta_n) \| x_n - T(\mu_n)x_n \| \| x_{n+1} - x_n \| \\
&\leq (1-\beta_n) [\| x_n - p \| + \| p - T(\mu_n)x_n \|]^2 \\
&\quad + 2(1-\beta_n) [\| x_n - p \| + \| p - T(\mu_n)x_n \|] \| x_{n+1} - x_n \| \\
&\leq 2(1-\beta_n) \| x_n - p \|^2 + 2(1-\beta_n) \| x_n - p \| \| x_{n+1} - x_n \| \\
&\leq 2(1-\beta_n) [\| x_n - x_0 \| + \| x_0 - p \|]^2 \\
&\quad + 2(1-\beta_n) [\| x_n - x_0 \| + \| x_0 - p \|] \| x_{n+1} - x_n \| \\
&\leq 8(1-\beta_n) \| x_0 - p \|^2 + 4(1-\beta_n) \| x_0 - p \| \| x_{n+1} - x_n \| . \qquad (17)
\end{aligned}
$$

Note $\lim_{n\to\infty}\beta_n = 0$. From (15), (16) and (17), we get

$$\lim_{n\to\infty} \| y_n - x_{n+1} \| = 0. \tag{18}$$

Since $\| y_n - x_n \| \leq \| y_n - x_{n+1} \| + \| x_{n+1} - x_n \|$, it follows from (15) and (18) that

$$\lim_{n\to\infty} \| y_n - x_n \| = 0. \tag{19}$$

For $p \in \mathscr{F}$, from (11) and Lemma 2.1, we have

$$\begin{aligned}
\| z_n - x_n \| &= (1-\beta_n) \| x_n - T(\mu_n)x_n \| \\
&\leq (1-\beta_n)[\| x_n - p \| + \| p - T(\mu_n)x_n \|] \\
&\leq 2(1-\beta_n) \| x_n - p \| \\
&\leq 2(1-\beta_n)[\| x_n - x_0 \| + \| x_0 - p \|] \\
&\leq 4(1-\beta_n) \| x_0 - p \| .
\end{aligned}$$

Since $\lim_{n\to\infty}\beta_n = 0$, we get

$$\lim_{n\to\infty} \| z_n - x_n \| = 0. \tag{20}$$

Noticing that $y_n = \alpha_n x_n + (1-\beta_n)T(\mu_n)z_n$, we have $y_n - x_n = (1-\alpha_n)[T_{\mu_n}z_n - x_n]$. It follow that $\| T(\mu_n)z_n - x_n \| = \frac{1}{1-\alpha_n} \| y_n - x_n \|$. Note $\alpha_n \leq 1-\delta$. From (19), we obtain

$$\lim_{n\to\infty} \| T(\mu_n)z_n - x_n \| = 0. \tag{21}$$

From Lemma 2.1, we obtain

$$\begin{aligned}
\| x_n - T(\mu_n)x_n \| &\leq \| x_n - T(\mu_n)z_n \| + \| T(\mu_n)z_n - T(\mu_n)x_n \| \\
&\leq \| x_n - T(\mu_n)z_n \| + \| z_n - x_n \| .
\end{aligned}$$

It follows from (20) and (21) that

$$\lim_{n\to\infty} \| x_n - T(\mu_n)x_n \| = 0. \tag{22}$$

Set $D = \{y \in C :\parallel y - p \parallel \leq 2 \parallel x_0 - p \parallel\}$, for $p \in \mathscr{F}$. We remark that D is bounded closed convex set, from (11) $\{x_n\} \subset D$ and it is invariant under ϕ. Now we prove that

$$\limsup_{n\to\infty} \sup_{x\in D} \parallel T(\mu_n)x - T(t)T(\mu_n)x \parallel = 0, \quad \forall t \in S. \tag{23}$$

Let $t \in S$ and $\varepsilon > 0$. By Lemma 1 in [13] , there exists $\delta > 0$ such that

$$\overline{co}F_\delta(T(t);D) + B_\delta \subset F_\varepsilon(T(t);D). \tag{24}$$

As in the proof of Theorem 4.1 in [7], there exists $N_1 \in \mathbb{N}$ such that

$$T(\mu_n)(x) \in \overline{co}F_\delta(T(t);D) + B_\delta, \tag{25}$$

for all $x \in D$ and $n \geq N_1$. It follows from (24) and (25) that

$$T(\mu_n)(x) \in F_\varepsilon(T(t);D), \quad \forall x \in D,\ \forall n \geq N_1.$$

Therefore, we have $\limsup_{n\to\infty} \sup_{x\in D} \parallel T(t)(T(\mu_n)x) - T(\mu_n)x \parallel \leq \varepsilon$. Since $t \in S$ and $\varepsilon > 0$ are arbitrary, we get (23). We now claim that

$$\lim_{n\to\infty} \parallel x_n - T(t)x_n \parallel = 0, \quad \forall t \in S. \tag{26}$$

Let $t \in S$ and $\varepsilon > 0$. Then there exists $\delta > 0$ which satisfies (24). Since $\beta_n \in [0,1]$ and $\{x_n\} \subset D$, from (22) and (23) there exists $N_2 \in \mathbb{N}$ such that

$$\beta_n(x_n - T(\mu_n)x_n) \in B_\delta,\ \textit{and}\ T(\mu_n)x_n \in F_\delta(T(t);D), \quad n \geq N_2. \tag{27}$$

Observe that

$$\begin{aligned} z_n &= \beta_n x_n + (1-\beta_n)T(\mu_n)x_n \\ &= \beta_n(x_n - T(\mu_n)x_n) + T(\mu_n)x_n. \end{aligned}$$

It follows from (24) and (27) that $z_n \in F_\varepsilon(T_t;D)$ for all $n \geq N_2$. Since $t \in S$ and $\varepsilon > 0$ are arbitrary, so we get

$$\lim_{n\to\infty} \parallel z_n - T(t)z_n \parallel = 0, \quad \forall t \in S. \tag{28}$$

Noticing that

$$\| x_n - T(t)x_n \| \leq \| x_n - z_n \| + \| z_n - T(t)z_n \| + \| T(t)z_n - T(t)x_n \|$$
$$\leq 2 \| x_n - z_n \| + \| z_n - T(t)z_n \| .$$

From (20) and (28), we get (26). By (26) and Lemma 2.3, we obtain that $\emptyset \neq \omega_\omega\{x_n\} \subset \mathscr{F}$. Since $x_n = P_{Q_n}x_0$ and $P_{\mathscr{F}}x_0 \subset \mathscr{F} \subset Q_n$, we have $\| x_n - x_0 \| \leq \| x_0 - P_{\mathscr{F}}x_0 \|$. By the lower semicontinuity of the norm, we have $\| w - x_0 \| \leq \| x_0 - P_{\mathscr{F}}x_0 \|$ for all $w \in \omega_\omega\{x_n\}$. However, since $\omega_\omega\{x_n\} \subset \mathscr{F}$, we must have $w = P_{\mathscr{F}}x_0$ for all $w \in \omega_\omega\{x_n\}$. Hence $x_n \rightharpoonup P_{\mathscr{F}}x_0$. To see that $x_n \to P_{\mathscr{F}}x_0$, we compute

$$\begin{aligned}\| x_n - P_{\mathscr{F}}x_0 \|^2 &= \| (x_n - x_0) + (x_0 - P_{\mathscr{F}}x_0) \|^2 \\ &= \| x_n - x_0 \|^2 + 2\langle x_n - x_0, x_0 - P_{\mathscr{F}}x_0\rangle + \| x_0 - P_{\mathscr{F}}x_0 \|^2 \\ &\leq 2\langle x_n - x_0, x_0 - P_{\mathscr{F}}x_0\rangle + 2 \| x_0 - P_{\mathscr{F}}x_0 \|^2 \\ &= -2\langle x_0 - x_n, x_0 - P_{\mathscr{F}}x_0\rangle + 2 \| x_0 - P_{\mathscr{F}}x_0 \|^2 \to 0.\end{aligned}$$

That is, $\{x_n\}$ converges to $P_{\mathscr{F}}x_0$.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, S a semigroup and $\varphi = \{T(t) : t \in S\}$ is a nonexpansive semigroup of C into itself such that $\mathscr{F} = Fix(\varphi) \neq \emptyset$. Let X be a left invariant subspace of $B(S)$ such that $1 \in X$, $t \to \langle T(t)x, y\rangle$ an element of X for each $x, y \in C$ and $\{\mu_n\}_{n=0}^{\infty}$ is a left regular sequence of means on X. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0,1]$ such that $\alpha_n \leq 1-\delta$ for some $\delta \in (0,1]$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in C by the iteration algorithm*

$$(29) \qquad \begin{cases} x_0 \in C \;\; chosen \;\; arbitrarily, \\ y_n = \alpha_n x_n + (1-\alpha_n)T_{\mu_n}x_n, \\ C_n = \{v \in C : \| y_n - v \| \leq \| x_n - v \|\}, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} - P_{C_n \cap Q_n}x_0, \end{cases}$$

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to $P_{\mathscr{F}}x_0$.

Proof. By taking $\beta_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.1, we have $z_n = x_n$ and $y_n = \alpha_n x_n + (1 - \alpha_n)T_{\mu_n} x_n$ and our set in (8) reduce to the set C_n in (29). So the proof is complete

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H, S a semigroup and $\varphi = \{T(t) : t \in S\}$ is a nonexpansive semigroup of C into itself such that $\mathscr{F} = Fix(\varphi) \neq \emptyset$. Let X be a left invariant subspace of $B(S)$ such that $1 \in X$, $t \to \langle T(t)x, y\rangle$ an element of X for each $x, y \in C$ and $\{\mu_n\}_{n=0}^{\infty}$ is a left regular sequence of means on X. Assume that $\{\alpha_n\}_{n=0}^{\infty} \subset (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in C by the iteration algorithm*

$$\begin{cases} x_0 \in C \ \ chosen \ \ arbitrarily, \\ y_n = \alpha_n x_0 + (1-\alpha_n)T(\mu_n)x_n, \\ C_n = \{v \in C : \| y_n - v \|^2 \leq \| x_n - v \|^2 \\ \qquad + \alpha_n(\| x_0 \|^2 + 2\langle x_n - x_0, v\rangle)\}, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases}$$

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to $P_{\mathscr{F}} x_0$.

Proof. From the definition of C_n and Q_n it is obvious that C_n is closed and Q_n is closed and for each $n \in \mathbb{N} \cup \{0\}$. Also from Lemma 2.6, we see that C_n is convex for each $n \in \mathbb{N} \cup \{0\}$. For any $p \in \mathscr{F}$, using convexity of $\| \, . \, \|$ and Lemma 2.1, we have

$$\begin{aligned} \| y_n - p \|^2 &= \| \alpha_n x_0 + (1-\alpha_n)T(\mu_n)x_n - p \|^2 \\ &= \| \alpha_n(x_0 - p) + (1-\alpha_n)(T(\mu_n)x_n - p) \|^2 \\ &= \alpha_n \| x_0 - p \|^2 + (1-\alpha_n) \| T(\mu_n)x_n - p \|^2 - \alpha_n(1-\alpha_n) \| x_0 - T(\mu_n)x_n \| \\ &\leq \alpha_n \| x_0 - p \|^2 + (1-\alpha_n) \| T(\mu_n)x_n - p \|^2 \\ &\leq \alpha_n \| x_0 - p \|^2 + (1-\alpha_n) \| x_n - p \|^2 \\ &= \| x_n - p \|^2 + \alpha_n \left(\| x_0 - p \|^2 - \| x_n - p \|^2 \right) \\ &\leq \| x_n - p \|^2 + \alpha_n \left(\| x_0 \|^2 + 2\langle x_n - x_0, p\rangle \right). \end{aligned}$$

Hence $p \in C_n$ and $\mathscr{F} \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$. Similar to the proof of the Theorem 3.1, we also have $\mathscr{F} \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$ and hence x_n is well-defined for all $n \in \mathbb{N} \cup \{0\}$. It follows from the definition of Q_n and Lemma 2.4 that $x_n = P_{Q_n}x_0$. Therefore $\| x_n - x_0 \| \leq \| z - x_0 \|$, for all $z \in Q_n$ and for all $n \in \mathbb{N} \cup \{0\}$. Since $\mathscr{F} \subset C_n \cap Q_n$, for all $n \in \mathbb{N} \cup \{0\}$. Then for $p \in \mathscr{F}$, we have

$$\| x_n - x_0 \| \leq \| p - x_0 \|, \quad n \in \mathbb{N} \cup \{0\}. \tag{30}$$

That $x_{n+1} \in Q_n$ asserts that

$$\langle x_n - x_{n+1}, x_n - x_0 \rangle \leq 0, \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{31}$$

It follows from (31) and Lemma 2.5 that

$$\begin{aligned} \| x_n - x_0 \|^2 &\leq \| x_{n+1} - x_n \|^2 + \| x_n - x_0 \|^2 \\ &= \| (x_{n+1} - x_0) + (x_0 - x_n) \|^2 + \| x_n - x_0 \|^2 \\ &= \| x_{n+1} - x_0 \|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \\ &\leq \| x_{n+1} - x_0 \|^2 . \end{aligned} \tag{32}$$

It follows from (30) and (32) that $\{\| x_n - x_0 \|\}$ is nondecreasing sequence of real numbers which is bounded from above, so it is convergence. we assume that

$$\lim_{n\to\infty} \| x_n - x_0 \| = r \in \mathbb{R}. \tag{33}$$

On the other hand, from (30) and Lemma 2.5, we get

$$\begin{aligned} \| x_{n+1} - x_n \|^2 &= \| (x_{n+1} - x_0) + (x_0 - x_n) \|^2 \\ &= \| x_{n+1} - x_0 \|^2 - \| x_0 - x_n \|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \\ &\leq \| x_{n+1} - x_0 \|^2 - \| x_0 - x_n \|^2 . \end{aligned} \tag{34}$$

It follows from (33) and (34) that

$$\lim_{n\to\infty} \| x_{n+1} - x_n \| = 0. \tag{35}$$

The fact that $x_{n+1} \in C_n$ implies that

$$\| y_n - x_{n+1} \|^2 \leq \| x_n - x_{n+1} \|^2 + \alpha_n [\| x_0 \|^2 + 2\langle x_n - x_0, x_{n+1} \rangle]. \tag{36}$$

It follows from $\lim_{n\to\infty}\alpha_n = 0$, (35) and (36) that

$$\lim_{n\to\infty} \| y_n - x_{n+1} \| = 0. \tag{37}$$

Noticing that $\| y_n - x_n \| \le \| y_n - x_{n+1} \| + \| x_{n+1} - x_n \|$. From (35) and (37), we obtain

$$\lim_{n\to\infty} \| y_n - x_n \| = 0. \tag{38}$$

Let $p \in \mathscr{F}$ and set $D = \{y \in C : \| y - x_0 \| \le 2 \| p - x_0 \|\}$. We remark that D is bounded closed convex set, from (30), $\{x_n\} \subset D$ and it is invariant under ϕ. By using similar method as used in the proof of relation (23) of Theorem 3.1, we have

$$\lim_{n\to\infty} \sup \sup_{x\in D} \| T(\mu_n)x - T(t)T(\mu_n)x \| = 0, \quad \forall t \in S. \tag{39}$$

Now we claim that

$$\lim_{n\to\infty} \| x_n - T(t)x_n \| = 0, \quad \forall t \in S. \tag{40}$$

Let $t \in S$ and $\varepsilon > 0$. As in the proof of the Shioji and Takahashi [13, Lemma 1] , there exists $\delta > 0$ such that

$$\overline{co}F_\delta(T(t);D) + B_\delta \subset F_\varepsilon(T(t);D). \tag{41}$$

Since $\{T(\mu_n)\}$ is bounded. From (39) and (40), there exists $N \in \mathbb{N}$ such that

$$\alpha_n(x_0 - T(\mu_n)x_n) \in B_\delta, \ \ and \ \ T(\mu_n)x_n \in F_\delta(T(t);D), \quad n \ge N. \tag{42}$$

Observe that $y_n = \alpha_n x_0 + (1-\alpha_n)T(\mu_n)x_n = \alpha_n(x_0 - T(\mu_n)x_n) + T(\mu_n)x_n$. It follows from (41) and (42) that $y_n \in F_\varepsilon(T(t);D)$ for all $n \ge N$. Since $s \in S$ and $\varepsilon > 0$ are arbitrary, so we get

$$\lim_{n\to\infty} \| y_n - T(t)y_n \| = 0, \quad \forall t \in S. \tag{43}$$

Noticing that

$$\begin{aligned} \| x_n - T(t)x_n \| &\le \| x_n - y_n \| + \| y_n - T(t)y_n \| + \| T(t)y_n - T(t)x_n \| \\ &\le 2 \| x_n - y_n \| + \| y_n - T(t)y_n \| . \end{aligned}$$

From (38) and (43), we get (40). By (40) and Lemma 2.3, we obtain that $\emptyset \neq \omega_\omega\{x_n\} \subset \mathscr{F}$. Since $x_n = P_{Q_n}x_0$ and $P_{\mathscr{F}}x_0 \subset \mathscr{F} \subset Q_n$, we have $\| x_n - x_0 \| \leq \| x_0 - P_{\mathscr{F}}x_0 \|$. By the lower semicontinuity of the norm, we have $\| w - x_0 \| \leq \| x_0 - P_{\mathscr{F}}x_0 \|$ for all $w \in \omega_\omega\{x_n\}$. However, since $\omega_\omega\{x_n\} \subset \mathscr{F}$, we must have $w = P_{\mathscr{F}}x_0$ for all $w \in \omega_\omega\{x_n\}$. Hence $x_n \rightharpoonup P_{\mathscr{F}}x_0$. To see that $x_n \to P_{\mathscr{F}}x_0$, we compute

$$\begin{aligned}
\| x_n - P_{\mathscr{F}}x_0 \|^2 &= \| (x_n - x_0) + (x_0 - P_{\mathscr{F}}x_0) \|^2 \\
&= \| x_n - x_0 \|^2 + 2\langle x_n - x_0, x_0 - P_{\mathscr{F}}x_0\rangle + \| x_0 - P_{\mathscr{F}}x_0 \|^2 \\
&\leq 2\langle x_n - x_0, x_0 - P_{\mathscr{F}}x_0\rangle + 2 \| x_0 - P_{\mathscr{F}}x_0 \|^2 \\
&= 2\langle x_0 - x_n, x_0 - P_{\mathscr{F}}x_0\rangle + 2 \| x_0 - P_{\mathscr{F}}x_0 \|^2 \to 0.
\end{aligned}$$

That is, $\{x_n\}$ converges to $P_{\mathscr{F}}x_0$.

4. Applications

Corollary 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, S a semigroup and $\varphi = \{T(t) : t \in S\}$ is a nonexpansive semigroup of C into itself such that $\mathscr{F} = Fix(\varphi) \neq \emptyset$. Let X be a left invariant subspace of $B(S)$ such that $1 \in X$, $t \to \langle T(t)x, y\rangle$ an element of X for each $x, y \in C$ and $\{\mu_n\}_{n=0}^\infty$ is a left regular sequence of means on X. Assume that $\{\alpha_n\}_{n=0}^\infty$ and $\{\alpha_n\}_{n=0}^\infty$ are sequences in $[0,1]$ such that $\alpha_n \leq 1-\delta$ for some $\delta \in (0,1]$and* $\lim_{n\to\infty}\beta_n = 1$. *Define a sequence $\{x_n\}_{n=0}^\infty$ in C by the iteration algorithm*

$$(44)\qquad \begin{cases}
x_0 \in C \ \ chosen \ \ arbitrarily, \\
y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)z_n ds, \\
z_n = \beta_n x_n + (1-\beta_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\
C_n = \{v \in C : \| y_n - v \|^2 \leq \| x_n - v \|^2 \\
\qquad +(1-\alpha_n)(\| z_n \|^2 - \| x_n \|^2 + 2\langle x_n - z_n, v\rangle)\}, \\
Q_n = \{v \in C : \langle x_n - v, x_n - x_0\rangle \leq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n}x_0,
\end{cases}$$

where $\{t_n\}$ *is an increasing sequence in* $(0,1)$ *satisfying* $\lim_{n\to\infty} t_n = \infty$ *and* $\lim_{n\to\infty} \frac{t_n}{t_{n+1}} = 1$. *Then* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $P_{\mathscr{F}} x_0$.

Proof. For $n \in \mathbb{N}$, define $\mu_n(f) = \frac{1}{t_n}\int\limits_0^{t_n} f(t)\,\mathrm{d}t$ for each $f \in C(\mathbb{R}^+)$, where $C(\mathbb{R}^+)$ denotes the space of all real valued bounded continuous functions on $\mathbb{R}^+$ with supremum norm. Then, $\{\mu_n\}$ is strongly regular sequence of means; for more details, see [15]. Next for each $x \in C$, we have $T(\mu_n)(x) = \frac{1}{t_n}\int\limits_0^{t_n} T_s(x)\,\mathrm{d}s$. Therefore, it follows from Theorem 3.1 that the sequences $\{x_n\}$ converges strongly, as $n \to \infty$ to a point $P_{\mathscr{F}} x_0$.

Corollary 4.2. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H, S *a semigroup and* $\varphi = \{T(t) : t \in S\}$ *is a nonexpansive semigroup of* C *into itself such that* $\mathscr{F} = Fix(\varphi) \neq \emptyset$. *Let* X *be a left invariant subspace of* $B(S)$ *such that* $1 \in X$, $t \to \langle T(t)x, y\rangle$ *an element of* X *for each* $x, y \in C$ *and* $\{\mu_n\}_{n=0}^{\infty}$ *is a left regular sequence of means on* X. *Assume that* $\{\alpha_n\}_{n=0}^{\infty}$ *is a sequence in* $[0,1]$ *such that* $\alpha_n \le 1-\delta$ *for some* $\delta \in (0,1]$. *Define a sequence* $\{x_n\}_{n=0}^{\infty}$ *in* C *by the iteration algorithm*

$$(45)\qquad \begin{cases} x_0 \in C \ \ chosen \ \ arbitrarily, \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\ C_n = \{v \in C : \| y_n - v \| \le \| x_n - v \|, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0\rangle \le 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where $\{t_n\}$ *is an increasing sequence in* $(0,1)$ *satisfying* $\lim_{n\to\infty} t_n = \infty$ *and* $\lim_{n\to\infty} \frac{t_n}{t_{n+1}} = 1$. *Then* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $P_{\mathscr{F}} x_0$.

Proof. By taking $\beta_n = 1$ for all $n \in \mathbb{N}$ in Corollary 4.1, we have $z_n = x_n$, $y_n = \alpha_n x_n + (1 - \alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds$ and our set in (44) reduce to the set C_n in (45). So the proof is complete.

Corollary 4.3. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H, S *a semigroup and* $\varphi = \{T(t) : t \in S\}$ *is a nonexpansive semigroup of* C *into itself such that* $\mathscr{F} = Fix(\varphi) \neq \emptyset$. *Let* X *be a left invariant subspace of* $B(S)$ *such that* $1 \in X$, $t \to \langle T(t)x, y\rangle$ *an element of* X *for each* $x, y \in C$ *and* $\{\mu_n\}_{n=0}^{\infty}$ *is a left regular sequence of means on* X. *Assume that* $\{\alpha_n\}_{n=0}^{\infty} \subset (0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$. *Define a sequence* $\{x_n\}_{n=0}^{\infty}$ *in* C *by the iteration* *algorithm*

$$\begin{cases} x_0 \in C \ \ chosen \ \ arbitrarily, \\ y_n = \alpha_n x_0 + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\ C_n = \{v \in C : \| y_n - v \|^2 \leq \| x_n - v \|^2 \\ \qquad + \alpha_n(\| x_0 \|^2 + 2\langle x_n - x_0, v\rangle)\}, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where $\{t_n\}$ *is an increasing sequence in* $(0,1)$ *satisfying* $\lim_{n\to\infty} t_n = \infty$ *and* $\lim_{n\to\infty} \frac{t_n}{t_{n+1}} = 1$. *Then* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $P_{\mathscr{F}} x_0$.

Proof. For $n \in \mathbb{N}$, define $\mu_n(f) = \frac{1}{t_n}\int_0^{t_n} f(t)\ \mathrm{d}t$ for each $f \in C(\mathbb{R}^+)$, where $C(\mathbb{R}^+)$ denotes the space of all real valued bounded continuous functions on $\mathbb{R}^+$ with supremum norm. Then, $\{\mu_n\}$ is strongly regular sequence of means; for more details, see [15]. Next for each $x \in C$, we have $T(\mu_n)(x) = \frac{1}{t_n}\int_0^{t_n} T_s(x)\ \mathrm{d}s$. Therefore, it follows from Theorem 3.3 that the sequences $\{x_n\}$ converges strongly, as $n \to \infty$ to a point $P_{\mathscr{F}} x_0$.

Corollary 4.4. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H, S *a semigroup and* $\varphi = \{T(t) : t \in S\}$ *is a non-expansive semigroup of* C *into itself such that* $\mathscr{F} = Fix(\varphi) \neq \emptyset$. *Let* X *be a left invariant subspace of* $B(S)$ *such that* $1 \in X$, $t \to \langle T(t)x, y\rangle$ *an element of* X *for each* $x, y \in C$ *and* $\{\mu_n\}_{n=0}^{\infty}$ *is a left regular sequence of means on* X. *Assume that* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\alpha_n\}_{n=0}^{\infty}$ *are sequences in* $[0,1]$ *such that* $\alpha_n \leq 1-\delta$ *for some* $\delta \in (0,1]$*and* $\lim_{n\to\infty} \beta_n = 1$. *Define a sequence* $\{x_n\}_{n=0}^{\infty}$ *in* C *by the iteration algorithm*

$$\begin{cases} x_0 \in C \ \ chosen \ \ arbitrarily, \\ y_n = \alpha_n x_n + (1-\alpha_n) r_n \int_0^{\infty} e^{-r_n s} T(s) z_n ds, \\ z_n = \beta_n x_n + (1-\beta_n) r_n \int_0^{\infty} e^{-r_n s} T(s) x_n ds, \\ C_n = \{v \in C : \| y_n - v \|^2 \leq \| x_n - v \|^2 \\ \qquad + (1-\alpha_n)(\| z_n \|^2 - \| x_n \|^2 + 2\langle x_n - z_n, v\rangle)\}, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{46}$$

where $\{r_n\}$ *is a decreasing sequence in* $(0,\infty]$ *satisfying* $\lim_{n\to\infty} r_n = 0$. *Then* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $P_{\mathscr{F}}x_0$.

Proof. For $n \in \mathbb{N}$, define $\mu_n(f) = r_n \int\limits_o^\infty e^{-r_n s} f(s)\, \mathrm{d}t$ for each $f \in C(\mathbb{R}^+)$. Then, $\{\mu_n\}$ is strongly left regular sequence of means; for more details, see [15]. Next for each $x \in C$, we have $T(\mu_n(x)) = r_n \int\limits_o^\infty e^{-r_n s} T(s)(x)\, \mathrm{d}t$. It follows from Theorem 3.1 that the sequences $\{x_n\}$ converges strongly, as $n \to \infty$ to a point $P_{\mathscr{F}}x_0$.

Corollary 4.5. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H, S *a semigroup and* $\varphi = \{T(t) : t \in S\}$ *is a nonexpansive semigroup of* C *into itself such that* $\mathscr{F} = Fix(\varphi) \neq \emptyset$. *Let* X *be a left invariant subspace of* $B(S)$ *such that* $1 \in X$, $t \to \langle T_t x, y\rangle$ *an element of* X *for each* $x, y \in C$ *and* $\{\mu_n\}_{n=0}^{\infty}$ *is a left regular sequence of means on* X. *Assume that* $\{\alpha_n\}_{n=0}^{\infty}$ *is a sequence in* $[0,1]$ *such that* $\alpha_n \leq 1-\delta$ *for some* $\delta \in (0,1]$. *Define a sequence* $\{x_n\}_{n=0}^{\infty}$ *in* C *by the iteration algorithm*

$$
(47) \qquad \begin{cases} x_0 \in C \ \ chosen \ \ arbitrarily, \\ y_n = \alpha_n x_n + (1-\alpha_n) r_n \int_0^\infty e^{-r_n s} T(s) x_n ds, \\ C_n = \{v \in C : \| y_n - v \| \leq \| x_n - v \|, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0\rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}
$$

where $\{r_n\}$ *is a decreasing sequence in* $(0,\infty]$ *satisfying* $\lim_{n\to\infty} r_n = 0$. *Then* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $P_{\mathscr{F}}x_0$.

Proof. By taking $\beta_n = 1$ for all $n \in \mathbb{N}$ in Corollary 4.4, we have $z_n = x_n$, $y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s) x_n ds$ and our set in (46) reduce to the set C_n in (47). So the proof is complete.

Corollary 4.6. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H, S *a semigroup and* $\varphi = \{T(t) : t \in S\}$ *is a nonexpansive semigroup of* C *into itself such that* $\mathscr{F} = Fix(\varphi) \neq \emptyset$. *Let* X *be a left invariant subspace of* $B(S)$ *such that* $1 \in X$, $t \to \langle T_t x, y\rangle$ *an element of* X *for each* $x, y \in C$ *and* $\{\mu_n\}_{n=0}^{\infty}$ *is a left regular sequence of means on* X. *Assume that* $\{\alpha_n\}_{n=0}^{\infty} \subset (0,1)$ *such that* $\lim_{n\to\infty}\alpha_n = 0$. *Define a sequence* $\{x_n\}_{n=0}^{\infty}$ *in* C *by the iteration* *algorithm*

$$\begin{cases} x_0 \in C \ \ chosen \ \ arbitrarily, \\ y_n = \alpha_n x_0 + (1-\alpha_n) r_n \int_0^\infty e^{-r_n s} T(s) x_n ds, \\ C_n = \{v \in C : \| y_n - v \|^2 \le \| x_n - v \|^2 \\ \qquad +\alpha_n (\| x_0 \|^2 + 2\langle x_n - x_0, v \rangle)\}, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0 \rangle \le 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases}$$

where $\{r_n\}$ *is a decreasing sequence in* $(0,\infty]$ *satisfying* $\lim_{n\to\infty} r_n = 0$*. Then* $\{x_n\}_{n=0}^\infty$ *converges strongly to* $P_{\mathscr{F}} x_0$*.*

Proof. For $n \in \mathbb{N}$, define $\mu_n(f) = r_n \int_o^\infty e^{-r_n s} f(s)\, dt$ for each $f \in C(\mathbb{R}^+)$. Then, $\{\mu_n\}$ is strongly left regular sequence of means; for more details, see [15]. Next for each $x \in C$, we have $T(\mu_n(x)) = r_n \int_o^\infty e^{-r_n s} T(s)(x)\, dt$. It follows from Theorem 3.3 that sequences $\{x_n\}$ converges strongly, as $n \to \infty$ to a point $P_{\mathscr{F}} x_0$.

REFERENCES

[1] S. Atsushiba, W. Takahashi, a nonlinear ergodic theorem for nonexpansive mappings with compact domain. Math. Japon. 52 (2000), 183-195.

[2] B. Halpern, Fixed points of nonexpanding maps, Bull. Am. Math. Soc. 73 (1967), 957-961.

[3] N. Hirano, K. Kido, W. Takahashi, Nonexpansive retractions and nonlinear ergodic theorems in Banach spaces, Nonlinear Anal. 12 (1988), 1269-1281.

[4] S. Ishikawa, Fixed points by a new iteration method, Proc. Am. Math. Soc. 44 (1974), 147-150.

[5] O. Kada, W. Takahashi, Strong convergence and nonlinear ergodic theorems for commutative semigroups of nonexpansive mappings, Nonlinear Anal. 28 (1997), 495-511.

[6] P. Katchang, P. Kumam, A composite explicit iterative process with a viscosity method for Lipschitzian semigroup in smooth Banach space, Bull. Iranian Math. Soc. 37 (2011), 143-159.

[7] A. T. Lau, H. Miyake, W. Takahashi, Approximation of fixed points for amenable semigroups of nonexpansive mappings in Banach spaces, Nonlinear Anal. 67 (2007), 1211-1225.

[8] A. T. Lau, N. Shioji, and W. Takahashi, Existence of nonexpansive retractions for amenable semigroups of nonexpansive mappings and nonlinear ergodic theorems in Banach spaces, J. Funct. Anal. 161 (1999) 62-75.

[9] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 4 (1953) 506-510.

[10] H. Piri, Strong convergence for a minimization problem on solutions of systems of equilibrium problems and common fixed points of an infinite family and semigroup of nonexpansive mappings, Comput. Math. Appl. 61 (2011), 2562-2577.

[11] H. Piri, Hybrid pseudoviscosity approximation schemes for systems of equilibrium problems and fixed point problems of infinite family and semigroup of non-expansive mappings, Nonlinear Anal. 74 (2011), 6788-804.

[12] H. Piri and A. H. Badali, Strong convergence theorem for amenable semigroups of nonexpansive mappings and variational inequalities, Fixed Point Theory Appl. 2011 (2011), Article ID 55.

[13] N. Shioji, W. Takahashi, Strong convergence of averaged approximants for asymptotically nonexpansive mappings in Banach spaces, J. Approx. Theory 97 (1999), 53-64.

[14] W. Takahashi, A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space, Proc. Amer. Math. Soc. 81 (1981), 253-256.

[15] W. Takahashi, Nonlinear functional analysis, Yokohama Publishers, Yokohama, 2000.

[16] C. M. Yanes, H. K. Xu, Strong convergence of the CQ method for fixed point iteration processes, Nonlinear Anal. 64 (2006), 2400-2411.

[17] Y. Yao, Y.C. Liou, J. C. Yao, Convergence theorem for equilibrium problems and fixed point problems of infinite family of nonexpansive mappings, Fixed Point Theory Appl. 2007 (2007), Article ID 64363.