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FIXED POINT THEOREMS FOR CYCLIC CONTRACTIONS IN B-METRIC SPACES

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Abstract. Fixed point theorems for various cyclic contractions are proved in b-metric spaces. Our result generalizes many known results in fixed point theory. We use our results to obtain fixed points of certain contractions of integral type.

Keywords. Fixed point; Cyclic contraction; Contraction; Operator; b-metric space.

1. Introduction

Since the introduction of Banach contraction principle in 1922, because of its wide applications, the study of existence and uniqueness of fixed points and common fixed points has become a subject of great interest. Many authors proved the Banach contraction principle in various generalized metric spaces. In [17], Bakhtin introduced b-metric space as a generalization of metric space. He proved the contraction mapping principle in b-metric space that generalized the famous Banach contraction principle in metric space. Since then, several papers have dealt with fixed point theory or the variational principle for single-valued and multi-valued operators in b-metric space (see [7-13] and the references therein).

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In 2003, Kirk *et al.* [23] reported cyclic contraction as a generalization of the usual contraction and proved fixed point results for this type of contraction. Let A and B be nonempty subsets of a metric space (X, d) and $T : A\cup B \to A\cup B$. T is called cyclic map if $T(A) \subseteq B$ and $T(B) \subseteq A$. A cyclic map $T : A\cup B \to A\cup B$ is said to be a cyclic contraction if there exist $k \in (0,1)$ such that $d(Tx,Ty) \leq kd(x,y) \forall x \in A, y \in B$. Since then many researchers continued investigation in this direction and obtained many result (see [1], [14], [15], [16], [21]).

Many authors have studied fixed point theorems which are based on altering distance functions (see [2-6], [19], [22], [24], [25]). Berinde ([3-6]) initiated the concept of almost contractions and studied many interesting fixed point theorems. Ciric et al. [19] introduced the concept of almost generalized contractive condition and they proved existence of fixed points for this type of contraction mapping. Shatanawi *et al.* [24] introduced the notion of almost generalized (ϕ,ψ)-contractive mapping and established some results in complete ordered metric spaces. Generalizing the concept of Berinde ([3-6]), Ciric *et al.* [19] and Shatanawi *et al.* [24], Roshan *et al.* [18] introduced the notion of almost generalized $(\phi,\psi)_s$ contractive mapping in ordered b-metric space and established some results in complete ordered b-metric space.

In this paper, we have proved the existence and uniqueness of fixed points for cyclic contraction mapping of Banach type, Kannan type, Chatterjee type and Ciric type in b-metric space. We have also introduced cyclic generalized (ψ,ϕ)-rational contraction and proved the existence and uniqueness of fixed points for this type of contraction mapping in b-metric space. We apply our results to obtain fixed points of certain contractions of integral type.

2. Preliminaries

Definition 2.1. [17] Let X be a nonempty set and the mapping $d\colon X\times X \to [0,\infty)$ satisfies:

(bM1) $d(x,y) = 0$ if and only if $x = y$ for all $x,y \in X$;
(bM2) $d(x,y) = d(y,x)$ for all $x,y \in X$;
(bM3) there exist a real number $s \geq 1$ such that $d(x,y) \leq s[d(x,z)+d(z,y)]$ for all $x,y,z \in X$.

Then d is called a b-metric on X and (X,d) is called a b-metric space (in short bMS) with coefficient s.

Example 2.1. [18] Let (X,d) be a metric space and $\rho(x,y) = (d(x,y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$.

Now we define convergence and Cauchy sequence in b-metric space and completeness of b-metric spaces.

Definition 2.2. [17] Let (X,d) be a b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then

(a) The sequence $\{x_n\}$ is said to be convergent in (X,d) and converges to x, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n,x) < \varepsilon$ for all $n > n_0$ and this fact is represented by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(b) The sequence $\{x_n\}$ is said to be Cauchy sequence in (X,d) if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n,x_{n+p}) < \varepsilon$ for all $n > n_0, p > 0$ or equivalently, if $\lim_{n\to\infty} d(x_n,x_{n+p}) = 0$ for all $p > 0$.

(c) (X,d) is said to be a complete b-metric space if every Cauchy sequence in X converges to some $x \in X$.

Definition 2.3. [23] Let (X,d) be a metric space. Let p be a positive integer, $A_1,A_2,...,A_p$ be nonempty subsets of $X, Y = \cup_{i=1}^{p} A_i$, and $T : Y \to Y$. Then T is called a cyclic operator if

(1) $A_i, i = 1,2,...,p$ are nonempty subsets, and

(2) $T(A_1) \subseteq A_2,...,T(A_{p-1}) \subseteq A_p, T(A_p) \subseteq A_1$.

3. Main results

The following result is the analogue of Banach contraction principle for cyclic contraction in b-metric space.

Theorem 3.1. *Let $\{A_i\}_{i=1}^{p}$ where p is a positive integer, be non empty closed subsets of a complete b-metric space (X,d) with coefficient $s \geq 1$ and suppose $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is a cyclical operator that satisfies the condition*

$$T(A_i) \subseteq A_{i+1} \; for \; all \; i \in \{1,2,...,p\} \tag{3.1}$$

such that

$$d(Tx,Ty) \leq \lambda d(x,y) \tag{3.2}$$

for all $x \in A_i, y \in A_{i+1}$ *and* $\lambda \in (0, \frac{1}{s})$. *Then* T *has a unique fixed point.*

Proof. Let $x_0 \in \cup_{i=1}^{p} A_i$. So there exist $i \in \{1,2,...,p\}$ such that $x_0 \in A_i$ and from (3.1) we have $x_1 = Tx_0 \in A_{i+1}$. Thus we define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. We shall show that $\{x_n\}$ is a Cauchy sequence. If $x_n = x_{n+1}$ then x_n is a fixed point of T. So, suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. Setting $d(x_n, x_{n+1}) = d_n$, it follows from (3.2) that

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \lambda d(x_{n-1}, x_n) \\ d_n &\leq \lambda d_{n-1}. \end{aligned}$$

Repeating this process, we obtain

$$d_n \leq \lambda^n d_0. \tag{3.3}$$

Also, we can assume that x_0 is not a periodic point of T. Indeed, if $x_0 = x_n$ then using (3.3), for any $n \geq 2$, we have

$$\begin{aligned} d(x_0, Tx_0) &= d(x_n, Tx_n) \\ d(x_0, x_1) &= d(x_n, x_{n+1}) \\ d_0 &= d_n \\ d_0 &\leq \lambda^n d_0, \end{aligned}$$

a contradiction. Therefore, we must have $d_0 = 0$, i.e., $x_0 = x_1$, and so x_0 is a fixed point of T. Thus we assume that $x_n \neq x_m$ for all distinct $n, m \in \mathbb{N}$. For any $m \geq 1, p \geq 1$ it follows from (3.3)

$$\begin{aligned}
d(x_m, x_{m+p}) &\leq s[d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+p})] \\
&= sd(x_m, x_{m+1}) + sd(x_{m+1}, x_{m+p}) \\
&\leq sd(x_m, x_{m+1}) + s^2[d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_{m+p})] \\
&= sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^2 d(x_{m+2}, x_{m+p}) \\
&\leq sd(x_m, x_{m+1}) + s^2 d(x_{m+1}, x_{m+2}) + s^3 d(x_{m+2}, x_{m+3}) + \ldots \\
&\quad + s^{p-1} d(x_{m+p-2}, x_{m+p-1}) + s^{p-1} d(x_{m+p-1}, x_{m+p}) \\
&\leq s\lambda^m d(x_1, x_0) + s^2 \lambda^{m+1} d(x_1, x_0) + s^3 \lambda^{m+2} d(x_1, x_0) + \ldots \\
&\quad + s^{p-1} \lambda^{m+p-2} d(x_1, x_0) + s^{p-1} \lambda^{m+p-1} d(x_1, x_0) \\
&\leq s\lambda^m d(x_1, x_0) + s^2 \lambda^{m+1} d(x_1, x_0) + s^3 \lambda^{m+2} d(x_1, x_0) + \ldots \\
&\quad + s^{p-1} \lambda^{m+p-2} d(x_1, x_0) + s^{p} \lambda^{m+p-1} d(x_1, x_0) \\
&= s\lambda^m (1 + s\lambda + s^2\lambda^2 + \cdots + s^{p-2}\lambda^{p-2} + s^{p-1}\lambda^{p-1}) d(x_1, x_0) \\
&\leq s\lambda^m (1 + s\lambda + (s\lambda)^2 + \cdots + (s\lambda)^{p-2} + (s\lambda)^{p-1} + \cdots) d(x_1, x_0) \\
&= s\lambda^m \frac{1}{1 - s\lambda} d(x_1, x_0) \text{ where } s\lambda < 1.
\end{aligned}$$

Therefore, we have

$$\lim_{m \to \infty} d(x_m, x_{m+p}) = 0 \text{ for all } p > 0. \tag{3.4}$$

Thus $\{x_n\}$ is a Cauchy sequence in $\cup_{i=1}^{p} A_i$, a subspace of a complete b-metric space (X, d). Therefore there exists $u \in \cup_{i=1}^{p} A_i$ such that

$$\lim_{n \to \infty} x_n = u. \tag{3.5}$$

The sequence $\{x_n\}$ has infinite number of terms in each A_i for all $i \in \{1,2,...,p\}$. Therefore $u \in \cap_{i=1}^{p} A_i$. We shall show that u is a fixed point of T. Again, for any $n \in \mathbb{N}$ we have

$$\begin{aligned} d(u,Tu) &\leq s[d(u,x_{n+1}) + d(x_{n+1},Tu)] \\ &= s[d(u,x_{n+1}) + d(Tx_n,Tu)] \\ &\leq s[d(u,x_{n+1}) + \lambda d(x_n,u)]. \end{aligned}$$

Using (3.5) it follows from above inequality that $d(u,Tu) = 0$, i.e., $Tu = u$. Thus u is a fixed point of T.

For uniqueness, let v be another fixed point of T. Then it follows from (3.2) that $d(u,v) = d(Tu,Tv) \leq \lambda d(u,v) < d(u,v)$, a contradiction. Therefore, we must have $d(u,v) = 0$, i.e., $u = v$. Thus the fixed point is unique.

Example 3.2. Let $X = [0,1]$ and $d(x,y) = \mid x-y \mid^2$ for all $x,y \in X$. Then (X,d) is a complete b-metric space with s=2. Let $A_1 = [0,\frac{1}{2}], A_2 = [0,\frac{1}{3}], A_3 = [0,\frac{1}{4}], A_4 = [0,\frac{1}{5}]$. Define $T : \cup_{i=1}^{4} A_i \to \cup_{i=1}^{4} A_i$ as $Tx = \frac{x}{2} - \frac{x^2}{4}$ for all $x \in X$. Then $\cup_{i=1}^{4} A_i$ is a cyclic representation of Y with respect to T.

$$\begin{aligned} d(Tx,Ty) &= \mid Tx - Ty \mid^2 = \mid \frac{x}{2} - \frac{x^2}{4} - \frac{y}{2} + \frac{y^2}{4} \mid^2 \\ & \mid \frac{x-y}{2} - \frac{x^2-y^2}{4} \mid^2 \\ &= \mid \frac{x-y}{2}[1 - \frac{x+y}{2}] \mid^2 \\ &< \mid \frac{x-y}{2} \mid^2 \\ &= \frac{1}{4} d(x,y) \end{aligned}$$

$0 \in \cap_{i=1}^{4} A_i$ is the unique fixed point of T.

The following Theorem is the fixed point result for Kannan type cyclic contraction in b-metric spaces.

Theorem 3.3. *Let $\{A_i\}_{i=1}^{p}$ where p is a positive integer, be non empty closed subsets of a complete b-metric space (X,d) with coefficient $s \geq 1$ and suppose $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is a* *cyclical operator that satisfies the condition*

$$T(A_i) \subseteq A_{i+1} \text{ for all } i \in \{1,2,...,p\} \tag{3.6}$$

such that

$$d(Tx,Ty) \leq \lambda[d(x,Tx)+d(y,Ty)] \tag{3.7}$$

for all $x \in A_i, y \in A_{i+1}$*, where* $\lambda \in (0, \frac{1}{s+1})$*. Then* T *has a unique fixed point.*

Proof. Let $x_0 \in \cup_{i=1}^{p} A_i$. So there exist $i \in \{1,2,...,p\}$ such that $x_0 \in A_i$ and from (3.6) we have $x_1 = Tx_0 \in A_{i+1}$. Thus we define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. We shall show that $\{x_n\}$ is a Cauchy sequence. If $x_n = x_{n+1}$ then x_n is fixed point of T. So, suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. Setting $d(x_n,x_{n+1}) = d_n$, it follows from (3.7) that

$$\begin{aligned}
d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n) \leq \lambda[d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)] \\
d(x_n,x_{n+1}) &\leq \lambda[d(x_{n-1},x_n)+d(x_n,x_{n+1})] \\
i.e\ d_n &\leq \lambda[d_{n-1}+d_n] \\
d_n &\leq \frac{\lambda}{1-\lambda}d_{n-1} = \beta d_{n-1},
\end{aligned}$$

where $\beta = \frac{\lambda}{1-\lambda} < \frac{1}{s}$ (as, $\lambda < \frac{1}{s+1}$). Repeating this process we obtain

$$d_n \leq \beta^n d_0. \tag{3.8}$$

Also, we can assume that x_0 is not a periodic point of T. Indeed, if $x_0 = x_n$ then using (3.8), for any $n \geq 2$, we have

$$\begin{aligned}
d(x_0,Tx_0) &= d(x_n,Tx_n) \\
d(x_0,x_1) &= d(x_n,x_{n+1}) \\
d_0 &= d_n \\
d_0 &\leq \beta^n d_0,
\end{aligned}$$

a contradiction. Therefore, we must have $d_0 = 0$, i.e., $x_0 = x_1$, and so x_0 is a fixed point of T. Thus we assume that $x_n \neq x_m$ for all distinct $n,m \in \mathbb{N}$.

Since $s\beta < 1$, following the argument similar to that given in Theorem 3.1 there exist $u \in \cup_{i=1}^{p} A_i$ such that

$$\lim_{n\to\infty} x_n = u. \tag{3.9}$$

The sequence $\{x_n\}$ has infinite number of terms in each A_i for all $i \in \{1,2,...,p\}$. Therefore $u \in \cap_{i=1}^{p} A_i$. We shall show that u is a fixed point of T. Again, for any $n \in \mathbb{N}$ we have

$$\begin{aligned} d(u,Tu) &\leq s[d(u,x_{n+1}) + d(x_{n+1},Tu)] \\ &= s[d(u,x_{n+1}) + d(Tx_n,Tu)] \\ &\leq s[d(u,x_{n+1}) + \lambda\{d(x_n,Tx_n) + d(u,Tu)\}] \\ &= s[d(u,x_{n+1}) + \lambda\{d(x_n,x_{n+1}) + d(u,Tu)\}] \end{aligned}$$

Using (3.9) and the fact that $\lambda < \frac{1}{s+1}$, it follows from the above inequality that

$$d(u,Tu) \leq s\lambda d(u,Tu) \Rightarrow d(u,Tu) = 0$$

i.e., $Tu = u$. Thus u is a fixed point of T. For uniqueness, let v be another fixed point of T. Then it follows from (3.7) that $d(u,v) = d(Tu,Tv) \leq \lambda[d(u,Tu) + d(v,Tv)] = \lambda[d(u,u) + d(v,v)] = 0$. Therefore, we have $d(u,v) = 0$, i.e., $u = v$. Thus fixed point is unique.

Following theorem is the fixed point result for Chatterjee type cyclic contraction in b-metric space.

Theorem 3.4. *Let $\{A_i\}_{i=1}^{p}$ where p is a positive integer, be non empty closed subsets of a complete b-metric space (X,d) with coefficient $s \geq 1$ and suppose $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is a cyclical operator that satisfies the condition*

$$T(A_i) \subseteq A_{i+1} \text{ for all } i \in \{1,2,...,p\} \tag{3.10}$$

such that

$$d(Tx,Ty) \leq \lambda[d(Tx,y) + d(Ty,x)] \tag{3.11}$$

for all $x \in A_i, y \in A_{i+1}$, where $\lambda \in (0, \frac{1}{s(s+1)})$. Then T has a unique fixed point.

Proof. Let $x_0 \in \cup_{i=1}^{p} A_i$. So there exist $i \in \{1,2,...,p\}$ such that $x_0 \in A_i$ and from (3.10) we have $x_1 = Tx_0 \in A_{i+1}$. Thus we define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. We shall show that $\{x_n\}$ is Cauchy sequence. If $x_n = x_{n+1}$ then x_n is fixed point of T. So, suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. Setting $d(x_n,x_{n+1}) = d_n$, it follows from (3.11) that

$$\begin{aligned} d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n) \leq \lambda[d(Tx_{n-1},x_n)+d(Tx_n,x_{n-1})] \\ d(x_n,x_{n+1}) &\leq \lambda[d(x_n,x_n)+d(x_{n+1},x_{n-1})] \\ i.e\ d_n &\leq \lambda s[d(x_{n+1},x_n)+d(x_n,x_{n-1})] \\ d_n &\leq \frac{\lambda s}{1-\lambda s} d_{n-1} = \beta d_{n-1}, \end{aligned}$$

where $\beta = \frac{\lambda s}{1-\lambda s} < 1$ (as, $\lambda < \frac{1}{s(s+1)}$). Repeating this process we obtain

$$d_n \leq \beta^n d_0. \tag{3.12}$$

Also, we can assume that x_0 is not a periodic point of T. Indeed, if $x_0 = x_n$ then using (3.12), for any $n \geq 2$, we have

$$\begin{aligned} d(x_0,Tx_0) &= d(x_n,Tx_n) \\ d(x_0,x_1) &= d(x_n,x_{n+1}) \\ d_0 &= d_n \\ d_0 &\leq \beta^n d_0, \end{aligned}$$

a contradiction. Therefore, we must have $d_0 = 0$, i.e., $x_0 = x_1$, and so x_0 is a fixed point of T. Thus we assume that $x_n \neq x_m$ for all distinct $n,m \in \mathbb{N}$.

Since $s\beta < 1$, following the argument similar to that given in Theorem 3.1 there exist $u \in \cup_{i=1}^{p} A_i$ such that

$$\lim_{n\to\infty} x_n = u. \tag{3.13}$$

The sequence $\{x_n\}$ has infinite number of terms in each A_i for all $i \in \{1,2,...,p\}$. Therefore $u \in \cap_{i=1}^{p} A_i$ We shall show that u is a fixed point of T. Again, for any $n \in \mathbb{N}$ we have

$$\begin{aligned} d(u,Tu) &\leq s[d(u,x_{n+1})+d(x_{n+1},Tu)] \\ &= s[d(u,x_{n+1})+d(Tx_n,Tu)] \\ &\leq s[d(u,x_{n+1})+\lambda\{d(x_n,Tu)+d(u,Tx_n)\}] \\ &= s[d(u,x_{n+1})+\lambda\{d(x_n,Tu)+d(u,x_{n+1})\}] \end{aligned}$$

Using(3.13) and the fact that $\lambda < \frac{1}{s(s+1)}$, it follows from the above inequality that

$$d(u,Tu) \leq s\lambda d(u,Tu) \Rightarrow d(u,Tu) = 0$$

i.e., $Tu = u$. Thus u is a fixed point of T.

For uniqueness, let v be another fixed point of T. Then it follows from (3.11) that $d(u,v) = d(Tu,Tv) \leq \lambda[d(u,Tv)+d(v,Tu)] = \lambda[d(u,v)+d(v,u)] = 2\lambda d(u,v)$. This is a contraction because $\lambda < \frac{1}{s(s+1)}$ and $s \geq 1$. Therefore, we have $d(u,v) = 0$, i.e., $u = v$. Thus fixed point is unique.

Following Theorem is the fixed point result for Ciric type cyclic contraction in b-metric space.

Theorem 3.5. *Let $\{A_i\}_{i=1}^{p}$ where p is a positive integer, be non empty closed subsets of a complete b-metric space (X,d) with coefficient $s \geq 1$ and suppose $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is a cyclical operator that satisfies the condition*

$$T(A_i) \subseteq A_{i+1} \; for \; all \; i \in \{1,2,...,p\} \tag{3.14}$$

such that

$$d(Tx,Ty) \leq \lambda max\{d(x,y), d(Tx,x), d(Ty,y), d(x,Ty), d(y,Tx)\} \tag{3.15}$$

for all $x \in A_i, y \in A_{i+1}$, where $\lambda \in (0, \frac{1}{s(s+1)})$. Then T has a unique fixed point.

Proof. Let $x_0 \in \cup_{i=1}^{p} A_i$. So there exist $i \in \{1,2,...,p\}$ such that $x_0 \in A_i$ and from (3.14) we have $x_1 = Tx_0 \in A_{i+1}$. Thus we define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. We shall show that $\{x_n\}$ is Cauchy sequence. If $x_n = x_{n+1}$ then x_n is fixed point of T. So, suppose that

$x_n \neq x_{n+1}$ for all $n \geq 0$. Setting $d(x_n, x_{n+1}) = d_n$, it follows from (3.15) that

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \lambda max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), \\ &\quad d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\} \\ &= \lambda max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), \\ &\quad d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n)\} \\ &= \lambda max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1})\} \end{aligned}$$

case 1: If $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1})\} = d(x_{n-1}, x_n)$, then

$$d_n \leq \lambda d(x_n, x_{n-1}) = \lambda d_{n-1} \text{ where } \lambda < 1 \tag{3.16}$$

case 2: If $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1})\} = d(x_n, x_{n+1})$, then

$$d_n \leq \lambda d_n \tag{3.17}$$

this is a contradiction

case 3: If $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1})\} = d(x_{n-1}, x_{n+1})$, then

$$d_n \leq \frac{\lambda s}{1 - \lambda s} d_{n-1} = \beta d_{n-1}, \tag{3.18}$$

where $\beta = \frac{\lambda s}{1-\lambda s} < 1$ (as, $\lambda < \frac{1}{s(s+1)}$). Repeating this process we obtain either

$$\lim_{n\to\infty} d_n = \lambda^n d_0. \tag{3.19}$$

or

$$\lim_{n\to\infty} d_n = \beta^n d_0.$$

Since $s\beta < 1$, following the argument similar to that given in Theorem 3.1 there exist $u \in \cup_{i=1}^{p} A_i$ such that

$$\lim_{n\to\infty} x_n = u. \tag{3.20}$$

The sequence $\{x_n\}$ has infinite number of terms in each A_i for all $i \in \{1,2,...,p\}$. Therefore $u \in \cap_{i=1}^{p} A_i$. We shall show that u is a fixed point of T. Again, for any $n \in \mathbb{N}$ we have

$$\begin{aligned} d(u,Tu) &\leq s[d(u,x_{n+1}) + d(x_{n+1},Tu)] \\ &= sd(u,x_{n+1}) + sd(Tx_n,Tu) \\ &\leq sd(u,x_{n+1}) + s\lambda max\{d(x_n,u), d(x_n,Tu), d(u,Tx_n), d(x_n,Tx_n), d(u,Tu)\} \\ &= sd(u,x_{n+1}) + s\lambda max\{d(x_n,u), d(x_n,Tu), d(u,x_{n+1}), d(x_n,x_{n+1}), d(u,Tu)\}. \end{aligned}$$

Using (3.20) and the fact that $\lambda < \frac{1}{s(s+1)}$, it follows from above inequality that

$$d(u,Tu) \leq s\lambda d(u,Tu) \Rightarrow d(u,Tu) = 0,$$

i.e., $Tu = u$. Thus u is a fixed point of T.

For uniqueness, let v be another fixed point of T. Then it follows from (3.15) that $d(u,v) = d(Tu,Tv) \leq \lambda max\{d(u,v), d(u,Tv), d(v,Tu), d(u,Tu), d(v,Tv)\} = \lambda d(u,v)$. This is a contradiction. Therefore, we have $d(u,v) = 0$, i.e., $u = v$. Thus fixed point is unique.

Khan et al. [20] introduced the concept of an altering distance function as follows:

Definition 3.6. The function $\phi : [0,\infty) \to [0,\infty)$ is called an altering distance function, if the following properties hold:

1. ϕ is continuous and nondecreasing,
2. $\phi(t) = 0$ if and only if t=0.

Definition 3.7. Let (X,d) be a complete b-metric space, $A_1, A_2, \cdots, A_p$ where p is a positive integer, be non empty closed subsets of X and $Y = \cup_{i=1}^{p} A_i$. An operator $T : Y \to Y$ is called a cyclic generalized (ψ,ϕ)-rational contraction if

(1) T is a cyclic operator,

(2) For any $x \in A_i, y \in A_{i+1}, i = 1,2,\cdots p$, ψ and ϕ are altering distance function, $L \geq 0, A_{p+1} = A_1$,

$$\psi(s^4 d(Tx,Ty)) \leq \psi(M(x,y)) - \phi(M(x,y)) + L\psi(m(x,y)) \tag{3.21}$$

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\frac{1+d(x,Tx)}{1+d(x,y)}, \frac{d(x,Ty)+d(y,Tx)}{2s}\} \tag{3.22}$$

$$m(x,y) = min\{d(x,Tx)+d(y,Ty), d(x,Ty), d(y,Tx))\}. \tag{3.23}$$

Theorem 3.8. *Let (X,d) be a complete b-metric space. $A_1,A_2,\cdots,A_p$ where p is a positive integer, be non empty closed subsets of X and $Y=\cup_{i=1}^m A_i$. Suppose that $T:Y\to Y$ is a cyclic generalized (ψ,ϕ)-rational contraction. Then T has a unique fixed point in $\cap_{i=1}^m A_i$*

Proof. Let x_0 be any arbitrary point in X. Consider the sequence $\{x_n\}$ such that $x_n=Tx_{n-1}$ for $n=1,2,3\cdots$. If $x_n=x_{n+1}$ for any n, then $\{x_n\}$ is a constant sequence and thus convergent. Suppose $x_n\neq x_{n+1}$ for all n. Then

$\psi(d(x_n,x_{n+1})\leq\psi(s^4d(x_n,x_{n+1})=\psi(s^4d(Tx_{n-1},Tx_n))$

$$\leq\psi(M(x_{n-1},x_n)-\phi(M(x_{n-1},x_n))+L\psi(m(x_{n-1},x_n)) \tag{3.24}$$

$$\begin{aligned}
m(x_{n-1},x_n) &= min\{d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n),d(x_{n-1},Tx_n),d(x_n,Tx_{n-1}\}\\
&= min\{d(x_{n-1},x_n)+d(x_n,x_{n+1}),d(x_{n-1},x_{n+1}),d(x_n,x_n\}\\
&= 0.
\end{aligned}$$

Also

$$\begin{aligned}
M(x_{n-1},x_n) &= max\{d(x_{n-1},x_n),d(x_{n-1},Tx_{n-1}),d(x_n,Tx_n)\frac{1+d(x_{n-1},Tx_{n-1})}{1+d(x_{n-1},x_n)},\\
&\quad \frac{d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})}{2s}\}\\
&= max\{d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_n,x_{n+1})\frac{1+d(x_{n-1},x_n)}{1+d(x_{n-1},x_n)},\frac{d(x_{n-1},x_{n+1})}{2s}\}\\
&\leq max\{d(x_{n-1},x_n),d(x_n,x_{n+1}),\frac{s[d(x_{n-1},x_n)+d(x_n,x_{n+1})]}{2s}\}\\
&= max\{d(x_{n-1},x_n),d(x_n,x_{n+1}),\frac{[d(x_{n-1},x_n)+d(x_n,x_{n+1})]}{2}\}\\
&= max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}.
\end{aligned}$$

Suppose, $M(x_{n-1},x_n)=d(x_n,x_{n+1})$, then

$$\begin{aligned}
\psi(d(x_n,x_{n+1}) &\leq \psi(d(x_n,x_{n+1}))-\phi(d(x_n,x_{n+1}))+0\\
&< \psi(d(x_n,x_{n+1})),
\end{aligned}$$

which is a contradiction. Therefore $M(x_{n-1},x_n) = d(x_{n-1},x_n)$. From (3.24), we have

$$\psi(d(x_n,x_{n+1})) \leq \psi(d(x_{n-1},x_n)) - \phi(d(x_{n-1},x_n)) \tag{3.25}$$

$$\psi(d(x_n,x_{n+1})) \quad < \quad \psi(d(x_{n-1},x_n))$$

since ψ is a non-decreasing mapping, $\{d(x_n,x_{n+1}) : n \in N \cup 0\}$ is a non increasing sequence of positive numbers. So, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = r$$

Letting $n \to \infty$ in (3.25), we get

$$\psi(r) \leq \psi(r) - \phi(r).$$

Therefore $\phi(r) = 0$, and hence $r = 0$. Thus we have

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{3.26}$$

Now, we shall prove that $\{x_n\}$ is a Cauchy sequence in (X,d). Suppose, on the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two sequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that for all positive integer k,

$$m(k) > n(k) \geq k, d(x_{m(k)},x_{n(k)}) \geq \varepsilon. \tag{3.27}$$

Further, corresponding to $n(k)$, we can choose $m(k)$ in a such a way that it is the smallest integer with $m(k) > n(k) \geq k$ satisfying (3.27). Then we have

$$d(x_{m(k)-1},x_{n(k)}) < \varepsilon. \tag{3.28}$$

From (3.27), (3.28) and using the triangular inequality, we get

$$\begin{aligned} \varepsilon &\leq d(x_{n(k)},x_{m(k))}) \\ &\leq s[d(x_{n(k)},x_{m(k)-1}) + d(x_{m(k)-1},x_{m(k)})] \\ &< s\varepsilon + sd(x_{m(k)-1},x_{m(k)}). \end{aligned}$$

Thus we have

$$\varepsilon \leq d(x_{m(k)},x_{n(k)}) < s\varepsilon + sd(x_{m(k)-1},x_{m(k)}).$$

As $k \to \infty$ in the above inequality and using (3.26), we obtain

$$\lim_{n\to\infty} d(x_{m(k)}, x_{n(k)}) = r\varepsilon, \tag{3.29}$$

where $0 < r < s$.

On the other hand, for all k, ther exists $j(k) \in \{1, 2, \cdots, p\}$ such that $n(k) - m(k) + j(k) \equiv 1[p]$. Then $x_{m(k)-j(k)}$(for k large enough, $m(k) > j(k)$) and $x_{n(k)}$ lie in different adjacently labeled sets A_i and A_{i+1} for certain $i \in \{1, 2, \cdots, p\}$.

Using the triangular inequality, we get

$$\begin{aligned}
d(x_{m(k)}, x_{m(k)-j(k)}) &\leq s[d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)-j(k)})] \\
&\leq sd(x_{m(k)}, x_{m(k)-1}) + s^2[d(x_{m(k)-1}, x_{m(k)-2}) + d(x_{m(k)-2}, x_{m(k)-j(k)})] \\
&\leq sd(x_{m(k)}, x_{m(k)-1}) + s^2 d(x_{m(k)-1}, x_{m(k)-2}) + \cdots \\
&+ s^{j(k)} d(x_{m(k)-(j(k)-1)}, x_{m(k)-j(k))}).
\end{aligned}$$

Taking the upper limit as $k \to \infty$ and using (3.26), we get

$$d(x_{m(k)}, x_{m(k)-j(k)}) = 0. \tag{3.30}$$

From (3.27) and using the triangular inequality, we get

$$\begin{aligned}
\varepsilon &\leq d(x_{m(k)}, x_{n(k))}) \\
&\leq s[d(x_{m(k)}, x_{m(k)-j(k)}) + d(x_{m(k)-j(k)}, x_{n(k)})] \\
&\leq sd(x_{m(k)}, x_{m(k)-j(k)}) + sd(x_{m(k)-j(k)}, x_{n(k)}).
\end{aligned}$$

Using (3.30) and taking the upper limit as $k \to \infty$, we get

$$\frac{\varepsilon}{s} \leq \lim_{k\to\infty} \sup d(x_{m(k)-j(k)}, x_{n(k)}).$$

On the other hand, we have

$$d(x_{m(k)-j(k)}, x_{n(k)}) \leq sd(x_{m(k)-j(k)}, x_{m(k)}) + sd(x_{m(k)}, x_{n(k)}).$$

Using (3.29), (3.30)and taking the upper limit as $k \to \infty$, we get

$$\lim_{k\to\infty} \sup d(x_{m(k)-j(k)}, x_{n(k)}) \leq sr\varepsilon.$$

So, we have

$$\frac{\varepsilon}{s} \leq \lim_{k\to\infty}\sup d(x_{m(k)-j(k)}, x_{n(k)}) \leq sr\varepsilon. \tag{3.31}$$

Again, using the triangular inequality, we have

$$d(x_{m(k)-j(k)}, x_{n(k)+1}) \leq sd(x_{m(k)-j(k)}, x_{n(k)}) + sd(x_{n(k)}, x_{n(k)+1})$$

$$\begin{aligned}\varepsilon &\leq d(x_{m(k)}, x_{n(k)}) \leq sd(x_{m(k)}, x_{m(k)-j(k)}) + sd(x_{m(k)-j(k)}, x_{n(k)}) \\ &\leq sd(x_{m(k)}, x_{m(k)-j(k)}) + s[s[d(x_{m(k)-j(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)})\end{aligned}$$

and

$$\varepsilon \leq d(x_{m(k)}, x_{n(k)}) \leq sd(x_{m(k)}, x_{m(k)-j(k)+1}) + sd(x_{m(k)-j(k)+1}, x_{n(k)}).$$

Taking the upper limit as $k \to \infty$ in the first and second inequalities above, and using (3.26), (3.30) and (3.31) we get

$$\frac{\varepsilon}{s^2} \leq \lim_{i\to\infty}\sup d(x_{m(k)-j(k)}, x_{n(k)+1}) \leq s^2 r\varepsilon. \tag{3.32}$$

Similarly, taking the upper limit as $i \to \infty$ in the third inequality above, and using (3.30) we get

$$\frac{\varepsilon}{s} \leq \lim_{i\to\infty}\sup d(x_{m(k)-j(k)+1}, x_{n(k)}).$$

Again using triangular inequality and (3.31),

$$d(x_{m(k)-j(k)+1}, x_{n(k)}) \leq s[d(x_{n(k)}, x_{m(k)-j(k)}) + d(x_{m(k)-j(k)}, x_{m(k)-j(k)+1})]$$

$\leq sd(x_{n(k)}, x_{m(k)-j(k)}) = s^2 r\varepsilon$. Hence we get

$$\frac{\varepsilon}{s} \leq \lim_{i\to\infty}\sup d(x_{m(k)-j(k)+1}, x_{n(k)}) < s^2 r\varepsilon. \tag{3.33}$$

Similarly, we have

$$\frac{\varepsilon}{s^2} \leq \lim_{i\to\infty}\sup d(x_{m(k)-j(k)+1}, x_{n(k)+1}) \leq s^2 r\varepsilon. \tag{3.34}$$

From (3.21), we have

$$\psi(s^4 d(x_{m(k)-j(k)}, x_{n(k)+1})) \leq \psi(M(x_{m(k)-j(k)}, x_{n(k)})) - \phi(M(x_{m(k)-j(k)}, x_{n(k)})) + \tag{3.35}$$

$L\psi(m(x_{m(k)-j(k)}, x_{n(k)}))$, where

$$M(x_{m(k)-j(k)}, x_{n(k)}) = \max\{d(x_{m(k)-j(k)}, x_{n(k)}), d(x_{m(k)-j(k)}, x_{m(k)-j(k)+1}),$$

$$d(x_{n(k)},x_{n(k)+1})\frac{1+d(x_{m(k)-j(k)},x_{m(k)-j(k)+1})}{1+d(x_{m(k)-j(k)},x_{n(k)})},\frac{d(x_{m(k)-j(k)},x_{n(k)+1})+d(x_{n(k)},x_{m(k)-j(k)+1})}{2s}\} \tag{3.36}$$

$$m(x_{m(k)-j(k)},x_{n(k)})=\min\{d(x_{m(k)-j(k)},x_{m(k)-j(k)+1})+d(x_{n(k)},x_{n(k)+1}),$$

$$d(x_{m(k)-j(k)},x_{n(k)+1}),d(x_{n(k)},x_{m(k)+1})\}. \tag{3.37}$$

Taking the upper limit as $k\to\infty$ in (3.36) and (3.37) and using (3.26), (3.31), (3.32) and (3.33), we get

$$\begin{aligned}
\frac{\varepsilon}{s} &\leq \lim_{k\to\infty}\sup d(x_{m(k)-j(k)},x_{n(k)})\\
&\leq \lim_{k\to\infty}\sup M(x_{m(k)-j(k)},x_{n(k)})\\
&= \max\{\lim_{k\to\infty}\sup d(x_{m(k)-j(k)},x_{n(k)}),0,0,\\
&\quad \frac{\lim_{k\to\infty}\sup d(x_{m(k)-j(k)},x_{n(k)+1})+\lim_{i\to\infty}\sup d(x_{n(k)},x_{m(k)-j(k)+1})}{2s}\\
&\leq \max\{sr\varepsilon,\frac{s^2r\varepsilon+s^2r\varepsilon}{2s}\}=sr\varepsilon.
\end{aligned}$$

So, we have

$$\frac{\varepsilon}{s}\leq \lim_{i\to\infty}\sup M(x_{m(k)-j(k)},x_{n(k)})\leq sr\varepsilon \tag{3.38}$$

and

$$\lim_{i\to\infty}\sup m(x_{m(k)-j(k)},x_{n(k)})=0. \tag{3.39}$$

Similarly, we can obtain

$$\frac{\varepsilon}{s}\leq \lim_{i\to\infty}\inf M(x_{m(k)-j(k)},x_{n(k)})\leq sr\varepsilon. \tag{3.40}$$

Now, taking the upper limit as $k\to\infty$ in (3.35) and using (3.34), (3.38) and (3.39), we have

$$\begin{aligned}
\psi(s^4\frac{\varepsilon}{s^2}) &\leq \psi(M(x_{m(k)-j(k)},x_{n(k)})))-\phi(M(x_{m(k)-j(k)},x_{n(k)}))+L\psi(m(x_{m(k)-j(k)},x_{n(k)}))\\
&= \psi((M(x_{m(k)-j(k)},x_{n(k)})))-\phi(M(x_{m(k)-j(k)},x_{n(k)}))\\
&< \psi(M(x_{m(k)-j(k)},x_{n(k)})\\
&\leq \psi(sr\varepsilon),
\end{aligned}$$

a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence in $\cup_{i=1}^{p} A_i$, a subspace of a complete b-metric space (X,d). Therefore there exists $u \in \cup_{i=1}^{p} A_i$ such that

$$\lim_{n\to\infty} x_n = u. \tag{3.41}$$

The sequence $\{x_n\}$ has infinite number of terms in each A_i for all $i \in \{1,2,...,p\}$. Therefore $u \in \cap_{i=1}^{p} A_i$, We shall show that u is a fixed point of T. Again, for any $n \in \mathbb{N}$ we have $\psi(s^4 d(Tx_{n-1}, Tu)$

$$\leq \psi(M(x_{n-1},u) - \phi(M(x_{n-1},u)) + L\psi(m(x_{n-1},u)) \tag{3.42}$$

$$\begin{aligned} m(x_{n-1},u) &= min\{d(x_{n-1},Tx_{n-1}) + d(u,Tu), d(x_{n-1},Tu), d(u,Tx_{n-1}\} \\ &= min\{d(x_{n-1},x_n) + d(u,Tu), d(x_{n-1},Tu), d(u,x_n\}. \end{aligned}$$

Letting $n \to \infty$, we get

$$m(x_{n-1},u) = min\{d(u,u) + d(u,Tu), d(u,Tu), d(u,u\} = 0$$

Also

$$\begin{aligned} M(x_{n-1},u) &= max\{d(x_{n-1},u), d(x_{n-1},Tx_{n-1}), d(u,Tu)\frac{1+d(x_{n-1},Tx_{n-1})}{1+d(x_{n-1},u)}, \\ &\quad \frac{d(x_{n-1},Tu)+d(u,Tx_{n-1})}{2s}\} \\ &= max\{d(x_{n-1},u), d(x_{n-1},x_n), d(u,Tu)\frac{1+d(x_{n-1},x_n)}{1+d(x_{n-1},u)}, \\ &\quad \frac{d(x_{n-1},Tu)+d(u,x_n)}{2s}\}. \end{aligned}$$

Letting $n \to \infty$, we get

$$M(x_{n-1},u) = max\{d(u,u), d(u,u), d(u,Tu), \frac{d(u,Tu)}{2s}\} = d(u,Tu).$$

Letting $n \to \infty$ in (3.42) we have

$$\psi(s^4 d(u,Tu)) \leq \psi(d(u,Tu) - \phi(d(u,Tu)) + 0.$$

This implies

$$\psi(s^4 d(u,Tu)) < \psi(d(u,Tu).$$

This is a contraction, since ψ is a nondecreasing. Hence $d(u,Tu) = 0$, i.e $u = Tu$. Thus u is a fixed point of T. Assume u^* is another fixed point of T, that is, $Tu^* = u^*$. Then $u^* \in \cap_{i=1}^{p} A_i$. Then apply condition

$$\psi(s^4 d(Tx,Ty)) \leq \psi(M(x,y)) - \phi(M(x,y)) + L\psi(m(x,y))$$

for $x = u$ and $y = u^*$. We obtain

$$\psi(d(u,u^*)) \leq \psi(s^4 d(u,u^*)) = \psi(s^4 d(Tu,Tu^*)) \leq \psi(M(u,u^*)) - \phi(M(u,u^*)) + L\psi(m(u,u^*)).$$

Since u and u^* are fixed points of T we can easily show that $M(u,u^*) = d(u,u^*)$ and $m(u^*,u^*) = 0$. If $d(u,u^*) > 0$, we get

$$\psi(d(u,u^*)) \leq \psi(d(u,u^*)) - \phi(d(u,u^*))$$

a contradiction. Thus we have $d(u,u^*) = 0$, that is $u = u^*$. Thus we proved the uniqueness of fixed point theorem.

Putting $s = 1, p = 1$ and $A_1 = X$ in Theorem 3.8, we get the following corollary.

Corollary 3.9. *Let (X,d) be a complete metric space and T is a mapping from X to X such that for all $x,y \in X$ and $L \geq 0$*

$$\psi(d(Tx,Ty)) \leq \psi(M(x,y)) - \phi(M(x,y)) + L\psi(m(x,y)),$$

where ψ and ϕ are altering distance function and

$$M(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty)\frac{1+d(x,Tx)}{1+d(x,y)}, \frac{d(x,Ty)+d(y,Tx)}{2}\}$$
$$m(x,y) = min\{d(x,Tx)+d(y,Ty), d(x,Ty), d(y,Tx))\}.$$

Then T has a unique fixed point in X.

Remark 3.10. Corollary 3.9 is a proper generalization of corresponding results of [18], [19] and [24].

4. Applications

In this section, we apply our results to prove fixed points of certain contractions of integral type. let $\wedge$ be the set of functions $f : [0,\infty) \to [0,\infty)$ such that:

(1) f is Lebesgue Integrable on each compact subset of $[0,\infty)$,

(2) $\int_0^{\varepsilon} f(t)dt > 0$ for every $\varepsilon > 0$.

It is an easy matter to check that the mapping $\psi \to [0,\infty) \to [0,\infty)$ defined by

$$\psi(t) = \int_0^{\varepsilon} f(t)dt > 0$$

is an altering distance function. Therefore, we have the following result.

Theorem 3.11. *Let (X,d) be a complete b-metric space, $A_1, A_2, \cdots, A_p$ be non empty closed subsets of X and $Y = \cup_{i=1}^{p} A_i$. Suppose that $T : Y \to Y$ is a mapping satisfying the following conditions:*

$$\int_0^{s^4 d(Tx,Ty)} f(t)dt \leq \int_0^{M(x,y)} f(t)dt - \int_0^{m(x,y)} g(t)dt + Lm(x,y).$$

For all $x \in A_i, y \in A_{i+1}, i = 1,2,\cdots p, f,g \in \wedge, L \geq 0, A_{p+1} = A_1$. where

$$M(x,y) = max\{(d(x,y), d(x,Tx), d(y,Ty)\frac{1+d(x,Tx)}{1+d(x,y)}, \frac{d(x,Ty)+d(y,Tx)}{2s}\}$$

$$m(x,y) = min\{(d(x,Tx)+d(y,Ty), d(x,Ty), d(y,Tx))\}.$$

Then T has a unique fixed point in X.

Proof. Follows from Theorem 3.8 by taking $\psi(t) = \int_0^t f(u)du$ and $\phi(t) = \int_0^t g(u)du$.

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