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INEQUALITIES FOR ALGEBRAIC POLYNOMIALS IN REGIONS WITH EXTERIOR CUSPS

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Abstract. In this paper, we study the estimation for algebraic polynomials in the bounded and unbounded regions with piecewise smooth boundary having exterior zero angles at the boundary.

Keywords. Algebraic polynomials; Conformal mapping; Smooth curve.

1. Introduction and main results

Let $G \subset \mathbb{C}$ be a finite region, with $0 \in G$, bounded by a Jordan curve $L := \partial G$, $\Omega := extL := \overline{\mathbb{C}} \setminus \overline{G}$, where $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$; $\Delta := \Delta(0,1) := \{w : |w| > 1\}$. Let function $w = \Phi(z)$ be the univalent conformal mapping of Ω onto the Δ normalized by $\Phi(\infty) = \infty$, $\Phi^{'}(\infty) > 0$, and $\Psi := \Phi^{-1}$.

Let $\wp_n$ denote the class of arbitrary algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N} := \{1, 2, ...\}$.

Let $h(z)$ be a weight function. For any $p > 0$ we introduce:

$$\|P_n\|_{A_p(h,G)} := \left(\iint_G h(z) \left| P_n(z) \right|^p d\sigma_z \right)^{1/p} < \infty, \tag{1}$$

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where σ_z is the two-dimensional Lebesgue measure,

$$\|P_n\|_{L_p(h,L)} := \left(\int_L h(z) \left| P_n(z) \right|^p |dz| \right)^{1/p} < \infty, \tag{2}$$

when L is rectifiable and

$$\|P_n\|_{C(\overline{G})} := \max_{z \in \overline{G}} |P_n(z)| . \tag{3}$$

Well known Bernstein -Walsh Lemma [24] shows that:

$$|P_n(z)| \leq |\Phi(z)|^n \|P_n\|_{C(\overline{G})}, \;\; z \in \Omega. \tag{4}$$

For $R > 1$ let us set $L_R := \{z : \; |\Phi(z)| = R\}, G_R := intL_R$, $\Omega_R := extL_R$. Then, (4) can be written as following:

$$\|P_n\|_{C(\overline{G}_R)} \leq R^n \|P_n\|_{C(\overline{G})} . \tag{5}$$

Hence, if we take $R = 1 + \frac{1}{n}$, according to (5), we see that the $C-$norm of polynomials $P_n(z)$ in $\overline{G}_R$ and $\overline{G}$ are equivalent, i.e. the norm $\|P_n\|_{C(\overline{G})}$ increases with at most by a constant.

The same effect is observed for the norm (2) according to the following estimate [13]:

$$\|P_n\|_{L_p(L_R)} \leq R^{n+\frac{1}{p}} \|P_n\|_{L_p(L)}, \;\; p > 0. \tag{6}$$

To give a similar estimation to (5) and (6) inequalities, firstly we will give some definitions and notations.

We can find a well known definition of a K-quasiconformal curve in [8], [15, p.97], [19, p.286] and [20] as following:

Definition 1.1. The Jordan curve (or arc) L is called $K-$quasiconformal ($K \geq 1$), if there is a $K-$quasiconformal mapping f of the region $D \supset L$ such that $f(L)$ is a circle (or line segment).

Let $F(L)$ denote the set of all sense preserving plane homeomorphisms f of the region $D \supset L$ such that $f(L)$ is a line segment (or circle) and let

$$K_L := \inf \{K(f) : f \in F(L)\},$$

where $K(f)$ is the maximal dilatation of a such mapping f. Then L is a quasiconformal curve, if $K_L < \infty$, and L is a $K-$quasiconformal curve, if $K_L \leq K$.

We well know that there exist quasiconformal curves which are not rectifiable [15, p.104].

Let $\{z_j\}_{j=1}^{m} \in L$ be a fixed system of distinct points. Consider a so-called generalized Jacobi weight function $h(z)$ defined as follows:

$$h(z) := \prod_{j=1}^{m} \left|z - z_j\right|^{\gamma_j}, \quad z \in G_R. \tag{7}$$

where $\gamma_j > -1$ for every $j = 1, 2, ..., m$.

The Bernstein-Walsh type estimation for the regions G with quasiconformal boundary and weight function $h(z)$, as defined in (7) with $\gamma_j > -2$ for any $p > 0$ was contained in [3] as follows:

$$\|P_n\|_{A_p(h,G_R)} \le c_1 \, R^{*^{n+\frac{1}{p}}} \|P_n\|_{A_p(h,G)}, \tag{8}$$

where $R^* := 1 + c_2(R-1)$, $c_2 > 0$ and $c_1 := c_1(G, p, c_2) > 0$ constants, independent from n and R. Therefore, if we choose $R = 1 + \frac{c_3}{n}$, for some $c_3 > 0$ absolute constants, then (8) shows that the $A_p(h,.)-$norm of polynomials $P_n(z)$ in G_R and G are equivalent.

The analogue of the (8) for $h(z) \equiv 1$ and Jordan regions was obtained in [7].

Stylianopoulos in [21] replaced the norm $\|P_n\|_{C(\overline{G})}$ with norm $\|P_n\|_{A_2(G)}$ on the right-hand side of (4) and found a new version of the Bernstein-Walsh Lemma:

Lemma A. [21] *Assume that L is quasiconformal and rectifiable. Then there exists a constant $c = c(L) > 0$ depending only on L such that*

$$|P_n(z)| \le c \frac{\sqrt{n}}{d(z,L)} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1}, \;\; z \in \Omega, \tag{9}$$

holds for every $P_n \in \wp_n$, where $d(z,L) := \inf\{|\zeta - z| : \; \zeta \in L\}$.

Analogous results to (9) with $\|.\|_{A_p(h,G)}$ at the right side for some regions and the weight function $h(z)$ defined in (7) with $\gamma_j > -2$ were obtained in [4], [6], [5] and in others.

In this work, firstly, we study similar problems for $z \in \Omega$ in regions with piecewise smooth boundary with exterior zero angles and generalized Jacobi weight function $h(z)$ defined in (7) through $\mathscr{L}_p(h,L)-$norm, $p > 1$, in the right side of (9). Secondly, for such regions we obtain estimate $|P_n(z)|$ for $z \in \overline{G}$ as follows:

$$\|P_n\|_{C(\overline{G})} \le c\mu_n(G,p) \|P_n\|_{L_p(h,L)}, \tag{10}$$

where $\mu_n(G,p) \to 0$, $n \to \infty$, $c = c(G,p) > 0$ is a constant independent from n and P_n. Finally, combining corresponding estimates, we obtain an estimate for $|P_n(z)|$ in whole complex plane when region G has exterior zero and nonzero angles (without interior cusps), depending on the geometrical properties of region G and weight function $h(z)$.

We note that, the first results of type (10), in case $h(z) \equiv 1$ for $L = \{z : |z| = 1\}$ and $0 < p < \infty$, was found by Jackson [14]. Suetin [22], [23] investigated this problem with sufficiently smooth Jordan curve. The estimate of (10) type for $0 < p < \infty$ and $h(z) \equiv 1$, when L is rectifiable Jordan curve, was investigated by Mamedhanov [16]. Same results of the (10) type were obtained by Nikol'skii [17] and others for the polynomials of several variables.

Let us give some definitions and notations that will be used later in the text.

Let S be rectifiable Jordan curve or arc and let $z = z(s),\ s \in [0,\ |S|],\ |S| := mes\,S$, denote the natural representation of S.

Definition 1.2. We say that a Jordan curve or arc $S \in C_\theta$, if S has a continuous tangent $\theta(z) := \theta(z(s))$ at every point $z(s)$. We will write a region $G \in C_\theta$ if $\partial G \in C_\theta$.

According to [20], we have the following facts:

Corollary 1.1. *If $S \in C_\theta$, then S is $(1+\varepsilon)-$quasiconformal for arbitrary small $\varepsilon > 0$.*

Corollary 1.2. *If S is an analytic curve or arc, then S is $1-$quasiconformal.*

Now, we shall define a new class of regions with piecewise smooth boundary, which have at the boundary points corners and exterior cusps simultaneously.

For any $j = 1, 2, \ldots$ and sufficiently small $\varepsilon_1 > 0$ we denote by $f_j : [0, \varepsilon_1] \to \mathbb{R}$ the twice differentiable functions such that $f_j(0) = 0,\ f_j^{(k)}(x) > 0,\ x > 0$ and $k = 0, 1, 2$.

Throughout this paper, $c, c_0, c_1, c_2, \ldots$ are positive and $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ are sufficiently small positive constants (generally, different in different relations), which depend on G in general.

Definition 1.2. We say that a Jordan region $G \in C_\theta(\lambda_i; f_j)$, $0 < \lambda_i < 2$, $i = \overline{1, m_1}$, $f_j = f_j(x)$, $j = \overline{m_1 + 1, m}$, if $L = \partial G$ consists of a union of finite number of $C_\theta-$ arcs $\{L_j\}_{j=0}^m$, connecting at the points $\{z_j\}_{j=0}^m \in L$, such that L is locally smooth at $z_0 \in L \backslash \ \{z_j\}_{j=1}^m$ and:

a) for every $z_i \in L,\ i = \overline{1, m_1},\ m_1 \le m$, the region G has exterior (with respect to $\overline{G}$) angles $\lambda_i \pi,\ 0 < \lambda_i < 2$, at the corner z_i;

b) for every $z_j \in L,\ j=\overline{m_1+1,m}$, in the local co-ordinate system (x,y) with origin at z_j the following conditions are satisfied:

$b_1)$ $\left\{z=x+iy : |z|<\varepsilon_1,\ c_1 f_j(x) \le y \le c_2 f_j(x),\ 0 \le x \le \varepsilon_1\right\} \subset \overline{\Omega}$,

$b_2)$ $\left\{z=x+iy : |z|<\varepsilon_1,\ |y| \ge \varepsilon_2 x,\ 0 \le x \le \varepsilon_1\right\} \subset \overline{G}$,

for some constants $-\infty < c_1 < c_2 < +\infty,\ 0<\varepsilon_i<1,\ i=1,2$.

Here and in further, for any $k \ge 0$ and $m>k$, notation $j=\overline{k,m}$ denotes $j=k,k+1,...,m$.

For any $z \in \mathbb{C}$ and sufficiently small $\varepsilon_1 > 0$ let

$$K(z,f_j,\varepsilon_1) := \left\{z=x+iy : 0 \le x \le \varepsilon_1,\ c_1 f_j(x) \le y \le c_2 f_j(x)\right\},$$

for some constants $-\infty < c_1 < c_2 < +\infty$. Let $f(x)=f_{j_0}(x)$, where $j_0,\ m_1+1 \le j_0 \le m$, is chosen such that $K(z_j,f_{j_0},\varepsilon_1) \subseteq K(z_j,f_j,\varepsilon_1)$ for all $m_1+1 \le j \le m$ and sufficiently small $\varepsilon_1 > 0$.

As it is clear from the Definition 1.3, each region $G \in C_\theta(\lambda_i;f_j)$ may have exterior nonzero $\lambda_i\pi,\ 0<\lambda_i<2$, angles at the points $z_i \in L,\ i=\overline{1,m_1}$, and exterior zero angles at which the boundary arcs are touching with $f_j(x)-$speed at the points $z_j \in L,\ j=\overline{m_1+1,m}$. If $m_1 = m=0$, then the region G doesn't have such angles, and in this case we will write: $G \in C_\theta$; if $m_1=m \ge 1$, then G has only $\lambda_i\pi,\ 0<\lambda_i<2,\ i=\overline{1,m_1}$, exterior nonzero angles, and in this case we will write: $G \in\ C_\theta(\lambda_i;0)$; if $m_1=0$ and $m \ge 1$, then G has only exterior zero angles, and in this case we will write: $G \in C_\theta(1;f_j))$.

According to the "three-point" criterion [8, p.100], every piecewise smooth curve (without any cusps) is quasiconformal.

Throughout this work, we will assume that the points $\left\{z_j\right\}_{j=1}^m \in L$ defined in (7) and Definition 1.3 are identical.

Without loss of generality, we assume that the points $\left\{z_j\right\}_{j=1}^m$ are ordered in the positive direction on curve L such that, G has $\lambda_j\pi,\ 0<\lambda_j<2,\ j=\overline{1,m_1}$, exterior nonzero angles at the points $\left\{z_j\right\}_{j=1}^{m_1}$, $m_1 \le m$, and has exterior zero angles on the points $\left\{z_j\right\}_{j=m_1+1}^{m}$ and $w_j := \Phi(z_j)$.

Let us set: $\Gamma := \Gamma_{1,m_1} \cup \Gamma_{2,m}$, where $\Gamma_{1,k} := \left\{\gamma_j,\ j=\overline{1,k},\ k \le m_1\right\}$

$$\Gamma_{2,k} := \left\{\gamma_j,\ j=\overline{m_1+1,\ m_1+k},\ 1 \le k \le m-m_1\right\};$$

$\Gamma_{1,k}^{(1)} := \left\{ \gamma_k \in \Gamma_{1,m_1} : \gamma_k > p-1,\ k \leq m_1 \right\}$ (with possible skips),

$$\Gamma_{1,k}^{(2)} := \left\{ \gamma_k \in \Gamma_{1,m_1} : \gamma_k = p-1,\ k \leq m_1 \right\},$$

$\Gamma_{1,k}^{(3)} := \left\{ \gamma_k \in \Gamma_{1,m_1} : \gamma_k < p-1,\ k \leq m_1 \right\}; \Gamma_{2,k}^{(1)} := \left\{ \gamma_k \in \Gamma_{2,m} : \gamma_k > p-1,\ m_1+1 \leq k \leq m \right\}$ (with possible skips); $\Gamma_{2,k}^{(2)} := \left\{ \gamma_k \in \Gamma_{2,m} : \gamma_k = p-1,\ m_1+1 \leq k \leq m \right\}$,

$$\Gamma_{2,k}^{(3)} := \left\{ \gamma_k \in \Gamma_{2,m} : \gamma_k < p-1,\ m_1+1 \leq k \leq m \right\};$$

$\gamma_k^1 = \max\left\{ \gamma_k : \gamma_k \in \Gamma_{1,k},\ k \leq\ m_1 \right\}$, $\gamma^1 := \gamma_{m_1}^1$; $\widetilde{\gamma}_k^1 = \max\left\{ 0; \gamma_k : \gamma_k \in \Gamma_{1,k},\ k \leq\ m_1 \right\}$, $\widetilde{\gamma}^1 := \widetilde{\gamma}_{m_1}^1$; $\gamma_j^2 = \max\left\{ \gamma_j : \gamma_j \in \Gamma_{2,k},\ m_1+1 \leq j \leq m_1+k,\ 1 \leq\ k \leq\ m-m_1 \right\}$, $\gamma^2 := \gamma_m^2$;

$$\widetilde{\gamma}_k^2 = \max\left\{ 0; \gamma_k : \gamma_k \in \Gamma_{2,k},\ k \leq\ m \right\},$$

$\widetilde{\gamma}^2 := \widetilde{\gamma}_m^2$; $\gamma_k^* = \max\{\gamma_k : \gamma_k \in \Gamma,\ k \leq\ m\}$, $\gamma^* := \gamma_m^*$; $\widetilde{\gamma}_k^* = \max\{0; \gamma_k : \gamma_k \in \Gamma,\ k \leq\ m\}$, $\widetilde{\gamma}^* := \widetilde{\gamma}_m^*$.
$\widetilde{\gamma}_{*,k}^2 := \left\{ \begin{array}{ll} 0; & -1 < \gamma_k \leq 0 \\ \min \gamma_k & \gamma_k > 0 \end{array} ,\gamma_k \in \Gamma_{2,k},\ k \leq\ m \right\}$; $\widetilde{\gamma}_{*,}^2 := \widetilde{\gamma}_{*,m}^2$. $\widetilde{\lambda}_k = \max\{\lambda_j; 1 : j = \overline{1,k},\ k \leq m_1\}$, $\tilde{\lambda} := \tilde{\lambda}_{m_1}$; $\alpha_* := \min\left\{ \alpha_j,\ j = \overline{m_1+1,m} \right\}$, $\alpha^* := \max\left\{ \alpha_j,\ j = \overline{m_1+1,m} \right\}$.

Now we can state our new results.

Theorem 1.1. *Let* $p > 1$; $G \in C_\theta(\lambda_i; f_j)$, *for some* $0 < \lambda_i < 2$, $i = \overline{1,m_1}$ *and* $f_j(x) = cx^{1+\alpha_j}$, $\alpha_j > 0$, $j = \overline{m_1+1,m}$; $h(z)$ *be defined as in (7).Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *and arbitrary small* $\varepsilon > 0$, *we have:*

$$|P_n(z)| \leq c_1 \frac{B_{n;1}}{d(z,L)} \|P_n\|_{\mathcal{L}_p(h,L)} \cdot \left\{ \begin{array}{ll} |\Phi(z)|^{n+1}, & z \in \Omega, \\ 1, & z \in G, \end{array} \right. \tag{11}$$

where $c_1 = c_1(G,\ p,\varepsilon) > 0$ *is the constant, independent from* z *and* n, *and*

$$B_{n;1} := \begin{cases} \sum\limits_{i=1}^{m_1} n^{\frac{(\gamma_i+1-p)\widetilde{\lambda}_i}{p}+\varepsilon} & \textit{if } \gamma_i \in \Gamma_{1,k}^{(1)} \\ & \textit{for all } i=\overline{1,k},\ 1\le k\le m_1 \\ +\sum\limits_{j=m_1+1}^{m} n^{\frac{\gamma_j+1-p}{p(1+\alpha_j)}+\varepsilon}, & \textit{and if } \gamma_j \in \Gamma_{2,k}^{(1)} \\ & \textit{for all } j=\overline{m_1+1,k},\ m_1+1\le k\le m; \\ (\ln n)^{1-\frac{1}{p}}, & \textit{if } \gamma_i \in \Gamma_{1,m_1}^{(2)} \textit{ for all } i=\overline{1,m_1} \textit{ and} \\ & \textit{if } \gamma_j \in \Gamma_{2,m}^{(2)} \textit{ for all } j=\overline{m_1+1,m}; \\ 1, & \textit{if } \gamma_i \in \Gamma_{1,m_1}^{(3)} \textit{ for all } i=\overline{1,m_1} \textit{ and} \\ & \textit{if } \gamma_j \in \Gamma_{2,m}^{(3)} \textit{ for all } j=\overline{m_1+1,m}. \end{cases} \tag{12}$$

Theorem 1.1 is local, that is, each term in the sums on the right side of $B_{n;1}$ shows the local growth of $|P_n(z)|$, depending on the behavior of the weight function $h(z)$ and the boundary L in the neighborhood of each point $z_j \in L$ for any $j=\overline{1,m}$. Comparing the terms in the sums for each point $\{z_j\}$, $j=\overline{1,m}$, and using the above notations, we can obtain the following result of global character:

Theorem 1.2. *Let* $p>1$; $G\in C_\theta(\lambda_i;f_j)$, *for some* $0<\lambda_i<2$, $i=\overline{1,m_1}$ *and* $f_j(x)=cx^{1+\alpha_j}$, $\alpha_j>0$, $j=\overline{m_1+1,m}$; $h(z)$ *be defined as in (7). Then, for any* $P_n\in\wp_n, n\in\mathbb{N}$, *and arbitrary small* $\varepsilon>0$, *we have:*

$$|P_n(z)| \le c_2 \frac{B_{n;2}}{d(z,L)} \|P_n\|_{\mathscr{L}_p(h,L)} \cdot \begin{cases} |\Phi(z)|^{n+1}, & z\in\Omega, \\ 1 & z\in G, \end{cases} \tag{13}$$

where $c_2 = c_2(G,p,m,\varepsilon) > 0$ *is the constant, independent from z and n, and*

$$(14) \qquad B_{n;2} := \begin{cases} n^{\frac{\left(\gamma^1+1-p\right)\widetilde{\lambda}}{p}+\varepsilon} + n^{\frac{\gamma^2+1-p}{p(1+\alpha_*)}+\varepsilon}, & \begin{array}{c} \text{if there is at least one } \gamma_i \in \Gamma_{1,k}^{(1)} \\ \text{for some } i=\overline{1,k},\ 1\le k\le m_1 \\ \text{and if there is at least one } \gamma_j \in \Gamma_{2,k}^{(1)} \\ \text{for some } j=\overline{m_1+1,k};\ m_1+1\le k\le m; \end{array} \\ (\ln n)^{1-\frac{1}{p}}, & \begin{array}{c} \text{if there is at least one} \\ \gamma_j \in (\Gamma_{1,k}^{(2)} \cup \Gamma_{2,k}^{(2)}) \setminus (\Gamma_{1,m_1}^{(1)} \cup \Gamma_{2,m}^{(1)}) \\ \text{for some } j=\overline{1,m},\ k\le m; \end{array} \\ 1, & \begin{array}{c} \text{if } \gamma_j \in \Gamma_{1,k}^{(3)} \cup \Gamma_{2,k}^{(3)} \\ \text{for all } j=\overline{1,m}. \end{array} \end{cases}$$

In particular, in case of L having two singular points $z_1 \in L$ and $z_2 \in L$ (i.e. $m_1 = 1,\ m = 2$), we obtain the following:

Corollary 1.3. *Let* $p > 1;\ G \in C_\theta(\lambda_1; cx^{1+\alpha_2})$, *for some* $0 < \lambda_1 < 2,\ \alpha_2 > 0;\ h(z)$ *be defined as in (7) for* $j = 2$. *Then, for any* $P_n \in \wp_n,\ n \in \mathbb{N}$, *and arbitrary small* $\varepsilon > 0$, *we have:*

$$(15) \qquad |P_n(z)| \le c_3 \frac{B_{n;3}}{d(z,L)} \|P_n\|_{\mathscr{L}_p(h,L)} \begin{cases} |\Phi(z)|^{n+1}, & z \in \Omega, \\ 1 & z \in G, \end{cases}$$

where $c_3 = c_3(G,p,\varepsilon) > 0$ *is the constant, independent from z and n, and*

$$(16) \qquad B_{n,3} := \begin{cases} n^{\frac{(\gamma_1+1-p)\widetilde{\lambda}_1}{p}+\varepsilon} + n^{\frac{\gamma_2+1-p}{p(1+\alpha_2)}+\varepsilon}, & \text{if } \gamma_1,\gamma_2 > p-1, \\ (\ln n)^{1-\frac{1}{p}}, & \begin{array}{c} \text{if } \gamma_1 = p-1,\ -1<\gamma_2\le p-1 \\ \text{or } -1<\gamma_1\le p-1,\ \gamma_2 = p-1 \end{array} \\ 1, & \text{if } -1<\gamma_1,\gamma_2<p-1. \end{cases}$$

We can take individual cases when the region G has only one singular point on the boundary L: exterior nonzero angle or exterior zero angle. In this case, from Corollary 1.3, we obtain the following:

Corollary 1.4. *Under the conditions of Corollary 1.3, we have:*

$$|P_n(z)| \leq c_4 \frac{B_{n;4}}{d(z,L)} \|P_n\|_{\mathscr{L}_p(h,L)} \begin{cases} |\Phi(z)|^{n+1}, & z \in \Omega, \\ 1 & z \in G, \end{cases} \tag{17}$$

where $c_4 = c_4(G,p,\varepsilon) > 0$ *is the constant, independent from* z *and* n*, and*

$$B_{n,4} := \begin{cases} n^{\frac{(\gamma_1+1-p)\widetilde{\lambda}_1}{p}+\varepsilon}, & \text{if } \alpha_2 = 0,\ \gamma_1 > p-1, \\ n^{\frac{\gamma_2+1-p}{p(1+\alpha_2)}+\varepsilon}, & \text{if } \lambda_1 = 1, \alpha_2 > 0,\ \gamma_2 > p-1; \\ (\ln n)^{1-\frac{1}{p}}, & \text{if } \alpha_2 = 0,\ \gamma_1 = p-1, \\ & \text{or } \lambda_1 = 1,\ \gamma_2 = p-1; \\ 1, & \text{if } \alpha_2 = 0,\ -1 < \gamma_1 < p-1, \\ & \text{or } \lambda_1 = 1,\ -1 < \gamma_2 < p-1. \end{cases} \tag{18}$$

As we can see from the above results, it remains to find an estimate on the $|P_n(z)|$ for the points $z \in \overline{G}$. In this case the following is true:

Theorem 1.3. *Let* $p > 1$; $G \in C_\theta(\lambda_i; f_j)$, *for some* $0 < \lambda_i < 2$, $i = \overline{1, m_1}$ *and* $f_j(x) = cx^{1+\alpha_j}$, $\alpha_j > 0$, $j = \overline{m_1+1, m}$; $h(z)$ *be defined as in (7). Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, *and arbitrary small* $\varepsilon > 0$, *we have:*

$$\|P_n\|_{C(\overline{G})} \leq c_5 \left(\sum_{i=1}^{m_1} n^{\frac{(\widetilde{\gamma}_i+1)\widetilde{\lambda}_i}{p}+\varepsilon} + \sum_{i=m_1+1}^{m} D^i_{n,m} \right) \|P_n\|_{L_p(h,\ L)}. \tag{19}$$

where $c_5 = c_5(G,p,\lambda_i,\alpha_i,\varepsilon) > 0$ *is the constant, independent from* z *and* n,

$$D^i_{n,m} := \begin{cases} n^{\frac{\widetilde{\gamma}_i+1}{p(1+\alpha_i)}+\frac{\alpha_i}{p(1+\alpha_i)}+\varepsilon}, & 1 < p < 2 + \frac{\widetilde{\gamma}_i}{1+\alpha_i}, \\ n^{1-\frac{1}{p}+\varepsilon}, & p \geq 2 + \frac{\widetilde{\gamma}_i}{1+\alpha_i}, \end{cases} \quad \text{for any } i = \overline{m_1+1, m}.$$

Corollary 1.5. *Under the conditions of Theorem 1.3, we have*

$$\|P_n\|_{C(\overline{G})} \leq c_6 \left(D_{n,m_1} + D_{n,m}\right) \|P_n\|_{L_p(h,\ L)}, \tag{20}$$

$$D_{n,m_1} := n^{\frac{(\widetilde{\gamma}^1+1)\widetilde{\lambda}}{p}+\varepsilon};\ D_{n,m} := \begin{cases} n^{\frac{\widetilde{\gamma}^2+1}{p(1+\alpha_*)}+\frac{\alpha_*}{p(1+\alpha_*)}+\varepsilon}, & 1<p<2+\frac{\widetilde{\gamma}_*^2}{1+\alpha^*}, \\ n^{1-\frac{1}{p}+\varepsilon}, & p \geq 2+\frac{\widetilde{\gamma}^2}{1+\alpha_*}, \end{cases}$$

where $c_6 = c_6(G,p,\lambda_i,\alpha_i,\varepsilon) > 0$ is the constant, independent from z and n.

In particular, when the region have one exterior nonzero angle $\lambda_1\pi,\ 0<\lambda_1<2$, at the boundary point z_1 and one exterior zero angle $x^{1+\alpha_2}, \alpha_2 > 0$, at the boundary point z_2, then from Theorem 1.3 we obtain the following:

Corollary 1.6. *Let $p>1$; $G \in C_\theta(\lambda_1; f_2)$, for some $0<\lambda_1<2$ and $f_2(x) = cx^{1+\alpha_2},\ \alpha_2 > 0$; $h(z)$ be defined as in (7). Then, for any $P_n \in \wp_n,\ n \in \mathbb{N}$, and arbitrary small $\varepsilon > 0$, we have:*

$$\|P_n\|_{C(\overline{G})} \leq c_7\left(D_{n,1}+D_{n,2}\right)\|P_n\|_{L_p(h,\,L)}, \tag{21}$$

where $c_7 = c_7(G,p,\lambda_1,\alpha_2,\varepsilon) > 0$ is the constant, independent from z and n, and

$$D_{n,1} := n^{\frac{(\widetilde{\gamma}_1+1)\widetilde{\lambda}}{p}+\varepsilon};\ D_{n,2} := \begin{cases} n^{\frac{\widetilde{\gamma}_2+1}{p(1+\alpha_2)}+\frac{\alpha_2}{p(1+\alpha_2)}+\varepsilon}, & 1<p<2+\frac{\widetilde{\gamma}_2}{1+\alpha_2}, \\ n^{1-\frac{1}{p}+\varepsilon}, & p \geq 2+\frac{\widetilde{\gamma}_2}{1+\alpha_2}. \end{cases}$$

Combining Corollary 1.6 with Corollary 1.4, we obtain an estimate at the behavior of $|P_n(z)|$ on the whole complex plane. For simplicity, in case $m_1 = 1,\ m = 2$ we obtain:

Corollary 1.7. *Let $p>1$; $G \in C_\theta(\lambda_1; f_2)$, for some $0<\lambda_1<2$ and $f_2(x) = cx^{1+\alpha_2},\ \alpha_2 > 0$; $h(z)$ be defined as in (7). Then, for any $P_n \in \wp_n,\ n \in \mathbb{N}$, and arbitrary small $\varepsilon > 0$, we have:*

$$|P_n(z)| \leq c_8\|P_n\|_{\mathscr{L}_p(h,L)} \begin{cases} \frac{B_{n,4}}{d(z,L)}\left|\Phi(z)\right|^{n+1}, & z \in \Omega, \\ D_{n,1}+D_{n,2}, & z \in \overline{G}, \end{cases} \tag{22}$$

where $c_8 = c_8(G,p,\varepsilon) > 0$ is the constant, independent from z and n; $B_{n;4}$ and $D_{n,1}, D_{n,2}$ defined as in (18) and (21), respectively.

Sharpness of estimates

The sharpness of the estimations (11)-(22) for some special cases can be discussed by comparing them with the following results:

Remark For any $n\in\mathbb{N}$ and $i=1,2$ there exist polynomials $P_n^{(i)}\in\wp_n$, regions $G^i\subset\mathbb{C}$ and constants $c_9=c_9(G^1)>0$, $c_{10}=c_{10}(G^2)>0$ such that

$$\left\|P_n^{(1)}\right\|_{C(\overline{G^1})}\geq c_9 n^{\frac{1}{p}}\left\|P_n^{(1)}\right\|_{L_p(\partial G^1)}, \tag{23}$$

and

$$\left|P_n^{(2)}(z)\right|\geq c_{10}\left|\Phi(z)\right|^n\left\|P_n^{(2)}\right\|_{L_2(\partial G^2)},\ \forall z\in F\Subset\overline{\mathbb{C}}\setminus\overline{G^2}. \tag{24}$$

2. Some auxiliary results

Recall that, as noted above throughout this work, $c,c_0,c_1,c_2,...$ are positive constants (generally, different in different relations), which depend on G in general. Further, for the nonnegative functions $a>0$ and $b>0$, we shall use the notations "$a\preceq b$" (order inequality), if $a\leq cb$ and "$a\asymp b$" are equivalent to $c_1a\leq b\leq c_2a$ for some constants c,c_1,c_2 (independent of a and b), respectively.

Lemma 2.1. [1],[2] *Let L be a $K-$quasiconformal curve,* $z_1\in L$, $z_2,z_3\in\Omega\cap\{z:|z-z_1|\preceq d(z_1,L_{r_0})\}$; $w_j=\Phi(z_j)$, $(z_2,z_3\in G\cap\{z:|z-z_1|\preceq d(z_1,L_{R_0})\};\ w_j=\varphi(z_j))$, $j=1,2,3$. *Then*

a) *The statements* $|z_1-z_2|\preceq|z_1-z_3|$ *and* $|w_1-w_2|\preceq|w_1-w_3|$ *are equivalent.*

So are $|z_1-z_2|\asymp|z_1-z_3|$ *and* $|w_1-w_2|\asymp|w_1-w_3|$;

b) *If* $|z_1-z_2|\preceq|z_1-z_3|$, *then*

$$\left|\frac{w_1-w_3}{w_1-w_2}\right|^{K^2}\preceq\left|\frac{z_1-z_3}{z_1-z_2}\right|\preceq\left|\frac{w_1-w_3}{w_1-w_2}\right|^{K^{-2}},$$

where $0<r_0<1,\ R_0:=r_0^{-1}$ *are constants, depending on G.*

Corollary 2.1. *Under the assumptions of Lemma 2.1, for* $z_3\in L_{r_0}$ $(z_3\in L_{R_0})$

$$|w_1-w_2|^{K^2}\preceq|z_1-z_2|\preceq|w_1-w_2|^{K^{-2}}$$

If $L\in C_\theta$, then for all $\varepsilon>0$

$$|w_1-w_2|^{1+\varepsilon}\preceq|z_1-z_2|\preceq|w_1-w_2|^{1-\varepsilon}.$$

Recall that for $0<\delta_j<\delta_0:=\frac{1}{4}\min\left\{\left|z_i-z_j\right|: i,j=1,2,...,m,\ i\neq j\right\}$, we put $\Omega(z_j,\ \delta_j):=\Omega\cap\left\{z:\left|z-z_j\right|\leq\delta_j\right\}$; $\delta:=\min\limits_{1\leq j\leq m}\delta_j$, $\Omega(\delta):=\bigcup\limits_{j=1}^{m}\Omega(z_j,\ \delta)$, $\widehat{\Omega}:=\Omega\setminus\Omega(\delta)$. Additionally, let $\Delta_j:=\Phi(\Omega(z_j,\ \delta))$, $\Delta(\delta):=\bigcup\limits_{j=1}^{m}\Phi(\Omega(z_j,\ \delta))$, $\widehat{\Delta}(\delta):=\Delta\backslash\Delta(\delta)$. Let $w_j:=\Phi(z_j)$ and for $\varphi_j:=\arg w_j,\ j=1,2,...,\ m$, we put $\Delta_j^{'}:=\left\{t=Re^{i\theta}:R>1,\ \frac{\varphi_{j-1}+\varphi_j}{2}\ \leq\theta<\frac{\varphi_j+\varphi_{j+1}}{2}\right\}$, where $\varphi_0\equiv\varphi_m,\ \varphi_1\equiv\varphi_{m+1}$; $\Omega_j:=\Psi(\Delta_j^{'})$, $L^j:=L\cap\overline{\Omega}_j,\ i=1,2,...,\ m$. Clearly, $\Omega=\bigcup\limits_{j=1}^{m}\Omega_j.L_R^j:=L_R\cap\overline{\Omega}^j$. $F^i:=\Phi(L^i)=\overline{\Delta}_i^{'}\cap\{\tau:|\tau|=1\}$, $F_R^i:=\Phi(L_R^i)=\overline{\Delta}_i^{'}\cap\{\tau:|\tau|=R\}$, $i=\overline{1,m}$.

The following lemma is a consequence of the results given in [12], [25], [18] and estimation for the $\left|\Psi^{'}\right|$ (see, for example, [11, Th.2.8]):

$$\left|\Psi'(\tau)\right|\asymp\frac{d(\Psi(\tau),L)}{|\tau|-1}. \tag{25}$$

Lemma 2.2. *Let $G\in C_\theta(\lambda_1,\lambda_2,...,\lambda_{m_1})$, $0<\lambda_j<2$, $j=\overline{1,m_1}$. Then for all $\varepsilon>0$:*
i) for any $w\in\Delta_j$, $|w-w_j|^{\lambda_j+\varepsilon}\preceq|\Psi(w)-\Psi(w_j)|\preceq|w-w_j|^{\lambda_j-\varepsilon}$, $|w-w_j|^{\lambda_j-1+\varepsilon}\preceq|\Psi^{'}(w)|\preceq|w-w_j|^{\lambda_j-1-\varepsilon}$,
ii) for any $w\in\overline{\Delta}\backslash\Delta_j$, $(|w|-1)^{1+\varepsilon}\preceq d(\Psi(w),L)\preceq(|w|-1)^{1-\varepsilon}$, $(|w|-1)^{\varepsilon}\preceq|\Psi^{'}(w)|\preceq(|w|-1)^{-\varepsilon}$.

Let L be a quasiconformal curve; $y(\zeta)$ denotes the regular quasiconformal reflection across L (see, for example, [15], [11], [2]),and for any $R>1$, let $L^*:=y(L_R)$, $G^*:=intL^*$, $\Omega^*:=extL^*$; $w=\Phi_R(z)$ be the conformal mapping of Ω^* onto the Δ normalized by $\Phi_R(\infty)=\infty$, $\lim_{z\to\infty}\frac{\Phi_R(z)}{z}>0$, and $\Psi_R:=\Phi_R^{-1}$. For $t>1$, let us set $L_t^*:=\{z:|\Phi_R(z)|=t\}$. According to [10], for all $z\in L^*$ and $t\in L$, such that $|z-t|=d(z,L)$, we have

$$d(z,L)\asymp d(t,L_R)\asymp d(z,L_R^*). \tag{26}$$

Lemma 2.3. [2] *Let L be a quasiconformal curve. Then, for any polynomial $P_n(z)\in\wp_n$, $n\in\mathbb{N}$ and $R=1+\frac{c}{n}$, we have*

$$\|P_n\|_{\mathscr{C}(\overline{G})}\preceq\|P_n\|_{\mathscr{C}(\overline{G}^*)}. \tag{27}$$

Let $\left\{z_j\right\}_{j=1}^{m}$ be a fixed system of the points on L and the weight function $h(z)$ be defined as in (7):

Lemma 2.4. *Let L be a rectifiable Jordan curve; $h(z)$ be defined as in (7). Then, for arbitrary $P_n(z) \in \wp_n$, any $R > 1$ and $n \in \mathbb{N}$, we have*

$$\|P_n\|_{L_p(h,L_R)} \leq R^{n+\frac{1+\gamma*}{p}} \|P_n\|_{L_p(h,L)}, \; p > 0, \tag{28}$$

where $\gamma^* = \max\{\gamma_k : \gamma_k \in \Gamma, \; k \leq \; m\}$.

Proof. Let us set:

$$f_n(t) := \prod_{j=1}^{m} \left[t \left(\Psi(\frac{1}{t}) - \Psi(w_j) \right) \right]^{\frac{\gamma_j}{p}} t^n P_n \left(\Psi \left(\frac{1}{t} \right) \right) \cdot (\Psi'(\frac{1}{t}))^{\frac{1}{p}} .$$

Since f_n is analytic in $B := \{t : |t| < 1\}$ and L is rectifiable, f_n belongs to Hardy class H_p, and then, by Hardy's convexity theorem, we have

$$\int\limits_{|t|=\frac{1}{R}} |f_n(t)|^p \frac{|dt|}{|t|} \leq \int\limits_{|t|=1} |f_n(t)|^p |dt|,$$

which implies that

$$\begin{aligned}
&\int\limits_{L_R} h(z) |P_n(z)|^p |dz| \\
= &\int\limits_{|w|=R} \prod_{j=1}^{m} |\Psi(w) - \Psi(w_j)|^{\gamma_j} \left| P_n(\Psi(w)) \left[\Psi'(w) \right]^{\frac{1}{p}} \right|^p |dw| \\
= &\int\limits_{|t|=\frac{1}{R}} \prod_{j=1}^{m} \left| \Psi(\frac{1}{t}) - \Psi(w_j) \right|^{\gamma_j} \left| P_n \left(\Psi \left(\frac{1}{t} \right) \right) \left[\Psi'(\frac{1}{t}) \right]^{\frac{1}{p}} \right|^p |dt| \\
\leq \; & R^{np+1+\gamma_j} \int\limits_{|t|=1} \prod_{j=1}^{m} \left| \Psi(\frac{1}{t}) - \Psi(w_j) \right|^{\gamma_j} \left| P_n \left(\Psi \left(\frac{1}{t} \right) \right) \left[\Psi'(\frac{1}{t}) \right]^{\frac{1}{p}} \right|^p |dt| \\
= \; & R^{np+1+\gamma_j} \int\limits_{L} h(z) |P_n(z)|^p |dz| .
\end{aligned}$$

We completed the proof of (28).

Remark 2.1. In case of $h(z) \equiv 1$, the estimate (28) has been prooved in [13].

3. Proofs

3.1. Proof of Theorem 1.1

Proof. Suppose that $G\in C_{\theta}(\lambda_i;f_j)$, for some $0<\lambda_i<2,\ i=\overline{1,m_1}$ and $f_j(x)=cx^{1+\alpha_j}$, $\alpha_j>0,\ j=\overline{m_1+1,m}$; $h(z)$ be defined as in (7). For any $z\in\Omega$ let

(29) $$T_n(z):=\frac{P_n(z)}{\Phi^{n+1}(z)}\ .$$

Cauchy integral representation for the unbounded region Ω gives:

(30) $$T_n(z)=-\frac{1}{2\pi i}\int\limits_L T_n(\zeta)\,\frac{d\zeta}{\zeta-z}\ ,\ z\in\Omega.$$

Since $|\ \Phi(\zeta)|=1$, for $\zeta\in L$, we have:

(31) $$|P_n(z)|=\frac{|\Phi(z)|^{n+1}}{2\pi}\int\limits_L|P_n(\zeta)|\,\frac{|d\zeta|}{|\zeta-z|}\ \leq\frac{|\Phi(z)|^{n+1}}{2\pi d(z,L)}\int\limits_L|P_n(\zeta)|\,|d\zeta|\,.$$

Let

(32) $$A_n:=\int\limits_L|P_n(\zeta)|\,|d\zeta|=\sum_{i=1}^{m}\int\limits_{L^i}|P_n(\zeta)|\,|d\zeta|\,.$$

Multiplying the numerator and denominator of the integrand by $\prod\limits_{j=1}^{m}\left|\zeta-z_j\right|^{\frac{\gamma_j}{p}}$, after applying the Holder inequality, we obtain:

(33) $$A_n\ \leq\ \sum_{i=1}^{m}\left(\int\limits_{L^i}\prod_{j=1}^{m}\left|\zeta-z_j\right|^{\gamma_j}|P_n(\zeta)|^p\,|d\zeta|\right)^{\frac{1}{p}}\times\left(\int\limits_{L^i}\frac{|d\zeta|}{\prod\limits_{j=1}^{m}\left|\zeta-z_j\right|^{\frac{q\gamma_j}{p}}}\right)^{\frac{1}{q}}=:\sum_{i=1}^{m}A_n^i,$$

where

$$A_n^i:=\left(\int\limits_{L^i}h(\zeta)\,|P_n(\zeta)|^p\,|d\zeta|\right)^{\frac{1}{p}}\left(\int\limits_{L^i}\frac{|d\zeta|}{\prod\limits_{j=1}^{m}\left|\zeta-z_j\right|^{\frac{q\gamma_j}{p}}}\right)^{\frac{1}{q}}=:J_{n,1}^i\cdot J_{n,2}^i,\ \frac{1}{p}+\frac{1}{q}=1.$$

For the $J_{n,1}^i$ we get:

(34) $$J_{n,1}^i\preceq\|P_n\|_{\mathscr{L}_p(h,L)}\,,\ i=\overline{1,m}.$$

Then, from (33) and (34) we have:

$$A_n \preceq \|P_n\|_{\mathscr{L}_p(h,L)} \sum_{i=1}^{m} \left(J^i_{n,2}\right)^{\frac{1}{q}}.$$

For the integral $J^i_{n,2}$ we obtain:

$$J^i_{n,2} := \int\limits_{L^i} \frac{|d\zeta|}{\prod\limits_{j=1}^{m} \left|\zeta - z_j\right|^{\frac{q\gamma_j}{p}}} \asymp \int\limits_{L^i} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i(q-1)}}, \tag{35}$$

since the points $\left\{z_j\right\}_{j=1}^{m}$ are distinct on L. To simplify further calculations, we assume that $m_1 = 1,\ m = 2, z_1 = -1, z_2 = 1; (-1,1) \subset G$ and let local co-ordinate axis in Definition 1.3 be parallel to OX and OY in the co-ordinate system; $L = L^+ \cup L^-$, where $L^+ := \{z \in L : \mathtt{Im}z \geq 0\}$, $L^- := \{z \in L : \mathtt{Im}z < 0\}$; Let $w^\pm := \left\{w = e^{i\theta} : \theta = \frac{\varphi_1 \pm \varphi_2}{2}\right\}$, $z^\pm \in \Psi(w^\pm)$, $l^\pm_i(z_i, z^\pm)$ denote the arcs, connecting the points z_i with $z^\pm$, respectively; $\left| l^\pm_i \right| := mes\ l^\pm_i(z_i, z^\pm)$, $i = 1,2$. Let z_0 be taken as an arbitrary point on L^+ (or on L^- subject to the chosen direction). Then, from (35), we have:

$$A_n \preceq \|P_n\|_{\mathscr{L}_p(h,L)} \sum_{i=1}^{2} \left(J^i_{n,2}\right)^{\frac{1}{q}}, \tag{36}$$

where

$$J^1_{n,2} = \int\limits_{L^1} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1(q-1)}}\ ;\ J^2_{n,2} = \int\limits_{L^2} \frac{|d\zeta|}{|\zeta - z_2|^{\gamma_2(q-1)}}. \tag{37}$$

For estimation of these integrals, we give some notations: $R = 1 + \frac{1}{n}$; $d_{i,R} := d(z_i, L_R)$;

$E^{1,\pm}_1 := \left\{\zeta \in L^1 :\ |\zeta - z_1| < c_1 d_{1,R}\right\}, E^{1,\pm}_2 := \left\{\zeta \in L^1 :\ c_1 d_{1,R} \leq |\zeta - z_1| \leq \left| l^\pm_1 \right|\right\}, E^{2,\pm}_1 := \left\{\zeta \in L^2 :\ |\zeta - z_2| < c_2 d_{2,R}\right\}, E^{2,\pm}_2 := \left\{\zeta \in L^2 :\ c_2 d_{2,R} \leq |\zeta - z_2| \leq \left| l^\pm_2 \right|\right\}; I^{i,\pm}_{n,k} := I^i_{n,k}(E^{i,\pm}_k) := \int\limits_{E^{i,\pm}_k} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i(q-1)}}\ ;\ i,\ k = 1,2.$

Taking into consideration these notations, (36) can be written as:

$$\begin{aligned} A_n &\preceq \|P_n\|_{\mathscr{L}_p(h,L)} \sum_{i=1}^{2} \left(J^i_{n,2}\right)^{\frac{1}{q}} \\ &=: \|P_n\|_{\mathscr{L}_p(h,L)} \sum_{i=1}^{2} \left[I^i_{n,1}(E^{i,\pm}_1) + I^i_{n,2}(E^{i,\pm}_2)\right]^{\frac{1}{q}} \\ &=: \|P_n\|_{\mathscr{L}_p(h,L)} \sum_{i=1}^{2} \left[I^{i,\pm}_{n,1} + I^{i,\pm}_{n,2}\right]^{\frac{1}{q}},\ i = 1,2. \end{aligned} \tag{38}$$

According to (31) and (32), it is sufficient to estimate the integrals $I_{n,k}^{i,\pm}$ for each $i=1,2$ and $k=1,2,3$.

Let's start with the evaluation of the following integrals $J_{n,2}^{1}$ from (37) and (38):

$$J_{n,2}^{1} = \int\limits_{L^1} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1(q-1)}} \tag{39}$$

$$: = \left(\sum_{k=1}^{2} \int\limits_{E_k^{1,\pm}} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_1(q-1)}} \right)^{\frac{1}{q}} =: \left[I_{n,1}^{1,\pm} + I_{n,2}^{1,\pm} \right]^{\frac{1}{q}}.$$

Given the possible values of γ_i $(-1<\gamma_i<0,\ \gamma_i \geq 0,\ i=1,2)$, we will consider the estimates for the $J_{n,2}^{1}$ separately.

Let $\gamma_1>0$ and $\gamma_2>0$. In this case for the integral $J_{n,2}^{1}$, we get:

$$I_{n,1}^{1,\pm} \preceq \int\limits_{E_1^{1,\pm}} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1(q-1)}} \tag{40}$$

$$\preceq \int\limits_{0}^{c_1 d_{1,R}} \frac{ds}{s^{\gamma_1(q-1)}} \preceq \begin{cases} d_{1,R}^{1-\gamma_1(q-1)}, & \gamma_1 > p-1, \\ 1, & \gamma_1 \leq p-1; \end{cases}$$

$$I_{n,2}^{1,\pm} \preceq \int\limits_{E_2^{1,\pm}} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1(q-1)}} \preceq \int\limits_{c_1 d_{1,R}}^{|l_1^{\pm}|} \frac{ds}{s^{\gamma_1(q-1)}} \preceq \begin{cases} d_{1,R}^{1-\gamma_1(q-1)}, & \gamma_1 > p-1, \\ \ln \frac{1}{d_{1,R}}, & \gamma_1 = p-1, \\ 1, & \gamma_1 < p-1. \end{cases}$$

Similar estimate for the integral J_n^2 is given:

$$I_{n,1}^{2} \preceq \int\limits_{E_1^{2}} \frac{|d\zeta|}{|\zeta - z_2|^{\gamma_2(q-1)}} \tag{41}$$

$$\preceq \int\limits_{0}^{c_2 d_{2,R}} \frac{ds}{s^{\gamma_2(q-1)}} \preceq \begin{cases} d_{2,R}^{1-\gamma_2(q-1)}, & \gamma_2 > p-1, \\ 1, & \gamma_2 \leq p-1; \end{cases}$$

$$I_{n,2}^{2,\pm} \preceq \int\limits_{E_2^{2,\pm}} \frac{|d\zeta|}{|\zeta - z_2|^{\gamma_2(q-1)}} \preceq \int\limits_{c_2 d_{2,R}}^{|l_2^{\pm}|} \frac{ds}{s^{\gamma_2(q-1)}} \preceq \begin{cases} d_{2,R}^{1-\gamma_2(q-1)}, & \gamma_2 > p-1, \\ \ln \frac{1}{d_{2,R}}, & \gamma_2 = p-1, \\ 1, & \gamma_2 < p-1. \end{cases}$$

Let $\gamma_1 \leq 0$ and $\gamma_2 \leq 0$. Then, analogously to the (40) and (41)

$$I_{n,1}^1 \preceq \int\limits_{E_1^1} |\zeta - z_1|^{(-\gamma_1)(q-1)} |d\zeta| \preceq d_{1,n}^{(-\gamma_1)(q-1)} mesE_1^1 \preceq 1, \tag{42}$$

$$I_{n,2}^1 \preceq \int\limits_{E_2^1} |\zeta - z_1|^{(-\gamma_1)(q-1)} |d\zeta| \preceq \left| l_1^{\pm} \right|^{(-\gamma_1)(q-1)+1} \preceq 1,$$

and

$$I_{n,1}^{2,\pm} \preceq \int\limits_{E_1^{2,\pm}} |\zeta - z_2|^{(-\gamma_2)(q-1)} |d\zeta| \preceq d_{2,R}^{(-\gamma_2)(q-1)} mesE_1^{2,\pm} \preceq 1, \tag{43}$$

$$I_{n,2}^{2,\pm} \preceq \int\limits_{E_2^{2,\pm}} |\zeta - z_2|^{(-\gamma_2)(q-1)} |d\zeta| \preceq \left| l_2^{\pm} \right|^{(-\gamma_2)(q-1)+1} \preceq 1.$$

Therefore, in this case, from (38) - (43), we obtain:

$$A_n \preceq \|P_n\|_{\mathscr{L}_p(h,L)} \begin{cases} d_{1,R}^{\frac{1-\gamma_1(q-1)}{q}} + d_{2,R}^{\frac{1-\gamma_2(q-1)}{q}}, & \gamma_1, \gamma_2 > p-1, \\ \left(\ln \frac{1}{d_{1,R}}\right)^{\frac{1}{q}} + \left(\ln \frac{1}{d_{2,R}}\right)^{\frac{1}{q}}, & \gamma_1 = \gamma_2 = p-1, \\ 1, & \gamma_1, \gamma_2 < p-1. \end{cases} \tag{44}$$

Comparing (31), (32) and (44), we have:

$$|P_n(z)| \leq c \frac{B_{n,1}^0}{d(z,L)} \|P_n\|_{\mathscr{L}_p(h,L)} |\Phi(z)|^{n+1}, \tag{45}$$

where $c = c(G, p, \gamma_i) > 0$, $i = \overline{1,p}$, is the constant independent from n and z, and

$$B_{n,1}^0 := \begin{cases} d_{1,R}^{1-\gamma_1(q-1)} + d_{2,R}^{1-\gamma_2(q-1)}, & \gamma_1, \gamma_2 > p-1, \\ \ln \frac{1}{d_{1,R}} + \ln \frac{1}{d_{2,R}}, & \gamma_1 = \gamma_2 = p-1, \\ 1, & \gamma_1, \gamma_2 < p-1. \end{cases} \tag{46}$$

According to Lemma 2.2, for the point z_1 we get:

$$d_{1,R} \succeq n^{-\widetilde{\lambda}_1 - \varepsilon}, \tag{47}$$

for arbitrary small $\varepsilon > 0$.

For the estimate $d_{2,R}$, let's set: $z_R \in L_R$, such that $d_{2,R} = |z_2 - z_R|$; $\zeta^{\pm} \in L^{\pm}$, such that $d(z_R, L^2 \cap L^{\pm}) := d(z_R, L^+)$; $z_2^{\pm} := \zeta \in L^2 : \ |\zeta - z_2| = c_2 d_{2,R}$. Under these notations, from Lemma 2.1, we obtain:

$$d_R^{\pm} := d(z_R, L^2 \cap L^{\pm}) \asymp \left| z_R - z_2^{\pm} \right| \asymp d_{2,R}^{1+\alpha_2}. \tag{48}$$

Hence, $d_{2,R} = \left(d_R^{\pm}\right)^{\frac{1}{1+\alpha_2}}$. On the other hand, according to Lemma 2.2 and [9, Corollary 2], we get: $d_R^{\pm} \succeq n^{-1-\varepsilon}$. Therefore,

$$d_{2,R} \succeq n^{\frac{\varepsilon - 1}{1+\alpha_2}}, \tag{49}$$

for arbitrary small $\varepsilon > 0$. Comparing (45)-(49), we get:

$$|P_n(z)| \preceq \frac{B_{n,1}^0}{d(z,L)} \|P_n\|_{\mathscr{L}_p(h,L)} |\Phi(z)|^{n+1},$$

where

$$B_{n,1}^0 \preceq \begin{cases} n^{\frac{(\gamma_1(q-1)-1)\widetilde{\lambda}_1}{q}+\varepsilon} + n^{\frac{\gamma_2(q-1)-1}{q(1+\alpha_2)}+\varepsilon}, & \gamma_1, \gamma_2 > p-1, \\ (\ln n)^{\frac{1}{q}}, & \gamma_1 = \gamma_2 = p-1, \\ 1, & \gamma_1, \gamma_2 < p-1. \end{cases} \tag{50}$$

Combining the corresponding estimates for each point $\{z_j\}$, $j = \overline{1,m}$, and taking into account the above notations, we complete the proof for the points $z \in \Omega$.

Let now $z \in G$ be arbitrary fixed. Cauchy integral representation for the G gives

$$P_n(z) = \frac{1}{2\pi i} \int_L P_n(\zeta) \frac{d\zeta}{\zeta - z}, \ z \in G. \tag{51}$$

In this, we have

$$\begin{aligned} |P_n(z)| &= \frac{1}{2\pi} \int_L |P_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|} \\ &\leq \frac{1}{2\pi d(z,L)} \int_L |P_n(\zeta)| |d\zeta| =: \frac{1}{2\pi d(z,L)} A_n, \end{aligned} \tag{52}$$

where A_n is defined as in (32). Combining relations (44), (46) and (52) we complete the proof.

3.2. Proof of Theorem 1.3

Proof. Let $G \in C_{\theta}(\lambda_i; f_j)$, for some $0 < \lambda_i < 2$, $i = \overline{1,m_1}$ and $f_j(x) = cx^{1+\alpha_j}$, $\alpha_j > 0$, $j = \overline{m_1+1,m}$, be given. If $G \in C_{\theta}(\lambda_i; 0)$, for some $0 < \lambda_i < 2$, $i = \overline{1,m_1}$, i.e. $\alpha_j = 0$ for all $j = \overline{m_1+1,m}$, the proof is obvious. Really, for any $z \in G$, according to the Theorem 1.1, (51) and (52), we have

$$|P_n(z)| \preceq \frac{B_{n,1}^{''}}{d(z,L)} \|P_n\|_{\mathscr{L}_p(h,L)}, \; z \in G, \tag{53}$$

where

$$B_{n;1}^{''} := \begin{cases} n^{\frac{\left(1-p+\widetilde{\gamma}^1\right)\widetilde{\lambda}}{p}+\varepsilon}, & \text{if there is at least one } \gamma_i \in \Gamma_{1,k}^{(1)} \\ & \text{for some } i = \overline{1,k},\; k \leq m_1; \\ (\ln n)^{1-\frac{1}{p}}, & \text{if there is at least one} \\ & \gamma_j \in \Gamma_{1,k}^{(2)} \backslash \Gamma_{1,m_1}^{(1)} \\ & \text{for some } j = \overline{1,k},\; k \leq m_1; \\ 1, & \text{if } \gamma_j \in \Gamma_{1,k}^{(3)} \\ & \text{for all } j = \overline{1,m}. \end{cases} \tag{54}$$

On the other hand, according to the "three-point" criterion [8, p.100], we see that $L := \partial G$ is $Q-$quasiconformal for some $Q > 1$. Let $z \in L^*$ be an arbitrary fixed and $R = 1 + \frac{1}{n}$. According to Lemma 2.2 and (26), for any arbitrary small $\varepsilon > 0$, we obtain:

$$d(z,L) \asymp d(t,L_R) \succeq n^{-\widetilde{\lambda}-\varepsilon}. \tag{55}$$

$$|P_n(z)| \preceq n^{\widetilde{\lambda}+\varepsilon} B_{n,1}^{''} \|P_n\|_{\mathscr{L}_p(h,L)}, \; z \in \overline{G}^*.$$

Combining (26), (53) and (55) from Lemma 2.2 we obtain the proof for this case.

Now, we consider the general case. Let $\alpha_j > 0$, for some $j = \overline{m_1+1,m}$. Cauchy integral representation for a region G_R gives:

$$P_n(z) = \frac{1}{2\pi i} \int_{L_R} P_n(\zeta) \frac{d\zeta}{\zeta - z}, \; z \in G_R.$$

Replacing the variable $\tau=\Phi(\zeta)$, multiplying the numerator and determinator of the integrant by $\prod_{i=1}^{m}|\Psi(\tau)-\Psi(w_i)|^{\frac{\gamma_i}{p}}\ |\Psi'(\tau)|^{\frac{1}{p}}$, according to the Hölder inequality, we obtain:

$$|P_n(z)|\leq\frac{1}{2\pi}\int_{L_R}|P_n(\zeta)|\frac{|d\zeta|}{|\zeta-z|} \tag{56}$$

$$\begin{aligned}
&= \frac{1}{2\pi}\int_{|\tau|=R}\frac{|P_n(\Psi(\tau))|\,|\Psi'(\tau)|}{|\Psi(\tau)-\Psi(w)|}|d\tau| \\
&\leq \frac{1}{2\pi}\left(\int_{|\tau|=R}\prod_{j=1}^{m}\left|\Psi(\tau)-\Psi(w_j)\right|^{\gamma_j}|P_n(\Psi(\tau))|^p\left|\Psi'(\tau)\right||d\tau|\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_{|\tau|=R}\frac{|\Psi'(\tau)|^{\left(1-\frac{1}{p}\right)q}}{\prod_{j=1}^{m}\left|\Psi(\tau)-\Psi(w_j)\right|^{\frac{q\gamma_j}{p}}|\Psi(\tau)-\Psi(w)|^q}|d\tau|\right)^{\frac{1}{q}} \\
&= \frac{1}{2\pi}\left(\int_{L_R}h(\zeta)|P_n(\zeta)|^p|d\zeta|\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_{|\tau|=R}\frac{|\Psi'(\tau)|}{\prod_{j=1}^{m}\left|\Psi(\tau)-\Psi(w_j)\right|^{\gamma_j(q-1)}|\Psi(\tau)-\Psi(w)|^q}|d\tau|\right)^{\frac{1}{q}} \\
&= \frac{1}{2\pi}J_{n,1}\times J_{n,2},
\end{aligned}$$

where

$$\begin{aligned}
J_{n,1} &:= \|P_n\|_{L_p(h,L_R)}, \\
J_{n,2} &:= \left(\int_{|\tau|=R}\frac{|\Psi'(\tau)|}{\prod_{j=1}^{m}\left|\Psi(\tau)-\Psi(w_j)\right|^{\gamma_j(q-1)}|\Psi(\tau)-\Psi(w)|^q}|d\tau|\right)^{\frac{1}{q}}.
\end{aligned}$$

Then, from Lemma 2.4, we have:

$$|P_n(z)|\preceq J_{n,1}\cdot J_{n,2}\preceq\|P_n\|_{L_p(h,\ L)}\cdot J_{n,2},\ z\in L,\ w=\Phi(z). \tag{57}$$

In order to evaluate the integral $J_{n,2}$, taking into account the estimation for the $|\Psi'|$ (25), we get

$$J_{n,2} = \left(\sum_{i=1}^{m} \int_{F_R^i} \frac{|\Psi'(\tau)|\,|d\tau|}{\prod_{j=1}^{m} \left|\Psi(\tau)-\Psi(w_j)\right|^{\gamma_j(q-1)} |\Psi(\tau)-\Psi(w)|^q} \right)^{\frac{1}{q}} =: \left(\sum_{i=1}^{m} J_{n,2}^i \right)^{\frac{1}{q}} \leq \sum_{i=1}^{m} \left(J_{n,2}^i\right)^{\frac{1}{q}}, \tag{58}$$

where

$$J_{n,2}^i := \int_{F_R^i} \frac{|\Psi'(\tau)|\,|d\tau|}{\prod_{j=1}^{m} \left|\Psi(\tau)-\Psi(w_j)\right|^{\gamma_j(q-1)} |\Psi(\tau)-\Psi(w)|^q},\ i=\overline{1,m}. \tag{59}$$
$$\asymp \int_{F_R^i} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w_i)|^{\gamma_i(q-1)} |\Psi(\tau)-\Psi(w)|^q} =: J(F_R^i),$$

since the points $w_i := \Phi(z_i)$ are distinct.

It remains to estimate the integrals $J(F_R^i)$ for each $i=\overline{1,m}$.

For simplicity of our next calculations, we assume that:

$$i=1,2;\ m_1=1,\ m=2;\ z_1=-1,\ z_2=1; R=1+\frac{1}{n}. \tag{60}$$

In addition to the previous notations, we introduce the following: $L_R = L_R^+ \cup L_R^-$, where $L_R^+ := \{z \in L_R : \mathtt{Im} z \geq 0\}$, $L_R^- := \{z \in L_R : \mathtt{Im} z < 0\}$; Let $w_R^{\pm} := \left\{ w = Re^{i\theta} : \theta = \frac{\varphi_1 \pm \varphi_2}{2} \right\}$, $z_R^{\pm} \in \Psi(w_R^{\pm})$. We set: $z_{i,R} \in L_R$, such that $d_{i,n} = |z_i - z_{i,R}|$ and $\zeta^{\pm} \in L^{\pm}$, such that $d(z_{2,R}, L^2 \cap L^{\pm}) := d(z_{2,R}, L^{\pm})$; $z_i^{\pm} := \left\{ \zeta \in L^i :\ |\zeta - z_i| = c_i d(z_i, L_R) \right\}$, $z_{i,R}^{\pm} := \left\{ \zeta \in L_R^i :\ |\zeta - z_{i,R}| = c_i d(z_{i,R}, L_R) \right\}$, $w_{i,R}^{\pm} = \Phi(z_{i,R}^{\pm})$. Let $l_{i,R}^{\pm}(z_{i,R}^{\pm}, z_R^{\pm})$ denote the arcs, connecting the points $z_{i,R}^{\pm}$ with $z_R^{\pm}$, respectively and $\left| l_{i,R}^{\pm} \right| := mes\ l_{i,R}^{\pm}(z_{i,R}^{\pm}, z_R^{\pm})$, $i=1,2$. Let z_0 be taken as an arbitrary fixed point on L^+distinct from z_1 and z_2 (or on L^- subject to the chosen direction). For simplicity, without loss of generality, we assume that $z_0 = z_R^+$ ($z_0 = z_R^-$). We denote: $E_{1,R}^{1,\pm} := \left\{ \zeta \in L_R^1 :\ |\zeta - z_1| < c_1 d_{1,R} \right\}$, $E_{2,R}^{1,\pm} := \left\{ \zeta \in L_R^1 : c_1 d_{1,R} \leq |\zeta - z_1| \leq \left| l_{1,R}^{\pm} \right| \right\}$, $E_{1,R}^{2,\pm} := \left\{ \zeta \in L_R^2 : |\zeta - z_2| < c_2 d_{2,R} \right\}$, $E_{2,R}^{2,\pm} := \left\{ \zeta \in L_R^2 : c_2 d_{2,R} \leq |\zeta - z_2| \leq \left| l_{2,R}^{\pm} \right| \right\}$ and $F_{j,R}^{i,\pm} := \Phi(E_{j,R}^{i,\pm})$, $i,j=1,2$.

Taking into consideration these designations, (59) can be written as:

$$J_{n,2}^{i} \asymp \sum_{i,j=1}^{2} \int\limits_{F_{j,R}^{i,\pm}} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w_i)|^{\gamma_i(q-1)}\,|\Psi(\tau)-\Psi(w)|^{q}} =: \sum_{i,j=1}^{2} J(F_{j,R}^{i,\pm}). \tag{61}$$

So, we need to evaluate the integrals $J(F_{j,R}^{i,\pm})$ for each $i,j=1,2$.

Let

$$\|P_n\|_{C(\overline{G})} =: \left|P_n\left(z'\right)\right|,\ z' \in L = L^1 \cup L^2, \tag{62}$$

and let $w' = \Phi(z')$. There are two possible cases: point z' may lie on L^1 or L^2.

1) Suppose first that $z' \in L^1$. In this case, from Lemma 2.2 and Lemma 2.1, we have:

1.1) If $z' \in E_1^{1,\pm}$, then

$$J(F_{1,R}^{1,+}) + J(F_{1,R}^{1,-}) \tag{63}$$

$$= \int\limits_{F_{1,R}^{1,+}\cup F_{1,R}^{1,-}} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1(q-1)}\,|\Psi(\tau)-\Psi(w')|^{q}} \preceq \int\limits_{F_{1,R}^{1,+}\cup F_{1,R}^{1,-}} \frac{|\tau-w_1|^{\lambda_1-1-\varepsilon}\,|d\tau|}{|\tau-w_1|^{\gamma_1(q-1)\lambda_1+\varepsilon}\,|\tau-w'|^{q\lambda_1+\varepsilon}}$$

$$\preceq \int\limits_{F_{1,R}^{1,+}\cup F_{1,R}^{1,-}} \frac{|d\tau|}{[\min\{|\tau-w_1|;|\tau-w'|\}]^{(\gamma_1(q-1)+q)\lambda_1-\lambda_1+1+\varepsilon}} \preceq n^{(\gamma_1+1)(q-1)\lambda_1+\varepsilon},\ \forall\varepsilon>0,$$

for $\gamma_1 > 0$, and

$$J(F_{1,R}^{1,+}) + J(F_{1,R}^{1,-}) \tag{64}$$

$$= \int\limits_{F_{1,R}^{1,+}\cup F_{1,R}^{1,-}} \frac{|\Psi(\tau)-\Psi(w_1)|^{(-\gamma_1)(q-1)}\,|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w')|^{q}} \preceq \int\limits_{F_{1,R}^{1,+}\cup F_{1,R}^{1,-}} \frac{|\tau-w_1|^{(-\gamma_1)(q-1)\lambda_1+\lambda_1-1-\varepsilon}\,|d\tau|}{|\tau-w'|^{q\lambda_1+\varepsilon}}$$

$$\preceq n^{(\gamma_1+1)(q-1)\lambda_1+\varepsilon},\ \forall\varepsilon>0,$$

for $-1 < \gamma_1 \le 0$;

1.2) If $z' \in E_2^{1,\pm}$, then

$$J(F_{1,R}^{1,+}) + J(F_{1,R}^{1,-}) \tag{65}$$

$$= \int\limits_{F_{1,R}^{1,+}\cup F_{1,R}^{1,-}} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1(q-1)}\,|\Psi(\tau)-\Psi(w')|^{q}} \preceq \int\limits_{F_{1,R}^{1,+}\cup F_{1,R}^{1,-}} \frac{|\tau-w_1|^{\lambda_1-1-\varepsilon}\,|d\tau|}{|\tau-w_1|^{\gamma_1(q-1)\lambda_1+q\lambda_1+\varepsilon}}$$

$$\preceq \int\limits_{F_{1,R}^{1,+}\cup F_{1,R}^{1,-}} \frac{|d\tau|}{|\tau-w_1|^{\gamma_1(q-1)\lambda_1+q\lambda_1-\lambda_1+1+\varepsilon}} \preceq n^{(\gamma_1+1)(q-1)\lambda_1+\varepsilon},\ \forall\varepsilon>0,$$

for all $\gamma_1>-1$;

1.3) If $z'\in E_1^{1,\pm}$, then

$$J(F_{2,R}^{1,+})+J(F_{2,R}^{1,-}) \tag{66}$$

$$= \int\limits_{F_{2,R}^{1,+}\cup F_{2,R}^{1,-}} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1(q-1)}\,|\Psi(\tau)-\Psi(w')|^{q}}$$

$$\preceq \int\limits_{F_{2,R}^{1,+}\cup F_{2,R}^{1,-}} \frac{|\Psi'(\tau)|\,|d\tau|}{\min\{|\Psi(\tau)-\Psi(w_1)|\,;|\Psi(\tau)-\Psi(w')|\}^{\gamma_1(q-1)+q}}$$

$$\preceq \int\limits_{F_{2,R}^{1,+}\cup F_{2,R}^{1,-}} \frac{|d\tau|}{\min\{|\tau-w_1|\,;|\tau-w'|\}^{[\gamma_1(q-1)+q]\lambda_1-\lambda_1+1+\varepsilon}} \preceq n^{(\gamma_1+1)(q-1)\lambda_1+\varepsilon},\ \forall\varepsilon>0.$$

for $\gamma_1>0$ and

$$J(F_{2,R}^{1,+})+J(F_{2,R}^{1,-}) \tag{67}$$

$$= \int\limits_{F_{2,R}^{1,+}\cup F_{2,R}^{1,-}} \frac{|\Psi(\tau)-\Psi(w_1)|^{(-\gamma_1)(q-1)}\,|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w')|^{q}} \preceq \int\limits_{F_{2,R}^{1,+}\cup F_{2,R}^{1,-}} \frac{|\tau-w_1|^{(-\gamma_1)(q-1)+\lambda_1-1-\varepsilon}\,|d\tau|}{|\tau-w'|^{q\lambda_1+\varepsilon}}$$

$$\preceq \int\limits_{F_{2,R}^{1,+}\cup F_{2,R}^{1,-}} \frac{|d\tau|}{\min\{|\tau-w_1|\,;|\tau-w'|\}^{[\gamma_1(q-1)+q]\lambda_1-\lambda_1+1+\varepsilon}} \preceq n^{(\gamma_1+1)(q-1)\lambda_1+\varepsilon},\ \forall\varepsilon>0.$$

for $-1<\gamma_1\le 0$;

1.4) If $z'\in E_2^{1,\pm}$, then

$$J(F_{2,R}^{1,+})+J(F_{2,R}^{1,-}) \tag{68}$$

$$= \int\limits_{F_{2,R}^{1,+}\cup F_{2,R}^{1,-}} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1(q-1)}\,|\Psi(\tau)-\Psi(w')|^{q}}$$

$$\preceq \int\limits_{F_{2,R}^{1,+}\cup F_{2,R}^{1,-}} \frac{|\Psi'(\tau)|\,|d\tau|}{[\min\{|\Psi(\tau)-\Psi(w_1)|;|\Psi(\tau)-\Psi(w')|\}]^{\gamma_1(q-1)+q-1}}$$

$$\preceq \int\limits_{F_{2,R}^{1,+}\cup F_{2,R}^{1,-}} \frac{|\tau-w_1|^{\lambda_1-1-\varepsilon}\,|d\tau|}{[\min\{|\tau-w_1|;|\tau-w'|\}]^{\gamma_1(q-1)+q-1+\varepsilon}} \preceq n^{(\gamma_1+1)(q-1)\lambda_1+\varepsilon},\ \forall\varepsilon>0.$$

for $\gamma_1>0$, and

(69) $$J(F_{2,R}^{1,+})+J(F_{2,R}^{1,-})$$

$$= \int\limits_{F_{2,R}^{1,+}\cup F_{2,R}^{1,-}} \frac{|\Psi(\tau)-\Psi(w_1)|^{(-\gamma_1)(q-1)}\,|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w')|^{q}}$$

$$\preceq \int\limits_{F_{2,R}^{1,+}\cup F_{2,R}^{1,-}} \frac{|d\tau|}{\min\{|\tau-w_1|;|\tau-w'|\}^{[\gamma_1(q-1)+q]\lambda_1-\lambda_1+1+\varepsilon}} \preceq n^{(\gamma_1+1)(q-1)\lambda_1+\varepsilon},\ \forall\varepsilon>0.$$

for $-1<\gamma_1\le 0$. Combining the relations (63)-(69), for all $\gamma_1>-1,\ q>1$ and $0<\lambda_1<2$, we obtain:

(70) $$J(F_{1,R}^{1,+})+J(F_{1,R}^{1,-})\preceq n^{(\widetilde{\gamma}_1+1)(q-1)\widetilde{\lambda}_1+\varepsilon},\ \forall\varepsilon>0.$$

2) Now, suppose that $z'\in L^2$.In this case, according to (25):

2.1) If $z' \in E_1^{2,\pm}$, then

$$J(F_{1,R}^{2,+})+J(F_{1,R}^{2,-}) \tag{71}$$
$$= \int\limits_{F_{1,R}^{2,+}\cup F_{1,R}^{2,-}} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2(q-1)}\,|\Psi(\tau)-\Psi(w')|^{q}}$$
$$\asymp \int\limits_{F_{1,R}^{2,+}\cup F_{1,R}^{2,-}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2(q-1)}\,|\Psi(\tau)-\Psi(w')|^{q}\,(|\tau|-1)}$$
$$\preceq \int\limits_{F_{1,R}^{2,+}} \frac{|d\tau|}{d_{2,R}^{\gamma_2(q-1)-1}\,|\Psi(\tau)-\Psi(w')|^{q}\,(|\tau|-1)}$$
$$+\int\limits_{F_{1,R}^{2,-}} \frac{|d\tau|}{d_{2,R}^{\gamma_2(q-1)-1}\,|\Psi(\tau)-\Psi(w')|^{q}\,(|\tau|-1)},$$

for all $\gamma_2>-1$. The last two integrals are evaluated identically. Therefore, we evaluate one of them, say the first. When $\tau\in F_{1,R}^{2,+}$ for the $|\Psi(\tau)-\Psi(w')|$, we obtain:

$$\begin{aligned}\left|\Psi(\tau)-\Psi(w')\right| &\succeq \max\left\{|\Psi(\tau)-\Psi(w_2)|;\ \left|\Psi(\tau)-z_2^{+}\right|\right\}\\ &= |\Psi(\tau)-\Psi(w_2)| \succeq \left|\Psi(\tau)-z_2^{+}\right|^{\frac{1}{1+\beta_2}}.\end{aligned}$$

Then,

$$\begin{aligned} J(F_{1,R}^{2,+}) &\preceq n\int\limits_{F_{1,R}^{2,+}} \frac{|d\tau|}{\left|\Psi(\tau)-z_2^{+}\right|^{\frac{\gamma_2(q-1)+q-1}{1+\alpha_2}}}\\ &\preceq n\int\limits_{F_{1,R}^{2,+}} \frac{|d\tau|}{\left|\Psi(\tau)-z_2^{+}\right|^{\frac{(\gamma_2+1)(q-1)}{1+\alpha_2}}} \preceq n\int\limits_{F_{1,R}^{2,+}} \frac{|d\tau|}{\left|\tau-w_2^{+}\right|^{\frac{(\gamma_2+1)(q-1)}{1+\alpha_2}+\varepsilon}}\\ &\preceq \begin{cases} n^{\frac{(\gamma_2+1)}{1+\alpha_2}(q-1)+\varepsilon}, & \frac{(\gamma_2+1)}{1+\alpha_2}(q-1)>1,\\ n^{1+\varepsilon}\ln n, & \frac{(\gamma_2+1)}{1+\alpha_2}(q-1)=1,\\ n^{1+\varepsilon}, & \frac{(\gamma_2+1)}{1+\alpha_2}(q-1)<1,\end{cases}\end{aligned}$$

if $\gamma_2 > 0$, and

$$\begin{aligned} J(F_{1,R}^{2,+}) &\preceq n \int\limits_{F_{1,R}^{2,+}} \frac{|\Psi(\tau)-\Psi(w_2)|^{(-\gamma_2)(q-1)}|d\tau|}{\left|\Psi(\tau)-z_2^{+}\right|^{\frac{q-1}{1+\alpha_2}}} \\ &\preceq n \int\limits_{F_{1,R}^{2,+}} \frac{|d\tau|}{\left|\tau-w_2^{+}\right|^{\frac{q-1}{1+\alpha_2}+\varepsilon}} \\ &\preceq \begin{cases} n^{\frac{q-1}{1+\alpha_2}+\varepsilon}, & \frac{q-1}{1+\alpha_2} > 1, \\ n^{1+\varepsilon}\ln n, & \frac{q-1}{1+\alpha_2} = 1, \\ n^{1+\varepsilon}, & \frac{q-1}{1+\alpha_2} < 1, \end{cases} \end{aligned}$$

if $-1 < \gamma_2 \le 0$, and so, in this case we get:

$$J(F_{1,R}^{2,+})+J(F_{1,R}^{2,-}) \preceq \begin{cases} n^{\frac{(\widetilde{\gamma}_2+1)}{1+\alpha_2}(q-1)+\varepsilon}, & \frac{(\widetilde{\gamma}_2+1)}{1+\alpha_2}(q-1) > 1, \\ n^{1+\varepsilon}\ln n, & \frac{(\widetilde{\gamma}_2+1)}{1+\alpha_2}(q-1) = 1, \quad \forall\varepsilon > 0. \\ n^{1+\varepsilon}, & \frac{(\widetilde{\gamma}_2+1)}{1+\alpha_2}(q-1) < 1, \end{cases} \tag{72}$$

2.2) If $z' \in E_2^{2,\pm}$, then

$$\begin{aligned} &J(F_{1,R}^{2,+})+J(F_{1,R}^{2,-}) \\ = &\int\limits_{F_{1,R}^{2,+}\cup F_{1,R}^{2,-}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2(q-1)}\,|\Psi(\tau)-\Psi(w')|^{q}\,(|\tau|-1)} \\ \preceq\ & n \int\limits_{F_{1,R}^{2,+}\cup F_{1,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2(q-1)}\,|\Psi(\tau)-\Psi(w')|^{q-1}}, \end{aligned}$$

for all $\gamma_2 > -1$. When $\tau \in F_{1,R}^{2,+}$ for the $|\Psi(\tau)-\Psi(w')|$, we obtain:

$$\left|\Psi(\tau)-\Psi(w')\right| \succeq \left|\Psi(\tau)-z_2^{+}\right|,$$

and analogous to previous case, we get:

$$\begin{aligned} J(F_{1,R}^{2,+}) &\preceq n \int\limits_{F_{1,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2(q-1)}\left|\Psi(\tau)-z_2^{+}\right|^{q-1}} \\ &\preceq n \int\limits_{F_{1,R}^{2,+}} \frac{|d\tau|}{\left|\Psi(\tau)-z_2^{+}\right|^{\frac{\gamma_2(q-1)}{1+\alpha_2}+q-1}} \\ &\preceq n \int\limits_{F_{1,R}^{2,+}} \frac{|d\tau|}{\left|\tau-w_2^{+}\right|^{\frac{\gamma_2(q-1)}{1+\alpha_2}+q-1+\varepsilon}} \\ &\preceq \begin{cases} n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+q-1+\varepsilon}, & \frac{\gamma_2(q-1)}{1+\alpha_2}+q-1>1, \\ n^{1+\varepsilon}\ln n, & \frac{\gamma_2(q-1)}{1+\alpha_2}+q-1=1, \\ n^{1+\varepsilon}, & \frac{\gamma_2(q-1)}{1+\alpha_2}+q-1<1, \end{cases} \end{aligned}$$

if $\gamma_2>0$, and

$$\begin{aligned} J(F_{1,R}^{2,+}) &\preceq n \int\limits_{F_{1,R}^{2,+}} \frac{|\Psi(\tau)-\Psi(w_2)|^{(-\gamma_2)(q-1)}|d\tau|}{\left|\Psi(\tau)-z_2^{+}\right|^{q-1}} \\ &\preceq n \int\limits_{F_{1,R}^{2,+}} \frac{|d\tau|}{\left|\Psi(\tau)-z_2^{+}\right|^{q-1}} \\ &\preceq n \int\limits_{F_{1,R}^{2,+}} \frac{|d\tau|}{\left|\tau-w_2^{+}\right|^{q-1}} \\ &\preceq \begin{cases} n^{q-1}, & q>2, \\ n^{1+\varepsilon}\ln n, & q=2, \\ n^{1+\varepsilon}, & q<2, \end{cases} \end{aligned}$$

if $-1<\gamma_2\leq 0$. So, in this case we have:

$$J(F_{1,R}^{2,+})+J(F_{1,R}^{2,-}) \preceq \begin{cases} n^{\frac{\widetilde{\gamma}_2(q-1)}{1+\alpha_2}+q-1+\varepsilon}, & \frac{\widetilde{\gamma}_2(q-1)}{1+\alpha_2}+q>2, \\ n^{1+\varepsilon}\ln n, & \frac{\widetilde{\gamma}_2(q-1)}{1+\alpha_2}+q=2, \\ n^{1+\varepsilon}, & \frac{\widetilde{\gamma}_2(q-1)}{1+\alpha_2}+q<2, \end{cases} \quad \forall\varepsilon>0. \tag{73}$$

2.3) If $z'\in E_1^{2,\pm}$, then

$$J(F_{2,R}^{2,+})+J(F_{2,R}^{2,-}) \tag{74}$$

$$\begin{aligned}
&= \int\limits_{F_{2,R}^{2,+}\cup F_{2,R}^{2,-}} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2(q-1)}\,|\Psi(\tau)-\Psi(w')|^{q}} \\
&\preceq n \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2(q-1)}\,|\Psi(\tau)-\Psi(w')|^{q-1}} \\
&\quad + n \int\limits_{F_{2,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2(q-1)}\,|\Psi(\tau)-\Psi(w')|^{q-1}}
\end{aligned}$$

for $\gamma_2 > 0$. The last two integrals are evaluated identically. Therefore, we evaluate one of them, say the first. For $\tau \in F_{2,R}^{2,+}$ and $z' \in E_1^{2,\pm}$, we have:

$$\begin{aligned}
\left|\Psi(\tau)-\Psi(w')\right| &\succeq \left|\Psi(\tau)-z_2^+\right|; \\
|\Psi(\tau)-\Psi(w_2)| &\succeq d_{2,R} \succeq \left|z_{2,R}-z_2^+\right|^{\frac{1}{1+\alpha_2}} \succeq \left(\frac{1}{n}\right)^{\frac{1+\varepsilon}{1+\alpha_2}}
\end{aligned}$$

Then

$$\begin{aligned}
J(F_{2,R}^{2,+}) &\preceq n \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{\left|\Psi(\tau)-z_2^+\right|^{\gamma_2(q-1)+q-1}} \\
&\preceq n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+1+\varepsilon} \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{\left|\tau-w_2^+\right|^{q-1+\varepsilon}} \preceq \begin{cases} n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+q-1+\varepsilon}, & q>2, \\ n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+1+\varepsilon}\ln n, & q=2, \\ n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+1+\varepsilon}, & q<2, \end{cases}
\end{aligned}$$

and so, for $\gamma_2 > 0$ we obtain:

$$J(F_{2,R}^{2,+}) + J(F_{2,R}^{2,-}) \preceq \begin{cases} n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+q-1+\varepsilon}, & q>2, \\ n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+1+\varepsilon}\ln n, & q=2, \\ n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+1+\varepsilon}, & q<2. \end{cases}$$

For $-1 < \gamma_2 \leq 0$ we get:

$$J(F_{2,R}^{2,+}) + J(F_{2,R}^{2,-}) \tag{75}$$

$$= \int\limits_{F_{2,R}^{2,+}\cup F_{2,R}^{2,-}} \frac{|\Psi(\tau)-\Psi(w_2)|^{(-\gamma_2)(q-1)}|\Psi'(\tau)||d\tau|}{|\Psi(\tau)-\Psi(w')|^{q}} \preceq n\int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|^{q-1}}$$

$$\preceq n\int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{\left|\Psi(\tau)-z_2^{+}\right|^{q-1}} \preceq n\int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{\left|\tau-w_2^{+}\right|^{q-1+\varepsilon}} \preceq \begin{cases} n^{q+\varepsilon}, & q>2,\\ n^{1+\varepsilon}\ln n, & q=2,\\ n^{1+\varepsilon}, & q<2.\end{cases}$$

Then, in this case we have:

$$J(F_{2,R}^{2,+})+J(F_{2,R}^{2,-}) \preceq \begin{cases} n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+q-1+\varepsilon}, & q>2,\\ n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+1+\varepsilon}\ln n, & q=2,\\ n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+1+\varepsilon}, & q<2.\end{cases} \tag{76}$$

2.4) If $z'\in E_2^{2,+}$, then for $\gamma_2>0$

$$J(F_{2,R}^{2,+}) \tag{77}$$

$$= \int\limits_{F_{2,R}^{2,+}} \frac{|\Psi'(\tau)||d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2(q-1)}|\Psi(\tau)-\Psi(w')|^{q}} \preceq \frac{n}{d_{2,R}^{\gamma_2(q-1)}}\int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|^{q-1}}$$

$$\preceq n^{1+\frac{\gamma_2(q-1)}{1+\alpha_2}+\varepsilon}\int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{|\tau-w'|^{q-1}} \preceq \begin{cases} n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+q-1+\varepsilon}, & q>2,\\ n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+1+\varepsilon}\ln n, & q=2,\\ n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+1+\varepsilon}, & q<2,\end{cases}$$

and

$$J(F_{2,R}^{2,-}) \tag{78}$$

$$= \int\limits_{F_{2,R}^{2,-}} \frac{|\Psi'(\tau)||d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2(q-1)}|\Psi(\tau)-\Psi(w')|^{q}} \preceq \frac{n}{d_{2,R}^{\gamma_2(q-1)}}\int\limits_{F_{2,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|^{q-1}}$$

$$\preceq n^{1+\frac{\gamma_2(q-1)}{1+\alpha_2}+\varepsilon}\int\limits_{F_{2,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|^{q-1}} \preceq n^{1+\frac{\gamma_2(q-1)}{1+\alpha_2}+\varepsilon}\int\limits_{F_{2,R}^{2,-}} \frac{|d\tau|}{|\tau-w'|^{q-1+\varepsilon}}$$

$$\preceq \begin{cases} n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+q-1+\varepsilon}, & q>2,\\ n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+1+\varepsilon}\ln n, & q=2,\\ n^{\frac{\gamma_2(q-1)}{1+\alpha_2}+1+\varepsilon}, & q<2.\end{cases}$$

Case of $z' \in E_2^{2,-}$ is absolutely identical to the case $z' \in E_2^{2,+}$.

If $-1 < \gamma_2 \leq 0$, then

$$J(F_{2,R}^{2,+}) = \int\limits_{F_{2,R}^{2,+}} \frac{|\Psi(\tau)-\Psi(w_2)|^{(-\gamma_2)(q-1)}|\Psi'(\tau)||d\tau|}{|\Psi(\tau)-\Psi(w')|^q} \tag{79}$$

$$\preceq n \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|^{q-1}} \preceq \begin{cases} n^{q-1}, & q>2, \\ n^{1+\varepsilon}\ln n, & q=2, \\ n^{1+\varepsilon}, & q<2, \end{cases}$$

and

$$J(F_{2,R}^{2,-}) = \int\limits_{F_{2,R}^{2,-}} \frac{|\Psi(\tau)-\Psi(w_2)|^{(-\gamma_2)(q-1)}|\Psi'(\tau)||d\tau|}{|\Psi(\tau)-\Psi(w')|^q} \tag{80}$$

$$\preceq n \int\limits_{F_{2,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|^{q-1}} \preceq \begin{cases} n^{q-1}, & q>2, \\ n^{1+\varepsilon}\ln n, & q=2, \\ n^{1+\varepsilon}, & q<2. \end{cases}$$

Combining the estimations (72), (72)-(80), we obtain:

$$\left(J_{n,2}^2\right)^{\frac{1}{q}} \preceq \begin{cases} n^{\frac{1}{p}+\varepsilon}, & p<2, \\ n^{1-\frac{1}{p}+\varepsilon}, & p \geq 2, \end{cases} \tag{81}$$

for each $-1 < \gamma_2 \leq 0$,

$$\left(J_{n,2}^2\right)^{\frac{1}{q}} \preceq \begin{cases} n^{\frac{\gamma_2}{p(1+\alpha_2)}+\frac{1}{p}+\varepsilon}, & 1<p<2+\frac{\gamma_2}{1+\alpha_2}, \\ n^{1-\frac{1}{p}+\varepsilon}, & p \geq 2+\frac{\gamma_2}{1+\alpha_2}, \end{cases} . \tag{82}$$

for each $\gamma_2 > 0$.

Combining (59), (70), (81) and (82), from (61), for $i = 1,2;\ m_1 = 1,\ m_2 = 2$, we get:

$$J_{n,2} \preceq n^{\frac{(\widetilde{\gamma}_1+1)\widetilde{\lambda}_1}{p}+\varepsilon} + \begin{cases} n^{\frac{\gamma_2}{p(1+\alpha_2)}+\frac{1}{p}+\varepsilon}, & 1<p<2+\frac{\gamma_2}{1+\alpha_2}, \\ n^{1-\frac{1}{p}+\varepsilon}, & p \geq 2+\frac{\gamma_2}{1+\alpha_2}, \end{cases} ,\ \forall \varepsilon > 0.$$

From (57), (58) and (59), according to (60) and (62), we have:

$$\|P_n\|_{C(\overline{G})} \preceq \|P_n\|_{L_p(h,\ L)} \cdot J_{n,2}$$

$$
\begin{aligned}
&\preceq \quad \|P_n\|_{L_p(h,\,L)} \sum_{i=1}^{m} \left(J_{n,2}^{i}\right)^{\frac{1}{q}} \\
&\preceq \quad \|P_n\|_{L_p(h,\,L)} \sum_{i=1}^{m} \sum_{j=1}^{2} \cdot \left(J(F_{j,R}^{i,\pm})\right)^{\frac{1}{q}} \\
&\preceq \quad \|P_n\|_{L_p(h,\,L)} \left(\sum_{i=1}^{m_1} n^{\frac{(\widetilde{\gamma}_i+1)\widetilde{\lambda}_i}{p}+\varepsilon} + \begin{cases} \sum\limits_{i=m_1+1}^{m} n^{\frac{\widetilde{\gamma}_i+1}{p(1+\alpha_i)}+\frac{\alpha_i}{p(1+\alpha_i)}+\varepsilon} & 1<p<2+\frac{\widetilde{\gamma}_i}{1+\alpha_i}, \\ n^{1-\frac{1}{p}+\varepsilon}, & p \geq 2+\frac{\widetilde{\gamma}_i}{1+\alpha_i}, \end{cases} \right)
\end{aligned}
$$

for any $i=\overline{m_1+1,m}$ and $\forall \varepsilon>0$ and the proof is completed.

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