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MONOTONE PROJECTION ALGORITHMS, NONLINEAR OPERATORS AND EQUILIBRIUM PROBLEMS

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Abstract. A family of uncountable infinite many asymptotically quasi-*phi*-nonexpansive mappings and an equilibrium problem are investigated. Strong convergence theorems of common solutions to fixed point and equilibrium problems are established in a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property.

Keywords. Asymptotically nonexpansive mapping; Fixed point; Nonexpansive mapping; Variational inequality.

1. Introduction-preliminaries

Let E be a real Banach space and let E^* be the dual space of E. Let B_E be the unit sphere of E. Recall that E is said to be uniformly convex iff for any $\varepsilon \in (0,2]$ there exists $\delta > 0$ such that for any $x, y \in B_E$, $\|x-y\| \geq \varepsilon$ implies $\|x+y\| \leq 2-2\delta$. E is said to be strictly convex space iff $\|x+y\| < 2$ for all $x, y \in B_E$ and $x \neq y$. It is known that a uniformly convex Banach space is strictly convex and reflexive.

Recall that Banach space E is said to have a Gâteaux differentiable norm iff $\lim_{t\to 0} \frac{\|x\|-\|x+ty\|}{t}$ exists for each $x, y \in B_E$. In this case, we also say that E is smooth. Banach space E is said to have a uniformly Gâteaux differentiable norm if for each $y \in B_E$, the limit is attained uniformly for all $x \in B_E$. E is also said to have a uniformly Fréchet differentiable norm iff the above limit

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is attained uniformly for $x,y \in B_E$. In this case, we say that E is uniformly smooth. It is known that a uniformly smooth Banach space is smooth and reflexive.

Recall that normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{y \in E^* : \|x\|^2 = \langle x,y \rangle = \|y\|^2\}.$$

It is known if E is a strictly convex Banach space, then J is strictly monotone; if E is uniformly smooth, then J is uniformly norm-to-norm continuous on every bounded subset of E; if E is a reflexive and strictly convex Banach space with a strictly convex dual E^* and $J^* : E^* \to E$ is the normalized duality mapping in E^*, then $J^{-1} = J^*$; if E is a smooth Banach space, then J is single-valued and demicontinuous, i.e., continuous from the strong topology of E to the weak star topology of E; if E is a smooth, strictly convex and reflexive Banach space, then J is single-valued, one-to-one and onto.

Recall that E has the Kadec-Klee property if $\lim_{m\to\infty} \|x_m - x\| = 0$, for any sequence $\{x_m\} \subset E$, and $x \in E$ with $\{x_m\}$ converges weakly to x, and $\{\|x_m\|\}$ converges strongly to $\|x\|$. It is known that every uniformly convex Banach space or the space which has the normal structure has the Kadec-Klee property; see [1] and the references therein.

Let C be a nonempty convex and convex subset of E and let $B : C \times C \to \mathbb{R}$ be a bifunction. Recall that the following equilibrium problem in the terminology of Blum and Oettli [2]. Find $\bar{x} \in C$ such that $B(\bar{x} y) \geq 0$, $\forall y \in C$. We use $Sol(B)$ to denote the solution set of the equilibrium problem. That is, $Sol(B) = \{x \in C : B(x,y) \geq 0, \forall y \in C\}$.

The following restrictions are essential for study the equilibrium problem in this paper.

(R-1) $B(a,a) \equiv 0, \forall a \in C$;

(R-2) $B(b,a) + B(a,b) \leq 0, \forall a,b \in C$;

(R-3) $B(a,b) \geq \limsup_{t\downarrow 0} B(tc + (1-t)a, b)$, $\forall a,b,c \in C$;

(R-4) $b \mapsto B(a,b)$ is convex and weakly lower semi-continuous, $\forall a \in C$.

In many applied sciences, there are many problems, including power control in CDMA data networks, and image processing, are reduced to finding solutions of solutions of the equilibrium problem, which cover fixed point problems, variational inequaity problems, complementarity problems, saddle point problems and special cases; see [3-10] and the references therein.

Recently, many author have extensively investigated the equilibrium mean valued algorithms and projection algorithms. Mean valued algorithms, in particular, Mann and Ishikawa algorithms are not strongly convergent even in the framework of Banach spaces. In many modern disciplines, including image recovery [11], economics [12], control theory [13], and quantum physics [14], problems arises in the framework of infinite dimension spaces. In such nonlinear problems, strong convergence is often much more desirable than the weak convergence [15]. To guarantee the strong convergence of mean valued iteration processes, many authors use different regularization methods. The projection method which was first introduced by Haugazeau [16] has been considered for the approximation of solutions of the equilibrium problems. The advantage of projection methods is that strong convergence of iterative sequences can be guaranteed without any compact assumptions imposed on mappings or spaces.

Let T be a mapping on C. T is said to be closed iff for any sequence $\{x_m\} \subset C$ such that $\lim_{m\to\infty} x_m = x'$ and $\lim_{m\to\infty} Tx_m = y'$, then $x' \in C$ and $Tx' = y'$. Let B be a bounded subset of C. Recall that T is said to be uniformly asymptotically regular on C if and only if $\limsup_{n\to\infty} \sup_{x\in B}\{\|T^n x - T^{n+1}x\|\} = 0$.

Next, we use $\rightharpoonup$ and $\to$ to stand for the weak convergence and strong convergence, respectively. and use $Fix(T)$ to denote the fixed point set of mapping T. Recall that a point p is said to be an asymptotic fixed point of mapping T iff subset C contains a sequence $\{x_m\}$ which converges weakly to p such that $\lim_{m\to\infty} \|x_m - Tx_m\| = 0$. We use $\widetilde{Fix}(T)$ to stand for the asymptotic fixed point set in this paper.

Next, we assume that E is a smooth Banach space which means mapping J is single-valued. Study the functional defined on E by

$$\phi(x,y) := \|x\|^2 + \|y\|^2 - 2\langle x, Jy\rangle, \quad \forall x,y \in E.$$

Let H be a Hilbert space and let C be a closed convex subset of H. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x-y\| \geq \|x - P_C x\|$, for all $y \in C$. The operator P_C is called the metric projection from H onto C. It is known that P_C is firmly nonexpansive. In [17], Alber studied a new mapping $Proj_C$ in a Banach space E, which is an analogue of P_C, the metric projection, in Hilbert spaces. Recall that generalized projection

$Proj_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of $\phi(x,y)$, which implies from the definition of ϕ that $\phi(x,y) \geq (\|x\| - \|y\|)^2$, $\forall x, y \in E$.

T is said to be relatively nonexpansive [18] iff

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \forall p \in \widetilde{Fix}(T) = Fix(T) \neq \emptyset.$$

T is said to be relatively asymptotically nonexpansive [19] iff

$$\phi(p, T^n x) \leq (\xi_n + 1)\phi(p, x), \quad \forall x \in C, \forall p \in \widetilde{Fix}(T) = Fix(T) \neq \emptyset, \forall n \geq 1,$$

where $\{\xi_n\} \subset [0, \infty)$ is a sequence such that $\mu_n \to 0$ as $n \to \infty$.

Remark 1.1. The class of relatively asymptotically nonexpansive mappings, which was first considered in [19], covers the class of relatively nonexpansive mappings [18].

T is said to be quasi-ϕ-nonexpansive [20] iff

$$\phi(p, Tx) \leq \phi(p, x), \quad \forall x \in C, \forall p \in Fix(T) \neq \emptyset.$$

T is said to be asymptotically quasi-ϕ-nonexpansive [21] iff there exists a sequence $\{\xi_n\} \subset [0, \infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$\phi(p, T^n x) \leq (\xi_n + 1)\phi(p, x), \quad \forall x \in C, \forall p \in Fix(T) \neq \emptyset, \forall n \geq 1.$$

Remark 1.2. The class of quasi-ϕ-nonexpansive mappings [20] and the class of asymptotically quasi-ϕ-nonexpansive mappings [21] cover the class of relatively nonexpansive mappings and the class of relatively asymptotically nonexpansive mappings. Quasi-ϕ-nonexpansive mappings and asymptotically quasi-ϕ-nonexpansive mappings do not require the strong restriction that the fixed point set equals the asymptotic fixed point set.

The following lemmas also play an important role in this paper.

Lemma 1.3. ([2], [20]) *Let E be a strictly convex, smooth, and reflexive Banach space and let C be a closed convex subset of E. Let B be a function with the restrictions (R-1), (R-2), (R-3) and (R-4), from $C \times C$ to $\mathbb{R}$. Let $x \in E$ and let $r > 0$. Then there exists $z \in C$ such that $rB(z,y) + \langle z - y, Jz - Jx \rangle \leq 0$, $\forall y \in C$ Define a mapping $K^{B,r}$ by*

$$K^{B,r}x = \{z \in C : rB(z,y) + \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C\}.$$

The following conclusions hold:

(1) $K^{B,r}$ *is single-valued quasi-ϕ-nonexpansive;*

(2) $Sol(B) = Fix(K^{B,r})$ *is convex and closed.*

Lemma 1.4. [17] *Let E be a strictly convex, reflexive, and smooth Banach space and let C be a nonempty, closed, and convex subset of E. Let $x \in E$. Then*

$$\phi(y, \Pi_C x) \leq \phi(y, x) - \phi(\Pi_C x, x), \quad \forall y \in C,$$

$\langle y - x_0, Jx - Jx_0 \rangle \leq 0$, $\forall y \in C$ *if and only if* $x_0 = \Pi_C x$.

Lemma 1.5. [22] *Let E be a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. Let C be a convex and closed subset of E and let T be a closed asymptotically quasi-ϕ-nonexpansive mapping on C. $Fix(T)$ is convex and closed.*

Lemma 1.6. [23] *Let r be a positive real number and let E be uniformly convex. Then there exists a convex, strictly increasing and continuous function $conf : [0, 2r] \to \mathbb{R}$ such that $cof(0) = 0$ and*

$$\|(1-t)b + ta\|^2 + t(1-t)conf(\|b-a\|) \leq t\|a\|^2 + (1-t)\|b\|^2$$

for all $a, b \in B^r := \{a \in E : \|a\| \leq r\}$ and $t \in [0, 1]$.

2. Main results

Theorem 2.1. *Let E be an uniformly smooth and strictly convex Banach space which also has the Kadec-Klee property. Let C be a convex and closed subset of E and let Λ be an arbitrary index set. Let T_i be an asymptotically quasi-ϕ-nonexpansive mapping on C for every $i \in \Lambda$ such that T_i is uniformly asymptotically regular and closed for every $i \in \Lambda$. Let B be a bifunction with (R-1), (R-2), (R-3) and (R-4). Assume that $Sol(B) \bigcap \cap_{i \in \Lambda} Fix(T_i)$ is nonempty and bounded. Let*

$\{x_j\}$ be a sequence generated by

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_{(1,i)} = C, \forall i \in \Lambda, \\ C_1 = \cap_{i\in\Lambda} C_{(1,i)}, \\ x_1 = Proj_{C_1} x_0, \\ y_{(j,i)} = J^{-1}\big((1-\alpha_{(j,i)})Ju_j + \alpha_{(j,i)} JT_i^j x_j\big), \\ C_{(j+1,i)} = \{z \in C_{(j,i)} : \phi(z, y_{(j,i)}) - \phi(z, x_j) \le \alpha_{(j,i)} \xi_{(j,i)} D_{(j,i)}\}, \\ C_{j+1} = \cap_{i\in\Lambda} C_{(j+1,i)}, x_{j+1} = Proj_{C_{j+1}} x_1, \end{cases}$$

where $\{u_j\}$ is such that $\langle u_j - \mu, Ju_j - Jx_j\rangle \le r_j B(u_j, \mu)$ $\forall \mu \in C$, $\{\alpha_{(j,i)}\}$ is a real sequence in $(0,1)$ such that $\liminf_{j\to\infty} \alpha_{(j,i)}(1-\alpha_{(j,i)}) > 0$, $D_j = \sup\{\phi(z,x_j) : z \in \cap_{i\in\Lambda} Fix(T_i) \bigcap Sol(B)\}$, *and $\{r_j\} \subset [r, \infty)$ is a real sequence, where r is some positive real number. Then $\{x_j\}$ converges strongly to $Proj_{\cap_{i\in\Lambda} Fix(T_i) \bigcap Sol(B)} x_1$.*

Proof. Using Lemma 1.3 and Lemma 1.5, we obtain that the common solution set is convex and closed. So, $Proj_{Sol(B)\bigcap \cap_{i\in\Lambda} Fix(T_i)} x$ is well defined, for any element x in E.

Next, we prove that set C_j is convex and closed. It is obvious that $C_{(1,i)} = C$ is convex and closed. Assume that $C_{(m,i)}$ is convex and closed for some $m \ge 1$. Let $p_1, p_2 \in C_{(m+1,i)}$. It follows that $p = sp_1 + (1-s)p_2 \in C_{(m,i)}$, where $s \in (0,1)$. Notice that $\phi(p_1, y_{(m,i)}) - \phi(p_1, x_m) \le \alpha_{(m,i)} \xi_{(m,i)} D_m$, and $\phi(p_2, y_{(m,i)}) - \phi(p_2, x_m) \le \alpha_{(m,i)} \xi_{(m,i)} D_m$. Hence, one has

$$2\langle p_1, Jx_m - Jy_{(m,i)}\rangle - \|x_m\|^2 + \|y_{(m,i)}\|^2 \le \alpha_{(m,i)} \xi_{(m,i)} D_m,$$

and

$$2\langle p_2, Jx_m - Jy_{(m,i)}\rangle - \|x_m\|^2 + \|y_{(m,i)}\|^2 \le \alpha_{(m,i)} \xi_{(m,i)} D_m.$$

Using the above two inequalities, one has $\phi(p, y_{(m,i)}) - \phi(p, x_m) \le \alpha_{(m,i)} \xi_{(m,i)} D_m$. This shows that $C_{(m+1,i)}$ is closed and convex. Hence, $C_j = \cap_{i\in\Lambda} C_{(j,i)}$ is a convex and closed set. This proves that $Proj_{C_{j+1}} x_1$ is well defined.

Next, we show sequence $\{x_n\}$ is bounded. $Sol(B) \bigcap \cap_{i\in\Lambda} Fix(T_i) \subset C_1 = C$ is clear. Suppose that $Sol(B) \bigcap \cap_{i\in\Lambda} Fix(T_i) \subset C_{(m,i)}$ for some positive integer m. For any $w \in Sol(B) \bigcap \cap_{i\in\Lambda} Fix(T_i) \subset$

$C_{(m,i)}$, we see that

$$\begin{aligned}\phi(z,y_{(m,i)}) &= \|(1-\alpha_{(m,i)})Ju_m+\alpha_{(m,i)}JT_i^m x_m\|^2+\|z\|^2\\ &\quad -2\langle z,\alpha_{(m,i)}JT_i^m x_m+(1-\alpha_{(m,i)})Ju_m\rangle\\ &\le \|z\|^2+\alpha_{(m,i)}\|T_i^m x_m\|^2+(1-\alpha_{(m,i)})\|u_m\|^2\\ &\quad -2\alpha_{(m,i)}\langle z,JT_i^m x_m\rangle-2(1-\alpha_{(m,i)})\langle z,Ju_m\rangle\\ &\le \alpha_{(m,i)}(1+\xi_{(m,i)})\phi(z,x_m)+(1-\alpha_{(m,i)})\phi(z,u_m),\\ &\le \phi(z,x_m)+\alpha_{(m,i)}\xi_{(m,i)}\phi(z,x_m),\end{aligned}$$

where $D_m=\sup\{\phi(z,x_m): z\in\cap_{i\in\Lambda}Fix(T_i)\bigcap Sol(B)\}$. This shows that $z\in C_{(m+1,i)}$. This implies that $Sol(B)\bigcap\cap_{i\in\Lambda}Fix(T_i)\subset\cap_{i\in\Lambda}C_{(j,i)}=C_j$. It follows from Lemma 1.4 that

$$\langle z-x_j,Jx_1-Jx_j\rangle\le 0,\forall z\in C_j.$$

Hence, we have

$$\langle z-x_j,Jx_1-Jx_j\rangle\le 0,\quad \forall z\in Sol(B)\bigcap\cap_{i\in\Lambda}Fix(T_i)\subset C_j. \tag{2.1}$$

In view of Lemma 1.4 yields that

$$\phi(x_j,x_1)\le\phi(Proj_{\cap_{i\in\Lambda}Fix(T_i)\bigcap Sol(B)}x_1,x_1).$$

This implies that $\{\phi(x_j,x_1)\}$ is bounded. Hence, sequence $\{x_j\}$ is also a bounded sequence. Without loss of generality, we assume $x_j\rightharpoonup\bar{x}$. Since C_j is convex and closed, one has $\bar{x}\in C_j$. Hence $\phi(x_j,x_1)\le\phi(\bar{x},x_1)$. This implies that

$$\phi(\bar{x},x_1)\le\liminf_{j\to\infty}(\|x_j\|^2+\|x_1\|^2-2\langle x_j,Jx_1\rangle)=\limsup_{j\to\infty}\phi(x_j,x_1)\le\phi(\bar{x},x_1).$$

It follows that $\lim_{j\to\infty}\phi(x_j,x_1)=\phi(\bar{x},x_1)$. Hence, we have $\lim_{j\to\infty}\|x_j\|=\|\bar{x}\|$. Using the Kadec-Klee property, one obtains that $\{x_j\}$ converges strongly to $\bar{x}$ as $j\to\infty$. Since $\phi(x_{j+1},x_1)-\phi(x_j,x_1)\ge\phi(x_{j+1},x_j)$, one has $\lim_{j\to\infty}\phi(x_{j+1},x_j)=0$. Since $x_{j+1}\in C_{j+1}$, one sees that

$$\phi(x_{j+1},y_{(j,i)})-\phi(x_{j+1},x_j)\le\alpha_{(j,i)}\xi_{(j,i)}D_{(j,i)}.$$

It follows that

$$\lim_{j\to\infty}\phi(x_{j+1},y_{(j,i)})=0.$$

Hence, one has

$$\lim_{j\to\infty}(\|y_{(j,i)}\|-\|x_{j+1}\|)=0.$$

This implies that

$$\lim_{j\to\infty}\|Jy_{(j,i)}\|=\lim_{j\to\infty}\|y_{(j,i)}\|=\|\bar{x}\|=\|J\bar{x}\|.$$

It follows that $\{Jy_{(j,i)}\}$ is bounded. Without loss of generality, we assume that $\{Jy_{(j,i)}\}$ converges weakly to $y^{(*,i)}\in E^*$. In view of the reflexivity of E, we see that $J(E)=E^*$. This shows that there exists an element $y^i\in E$ such that $Jy^i=y^{(*,i)}$. It follows that

$$\phi(x_{j+1},y_{(j,i)})+2\langle x_{j+1},Jy_{(j,i)}\rangle=\|x_{j+1}\|^2+\|Jy_{(j,i)}\|^2.$$

Taking $\liminf_{j\to\infty}$, one has

$$0\geq\|\bar{x}\|^2-2\langle\bar{x},y^{(*,i)}\rangle+\|y^{(*,i)}\|^2=\|\bar{x}\|^2+\|Jy^i\|^2-2\langle\bar{x},Jy^i\rangle=\phi(\bar{x},y^i)\geq 0.$$

That is, $\bar{x}=y^i$, which in turn implies that $J\bar{x}=y^{(*,i)}$. Hence, $Jy_{(j,i)}\rightharpoonup J\bar{x}\in E^*$. Since E^* is uniformly convex. Hence, it has the Kadec-Klee property, we obtain $\lim_{i\to\infty}Jy_{(j,i)}=J\bar{x}$. Since $J^{-1}:E^*\to E$ is demi-continuous and E has the Kadec-Klee property, one gets that $y_{(j,i)}\to\bar{x}$, as $j\to\infty$. Using the fact

$$\phi(z,x_j)-\phi(z,y_{(j,i)})\leq(\|x_j\|+\|y_{(j,i)}\|)\|y_{(j,i)}-x_j\|+2\langle z,Jy_{(j,i)}-Jx_j\rangle,$$

we find

$$\lim_{j\to\infty}\big(\phi(z,x_j)-\phi(z,y_{(j,i)})\big)=0. \tag{2.2}$$

Using Lemma 1.7, one finds that

$$\begin{aligned}\phi(z,y_{(j,i)})&=\|(1-\alpha_{(j,i)})Ju_j+\alpha_{(j,i)}JT_i^jx_j\|^2+\|z\|^2\\&\quad-2\langle z,\alpha_{(j,i)}JT_i^jx_j+(1-\alpha_{(j,i)})Ju_j\rangle\\&\leq(1-\alpha_{(j,i)})\|u_j\|^2+\alpha_{(j,i)}\|T_i^jx_j\|^2+\|z\|^2\\&\quad-2\alpha_{(j,i)}\langle z,JT_i^jx_j\rangle-2(1-\alpha_{(j,i)})\langle z,Ju_j\rangle\\&\quad-\alpha_{(j,i)}(1-\alpha_{(j,i)})conf(\||JT_i^jx_j-Ju_j\|)\\&\leq\phi(z,x_j)-\alpha_{(j,i)}(1-\alpha_{(j,i)})conf(\||JT_i^jx_j-Ju_j\|)+\xi_{(j,i)}D_j.\end{aligned}$$

This implies

$$\alpha_{(j,i)}(1-\alpha_{(j,i)})conf(\||JT_i^jx_j - Ju_j\|) \leq \alpha_{(j,i)}\xi_{(j,i)}D_j + \phi(z,x_j) - \phi(z,y_{(j,i)}).$$

Using (2.2), one has $\lim_{j\to\infty}\||JT_i^jx_j - Ju_j\| = 0$. It follows that $JT_i^jx_j \to J\bar{x}$ as $j \to \infty$. Since $J^{-1} : E^* \to E$ is demi-continuous, one has $T_i^jx_j \rightharpoonup \bar{x}$. Using the fact

$$|\|T_i^jx_j\| - \|\bar{x}\|| = |\|JT_i^jx_j\| - \|J\bar{x}\|| \leq \|JT_i^jx_j - J\bar{x}\|,$$

one has $\|T_i^jx_j\| \to \|\bar{x}\|$ as $j \to \infty$. Since E has the Kadec-Klee property, one has $\lim_{j\to\infty}\||\bar{x} - T_i^jx_j\| = 0$. Since T_i is also uniformly asymptotically regular, one has $\lim_{j\to\infty}\|\bar{x} - T_i^{j+1}x_j\| = 0$. That is, $T_i(T_i^jx_j) \to \bar{x}$. Using the closedness of T_i, we find $T_i\bar{x} = \bar{x}$. This proves $\bar{x} \in Fix(T_i)$, that is, $\bar{x} \in \cap_{i\in\Lambda}Fix(T_i)$.

Next, we show $\bar{x} \in Sol(B)$. Since B is monotone, we find that

$$r_jB_i(\mu, u_j) \leq \|\mu - u_j\|\|Ju_j - Jx_j\|.$$

Therefore, one sees $B(\mu,\bar{x}) \leq 0$. For $0 < \lambda < 1$, define $\mu_\lambda = (1-\lambda)\bar{x} + \lambda\mu$. This implies that $0 \geq B(\mu_\lambda, \bar{x})$. Hence, we have $0 = B(\mu_\lambda, \mu_\lambda) \leq \lambda B(\mu_\lambda, \mu)$. It follows that $B(\bar{x},\mu) \geq 0$, $\forall \mu \in C$. This implies that $\bar{x} \in Sol(B)$.

Finally, we prove $\bar{x} = Proj_{\cap_{i\in\Lambda}Fix(T_i)\cap(B)}x_1$. Using (2.1), one has $\langle \bar{x} - z, Jx_1 - J\bar{x}\rangle \geq 0$ $z \in \cap_{i\in\Lambda}Fix(T_i)Sol(B)$. Using Lemma 1.4, we find $\bar{x} = Proj_{\cap_{i\in\Lambda}Fix(T_i)\cap Sol(B)}x_1$. This completes the proof.

From Theorem 2.1, we have the following results.

Corollary 2.2. *Let E be a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. Let C be a convex and closed subset of E and let Λ be an arbitrary index set. Let T_i be a quasi-ϕ-nonexpansive mapping on C for every $i \in \Lambda$ such that T_i is closed for every $i \in \Lambda$. Let B be a bifunction with (R-1), (R-2), (R-3) and (R-4). Assume that*

$Sol(B)\bigcap\cap_{i\in\Lambda}Fix(T_i)$ is nonempty. Let $\{x_j\}$ be a sequence generated by

$$\begin{cases} x_0\in E \text{ chosen arbitrarily}, \\ C_{(1,i)}=C, \forall i\in\Lambda, \\ C_1=\cap_{i\in\Lambda}C_{(1,i)}, x_1=Proj_{C_1}x_0, \\ y_{(j,i)}=J^{-1}\big((1-\alpha_{(j,i)})Ju_j+\alpha_{(j,i)}JT_ix_j\big), \\ C_{(j+1,i)}=\{z\in C_{(j,i)}:\phi(z,y_{(j,i)})\leq\phi(z,x_j)\}, \\ C_{j+1}=\cap_{i\in\Lambda}C_{(j+1,i)}, x_{j+1}=Proj_{C_{j+1}}x_1, \end{cases}$$

where $D_j=\sup\{\phi(z,x_j):z\in\cap_{i\in\Lambda}Fix(T_i)\bigcap Sol(B)\}$, $\{\alpha_{(j,i)}\}$ is a real sequence in $(0,1)$ such that $\liminf_{j\to\infty}\alpha_{(j,i)}(1-\alpha_{(j,i)})>0$, $\{u_j\}$ is such that $r_jB_i(u_j,\mu)\geq\langle u_j-\mu, Ju_j-Jx_j\rangle$, $\forall\mu\in C$, and $\{r_j\subset[r,\infty)$ is a real sequence, where r is some positive real number. Then $\{x_j\}$ converges strongly to $Proj_{\cap_{i\in\Lambda}Fix(T_i)\bigcap Sol(B)}x_1$.

Corollary 2.3. *Let E be a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. Let C be a convex and closed subset of E. Let B be a bifunction with (R-1), (R-2), (R-3) and (R-4) such that $Sol(B)\neq\emptyset$. Let $\{x_j\}$ be a sequence generated by*

$$\begin{cases} x_0\in E \text{ chosen arbitrarily}, \\ C_1=C, \forall i\in\Lambda, \\ x_1=Proj_{C_1}x_0, \\ y_j=J^{-1}\big((1-\alpha_j)Ju_j+\alpha_jJx_j\big), \\ C_{j+1}=\{z\in C_j:\phi(z,y_j)\leq\phi(z,x_j)\}, \\ x_{j+1}=Proj_{C_{j+1}}x_1, \end{cases}$$

where $\{\alpha_j\}$ is a real sequence in $(0,1)$ such that $\liminf_{j\to\infty}\alpha_j(1-\alpha_j)>0$, $\{u_j\}$ is such that $r_jB_i(u_j,\mu)\geq\langle u_j-\mu, Ju_j-Jx_j\rangle$, $\forall\mu\in C$, and $\{r_j\subset[r,\infty)$ is a real sequence, where r is some positive real number. Then $\{x_j\}$ converges strongly to $Proj_{Sol(B)}x_1$.

Next, we consider the problem of finding solutions of a variational inequalities in the framework Banach spaces. Let $A : C \to E^*$ be a single valued monotone operator which is continuous along each line segment in C with respect to the weak* topology of E^* (hemicontinuous). Recall the the following variational inequality. Finding a point $x \in C$ such that $\langle x-y, Ax\rangle \le 0$, $\forall y \in C$. The symbol $Nc(x)$ stand for the normal cone for C at a point $x \in C$; that is, $Nc(x) = \{x^* \in E^* : \langle x-y, x^*\rangle \ge 0,\ \forall y \in C\}$. From now on, we use $VI(C,A)$ to denote the solution set of the variational inequality.

Corollary 2.4. *Let E be a strictly convex and uniformly smooth Banach space which also has the Kadec-Klee property. Let C be a convex and closed subset of E. Let $A : C \to E^*$ be a single valued, monotone and hemicontinuous operator such that Assume that $VI(C,A)$ is not empty. Let $\{x_j\}$ be a sequence generated in the following process.*

$$\begin{cases} x_0 \in E \textit{ chosen arbitrarily}, \\ C_1 = C, \forall i \in \Lambda, \\ x_1 = Proj_{C_1} x_0, \\ u_j = VI(C, A + \frac{1}{r}(J - Jx_n)), \\ Jy_j = (1-\alpha_j)Ju_j + \alpha_j Jx_j, \\ C_{j+1} = \{z \in C_j : \phi(z, y_j) \le \phi(z, x_j)\}, \\ x_{j+1} = Proj_{C_{j+1}} x_1, \end{cases}$$

where $\{\alpha_j\}$ is a real sequence in $(0,1)$ such that $\liminf_{j\to\infty} \alpha_j(1-\alpha_j) > 0$. Then $\{x_j\}$ converges strongly to $Proj_{VI(C,A)}x_1$.

Proof. Define a new operator M by

$$Mx = \begin{cases} Ax + Nc(x), & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

Hence, M is maximal monotone and $M^{-1}(0) = VI(C,A)$ [24], where $M^{-1}(0)$ stand for the zero point set of M. For each $r > 0$, and $x \in E$, we see that there exists a unique x_r in the domain of M such that $Jx \in Jx_{r_i} + rM(x_r)$, where $x_r = (J + rM)^{-1}Jx$. Notice that $u_j = VI(C, \frac{1}{r}(J - Jx) + A)$, which is equivalent to $\langle u_j - y, Az_j + \frac{1}{r}(Jz_j - Jx_j)\rangle \leq 0, \forall y \in C$, that is, $\frac{1}{r}\big(Jx_j - Ju_j\big) \in Nc(u_j) + Az_j$. This implies that $u_j = (J + rM)^{-1}Jx_j$. From [20], we find that $(J + rM)^{-1}J$ is closed quasi-ϕ-nonexpansive with $Fix((J + rM)^{-1}J) = M^{-1}(0)$. Using Theorem 2.1, we find the desired conclusion immediately.

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