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SECOND-ORDER OPTIMALITY CONDITIONS FOR VECTOR EQUILIBRIUM PROBLEMS

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Abstract. In this paper, some relationships between the second-order contingent derivative and the second-order contingent epiderivative are obtained. By virtute of the second-order contingent derivative of single-valued map with a cone has a compact base, we establish some necessary and sufficient optimality conditions of order 2 for weakly efficient, Henig efficient, globally efficient and superefficient solutions to the vector equilibrium problem without constraints. Besides, we also investigate the second-order optimality conditions to the vector equilibrium problem with constraints by using the Frechet differentiable functions whose Frechet derivatives are locally Lipschitz.

Keywords. Second-order optimality conditions; Weakly efficient solution; Globally efficient solution; Henig efficient solution; Superefficient solution.

1. Introduction

The vector equilibrium problem provides an unified mathematical model including vector complementarity problems, vector saddle point problems, vector optimization problems and vector variational inequality problems as special cases. A large number of results for vector equilibrium problem have been investigated consists of existences of solutions [1,2,3,4] and optimality conditions [1, 2, 3, 4, 5, 6,7, 8,9, 10] and references therein.

In this paper, let us consider the following vector equilibrium problem: Let X, Y be Banach spaces, $K \subset X$ a nonempty subset in X, $Q \subset Y$ a closed pointed convex convex in Y. Let F :

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$X \times X \longrightarrow Y$ be mapping, where F be also called vector bifunction. Our problem is find $\bar{x} \in K$ such that

$$F(\bar{x}, y) \notin P \quad \forall y \in K.$$

Our problem is also called VEP. If $intQ \neq \emptyset$ and $P = -intQ$ then $\bar{x} \in K$ is called a weakly efficient solution to the VEP. If $P = -Q \setminus \{0\}$ then $\bar{x} \in K$ is called an efficient solution to the VEP. Next, let us denote Y^* stands for the topological dual space of Y and write Q^+ stands for the dual cone of Q. Let us denote $< .,. >$ the coupling between Y^* and Y and we have $Q^+ = \{\xi \in Y^* \,|\, < \xi, q > \geq 0 \,\forall\, q \in Q\}$. The quasi-interior of Q^+ is denoted as $Q^\sharp = \{\xi \in Y^* \,|\, < \xi, q >> 0 \,\forall\, q \in Q \setminus \{0\}\}$. A nonempty convex subset B of Q is called a base of Q, if $0 \notin cl(B)$ and $Q = \{tb \,|\, b \in B,\, t \geq 0\} := cone(B)$. Where $cone(B)$ is also called the cone hull of B. Setting $Q^\Delta(B) := \{\xi \in Q^\sharp \,|\, \exists t > 0 \; s.t. < \xi, b > \geq t \,\forall\, b \in B\}$ and

$$V_B = \left\{ y \in Y \,|\, | < \xi, y > | < \frac{inf\{< \xi, b >: b \in B\}}{2} \right\} \quad \text{for some } \xi \in Y^* \setminus \{0\}.$$

For each convex neighborhood U of the origin with $U \subset V_B$, one has $cone(U + B)$ is a pointed convex cone, and $Q \setminus \{0\} \subset int[cone(U + B)]$. According to Xun-Hua-Gong [5, 6], we recall some definitions as follows

Definition 1.1. ([5, 6]) A vector $\bar{x} \in K$ is called a globally efficient solution to the VEP if there exists a pointed convex cone $H \subset Y$ with $Q \setminus \{0\} \subset intH$ such that

$$F(\bar{x}, K) \cap -H \setminus \{0\} = \emptyset.$$

Definition 1.2. ([5, 6]) A vector $\bar{x} \in K$ is called a Henig efficient solution to the VEP if there exists some absolutely convex neighborhood U of 0 with $U \subset V_B$ such that

$$cone\, F(\bar{x}, K) \cap -int[cone(U + B)] = \emptyset.$$

Definition 1.3. ([5, 6]) A vector $\bar{x} \in K$ is called a superefficient solution to the VEP if for each neighborhood V of the origin, there exists some neighborhood U of the origin such that

$$cone\, F(\bar{x}, K) \cap (U - Q) \subset V.$$

To establish second-order optimality conditions to the VEP, we shall use to some main tools of the second-order contingent derivatives with a cone Q has a base compact B. And to the VEPC, by using the second-order contingent derivatives with Frechet differentiable functions whose Frechet derivatives are locally Lipschitz.

The remainder of this paper is organized as follows. After some preliminaries and notations, in Section 3 let us provide some necessary and sufficient optimality conditions of order 2 to the VEP related to the second-order contingent epiderivative, the second-order contingent derivative and cone Q with a base compact B. In this section, the relationship between contingent derivative of order 2 of single-valued mapping with efficient points set of contingent epiderivative of order 2 of set-valued mapping are presented. Finally, we give a condition for the existence of the second-order contingent epiderivative. In section 4, the second-order optimality conditions to the VEP for Frchet differentiable functions whose Frchet derivatives locally Lipschitz are established. In section 5, we investigate second-order optimality conditions to the VEPC.

2. Preliminaries

Let X and Y be Banach spaces where Y be partially ordered by a pointed closed convex cone Q and let F be a set-valued mapping defined on X with 2^Y- valued. Let us recall that the effective domain, the graph and the epigraph of F be given as:

$$dom(F) = \{x \in X \,|\, F(x) \neq \emptyset\},$$

$$graph(F) = \{(x,y) \in X \times Y \,|\, y \in F(x)\},$$

$$epi(F) = \{(x,y) \in X \times Y \,:\, x \in dom(F),\ y \in F(x) + Q\},$$

Let M be a subset of Y, $u \in Y$, by cl(M) stands for the topological closure of M and let $\bar{z} \in cl(M)$.

The second-order contingent set $T^2(M,\bar{z},u)$ of M at $(\bar{z},u)$ is defined as

$$T^2(M,\bar{z},u) = \{y \in Y : \exists t_n \longrightarrow 0^+, \exists y_n \longrightarrow y, s.t. \bar{z} + t_n u + t_n^2 y_n \in M \ \forall\, n \geq 1\}.$$

The second-order adjacent set $A^2(M,\bar{z},u)$ of M at $(\bar{z},u)$ is defined as

$$A^2(M,\bar{z},u) = \{y \in Y : \forall t_n \longrightarrow 0^+, \exists y_n \longrightarrow y, s.t. \bar{z} + t_n u + t_n^2 y_n \in M \ \forall\, n \geq 1\}.$$

Let $f : X \longrightarrow Y$ be a single-valued map and let $\bar{x} \in X$ and $(u,v) \in X \times Y$. The second-order contingent derivative of f at $(\bar{x},u,v)$ is the set-valued map $D_c^2 f(\bar{x},u,v)$ from X to 2^Y defined as

$$graph\Big(D_c^2 f(\bar{x},u,v)\Big) = T^2\Big(graph(f),(\bar{x},f(\bar{x})),(u,v)\Big).$$

The second-order contingent epiderivative of f at $(\bar{x},u,v)$ is the single-valued map $\underline{D}^2 f(\bar{x},u,v)$ from X to Y defined as

$$epi\Big(\underline{D}^2 f(\bar{x},u,v)\Big) = T^2\Big(epi\,(f),(\bar{x},f(\bar{x})),(u,v)\Big).$$

Let $x,y \in Y, M \subset Y$ and let us denote $x \geq y$ $(y \leq x)$ stands for $x-y \in Q$.

Definition 2.1. ([1,2,3, 7]) (i) $y \in M$ is an ideal minimal point (respectively ideal maximal point) of M with respect to Q if $a \geq y$ (respectively $y \geq a$) for any $a \in M$. The set of all ideal points is denoted by $IMin(M)$ (respectively $IMax(M)$).

In some other cases, L is a cone, let us denote by $IMin(M|L)$ $(resp., Min(M|L))$ instead of $\{m \in M : M \subset m+L\}$ (resp., $\{m \in M : M \cap (\{m\}-L) = \{m\}\}$).
(ii). $y \in M$ is an efficient point of M with respect to Q if $y \geq a$, for some $a \in M$, implies $a \geq y$. The set of all efficient points of M is denoted by $Min(M)$ and $Max(M)$ is defined similar.
(iii). We say that the domination property holds for the set $A \subset Y$ if $A \subset Min(A)+Q$.

Next we give two defines related to the stable function and the steady function. Following [10], f is called steady at $\bar{x}$ in the direction $v \in X$ (shortly, f is steady $(\bar{x},v)$), if

$$\lim_{(t,u)\longrightarrow(0^+,v)} \frac{f(\bar{x}+tu)-f(\bar{x}+tv)}{t} = 0.$$

Following [10], f is said to be stable at $\bar{x}$ if there exist a constant $L>0$ and a neighborhood U of $\bar{x}$ such that

$$||f(x)-f(\bar{x})|| \leq L||x-\bar{x}|| \quad \text{for every } x \in K \cap U.$$

For $A \subset X, intA, clA, coneA$ stand for the interior, closure and conical hull of A, respectively (shortly, resp.). Let us denote by $t_n \longrightarrow 0^+$ instead of a sequence of positive numbers with limit 0 and denote $F(\bar{x},K) = \underset{y\in K}{\cup} F(\bar{x},y)$.

3. Second-order optimality conditions to the VEP

In this section, we turn to Problem VEP. Sufficient and necessary optimality conditions of order 2 for weakly efficient solution, Henig efficient solution, globally efficient solution and superefficient solution can be stated as follows. Note that, from now on, unless otherwise specify we always assume that X and Y are Banach spaces and Q is a pointed closed convex cone in Y with its interior nonempty, $\bar{x} \in K$ and $F(\bar{x},\bar{x}) = 0$. Let us denote by $f(x)$ stands for $F(\bar{x},x)$ and $(f+Q)(x)$ stands for $f(x)+Q$ for any $x \in X$.

Proposition 3.1. *Let $\bar{x} \in K$ to the VEP and let $(u,v) \in X \times Y$. Then, if $\underline{D}^2 f(\bar{x},u,v)(x)$ exists then $IMin\Big(D_c^2(f+Q)(\bar{x},u,v)(x)\Big) \neq \emptyset$ and*

$$\underline{D}^2 f(\bar{x},u,v)(x) = IMin\Big(D_c^2(f+Q)(\bar{x},u,v)(x)\Big)$$

for any $x \in dom\Big(D_c^2(f+Q)(\bar{x},u,v)\Big)$.

Proof. For every $x \in dom\Big(D_c^2(f+Q)(\bar{x},u,v)\Big)$, we must show that

$$\underline{D}^2 f(\bar{x},u,v)(x) \in D_c^2(f+Q)(\bar{x},u,v)(x) \tag{3.1}$$

$$\text{and} \quad D_c^2(f+Q)(\bar{x},u,v)(x) \subset \underline{D}^2 f(\bar{x},u,v)(x) + Q. \tag{3.2}$$

Indeed, take any $y \in \underline{D}^2 f(\bar{x},u,v)(x)$ then $(x,y) \in epi\Big(\underline{D}^2 f(\bar{x},u,v)\Big)$. By the definition of contingent epiderivative of order 2, it yields that

$$(x,y) \in T^2\Big(epi(f),(\bar{x},f(\bar{x})),(u,v)\Big).$$

This is equivalent to there exist $t_n \longrightarrow 0^+$ and $(x_n,y_n) \longrightarrow (x,y)$ such that

$$(\bar{x},f(\bar{x})) + t_n(u,v) + t_n^2(x_n,y_n) \in epi(f) = graph(f+Q).$$

Where $(t_n)_{n\geq 1} \subset \{t \in \mathbb{R} \mid t > 0\}$ and $(x_n,y_n)_{n\geq 1} \subset X \times Y$.
Thus, $(x,y) \in graph(D_c^2(f+Q)(\bar{x},u,v))$. Consequently, $y \in D_c^2(f+Q)(\bar{x},u,v)(x)$ and the condition (3.1) is proven. On the other hand, for all y belong to the set $D_c^2(f+Q)(\bar{x},u,v)(x)$. By an argument similar as above, with note that $graph(f+Q) = epi(f)$, one obtains as follows

$$(x,y) \in T^2\Big(graph(f+Q),(\bar{x},f(\bar{x}),(u,v))\Big) = T^2\Big(epi(f),(\bar{x},f(\bar{x}),(u,v))\Big) = epi\Big(\underline{D}^2 f(\bar{x},u,v)\Big).$$

A consequence is $y \in \underline{D}^2 f(\bar{x},u,v)(x) + Q$. Thus, the condition (3.2) is proven. The proof is complete.

Remark 3.1. For every $(x,y) \in X \times Y$. One has $(x,y) \in epi(f) \Longleftrightarrow y \in f(x)+Q = (f+Q)(x) \Longleftrightarrow (x,y) \in graph(f+Q)$. This implies that $epi(f) = graph(f+Q)$.

Proposition 3.2. *Let $\bar{x} \in K$ to the VEP and let $(u,v) \in X \times Y$. Let us assume that Q has a compact base B. Then, if $\underline{D}^2 f(\bar{x},u,v)(x)$ exists then*

$$\underline{D}^2 f(\bar{x},u,v)(x) \in D_c^2 f(\bar{x},u,v)(x)$$

for any $x \in dom\Big(D_c^2(f+Q)(\bar{x},u,v)\Big)$.

Proof. According to Proposition 3.1, it follows that

$$IMin\Big(D_c^2(f+Q)(\bar{x},u,v)(x)\Big) \neq \emptyset.$$

Due to Luc [7, Proposition 2.2],

$$IMin\Big(D_c^2(f+Q)(\bar{x},u,v)(x)\Big) = Min\Big(D_c^2(f+Q)(\bar{x},u,v)(x)\Big).$$

By Wang and Li [4, Proposition 3.6],

$$Min\Big(D_c^2(f+Q)(\bar{x},u,v)(x)\Big) \subset D_c^2 f(\bar{x},u,v)(x).$$

Therefore, by Proposition 3.1, we conclude that $\underline{D}^2 f(\bar{x},u,v)(x) \in D_c^2 f(\bar{x},u,v)(x)$ for any $x \in dom\Big(D_c^2(f+Q)(\bar{x},u,v)\Big)$. The proof is complete.

Proposition 3.3. *Let $\bar{x} \in K$ to the VEP and let $(u,v) \in X \times Y$. Let us assume that Q has a compact base B. Then, if $\underline{D}^2 f(\bar{x},u,v)(x)$ exists then*

$$\underline{D}^2 f(\bar{x},u,v)(x) = IMin\Big(D_c^2 f(\bar{x},u,v)(x)\Big)$$

for any $x \in dom\Big(D_c^2(f+Q)(\bar{x},u,v)\Big)$.

Proof. We fixed an $x \in dom\Big(D_c^2(f+Q)(\bar{x},u,v)\Big)$. By the definition of ideal minimal point, we must show that

$$\underline{D}^2 f(\bar{x},u,v)(x) \in D_c^2 f(\bar{x},u,v)(x) \subset \underline{D}^2 f(\bar{x},u,v)(x) + Q.$$

The relation $\underline{D}^2 f(\bar{x},u,v)(x) \in D_c^2 f(\bar{x},u,v)(x)$ is obvious (can see, Proposition 3.2). Now, for any $y \in D_c^2 f(\bar{x},u,v)(x)$, which is equivalent to

$$(x,y) \in graph\Big(D_c^2 f(\bar{x},u,v)\Big) = T^2(graph(f),(\bar{x},f(\bar{x}),(u,v))).$$

As $graph(f) \subset epi(f)$, hence

$$T^2(graph(f),(\bar{x},f(\bar{x}),(u,v))) \subset T^2(epi(f),(\bar{x},f(\bar{x}),(u,v))).$$

Consequently,

$$(x,y) \in T^2(epi(f), (\bar{x}, f(\bar{x}), (u,v))) = epi\Big(\underline{D}^2 f(\bar{x},u,v)\Big).$$

Thus,

$$y \in \underline{D}^2 f(\bar{x},u,v)(x) + Q.$$

The proof is complete.

Proposition 3.4. *Under the assumptions of Proposition 3.3 and in addition the cone Q has a compact base B. Then for any $x \in dom\Big(D_c^2(f+Q)(\bar{x},u,v)\Big)$ and if $\underline{D}^2 f(\bar{x},u,v)(x)$ exists then*

$$IMin\Big(D_c^2 f(\bar{x},u,v)(x)\Big) = IMin\Big(D_c^2(f+Q)(\bar{x},u,v)(x)\Big) = Min\Big(D_c^2 f(\bar{x},u,v)(x)\Big)$$
$$= Min\Big(D_c^2(f+Q)(\bar{x},u,v)(x)\Big).$$

Furthermore, $D_c^2(f+Q)(\bar{x},u,v)(x)$ satisfying the domination property.

Proof. It is a direct consequence from Propositions 3.1 and 3.3. Furthermore, by the inclusion (3.2) in Proposition 3.1, it follows that $D_c^2(f+Q)(\bar{x},u,v)(x)$ has the domination property and the proof is complete.

Theorem 3.1. *Consider problem VEP and assume that $f : X \longrightarrow Y$ is steady at $\bar{x}$ in all the directions $w \in X$. Then if $\bar{x} \in K$ is a weakly efficient solution to the VEP then for any $(u,v) \in X \times (-Q)$,*

$$D_c^2 f(\bar{x},u,v)(x) \cap -intQ = \emptyset \quad \forall x \in A^2(K,\bar{x},u).$$

Proof. We suppose to the contrary that, for each $(u,v) \in X \times (-Q)$ there exists $x \in A^2(K,\bar{x},u)$ such that

$$D_c^2 f(\bar{x},u,v)(x) \cap -intQ \neq \emptyset.$$

From there we take $y \in D_c^2 f(\bar{x},u,v)(x)$ such that $y \in -intQ$. We have

$$(x,y) \in graph\Big(D_c^2 f(\bar{x},u,v)\Big) = T^2(graph(f), (\bar{x}, f(\bar{x})), (u,v)).$$

Therefore, there exist $t_n \longrightarrow 0^+$ and $(x_n,y_n) \longrightarrow (x,y)$ such that

$$(\bar{x}, f(\bar{x})) + t_n(u,v) + t_n^2(x_n,y_n) \in graph(f) \quad \forall\, n \in \mathbb{N}.$$

It follows that

$$f(\bar{x})+t_n^2 y_n \in (f+Q)(\bar{x}+t_n u+t_n^2 x_n) \quad \forall\, n\in\mathbb{N}.$$

Consequently,

$$y_n \in \frac{f(\bar{x}+t_n u+t_n^2 x_n)-f(\bar{x})}{t_n^2}+Q \qquad \forall\, n\in\mathbb{N}.$$

Letting $n\longrightarrow\infty$, we obtain

$$y\in \lim_{n\longrightarrow\infty}\frac{f(\bar{x}+t_n u+t_n^2 x_n)-f(\bar{x})}{t_n^2}+Q \quad \forall n\geq 1.$$

On the other hand, as $x\in A^2(K,\bar{x},u)$, there exists $x'_n\longrightarrow x$ satisfying $\bar{x}+t_n u+t_n^2 x'_n\in K$ for all $n\geq 1$. As f is steady at $\bar{x}$ in all the directions $w\in X$, hence f is stable at $\bar{x}$ (see, [10]). So that,

$$\lim_{n\longrightarrow\infty}\frac{f(\bar{x}+t_n u+t_n^2 x'_n)-f(\bar{x})}{t_n^2}=\lim_{n\longrightarrow\infty}\frac{f(\bar{x}+t_n u+t_n^2 x'_n)-f(\bar{x}+t_n u+t_n^2 x)}{t_n^2}$$

$$+\lim_{n\longrightarrow\infty}\frac{f(\bar{x}+t_n u+t_n^2 x)-f(\bar{x}+t_n u+t_n^2 x_n)}{t_n^2}+\lim_{n\longrightarrow\infty}\frac{f(\bar{x}+t_n u+t_n^2 x_n)-f(\bar{x})}{t_n^2}$$

$$=\lim_{n\longrightarrow\infty}\frac{f(\bar{x}+t_n u+t_n^2 x_n)-f(\bar{x})}{t_n^2}.$$

In fact, as f is stable at $\bar{x}\in K$, there exists a positive real number L such that

$$\left|\left|\frac{f(\bar{x}+t_n u+t_n^2 x'_n)-f(\bar{x}+t_n u+t_n^2 x)}{t_n^2}\right|\right|_Y \leq L||x'_n-x||_X\longrightarrow 0,\ as\ n\longrightarrow+\infty$$

and

$$\left|\left|\frac{f(\bar{x}+t_n u+t_n^2 x)-f(\bar{x}+t_n u+t_n^2 x_n)}{t_n^2}\right|\right|_Y \leq L||x_n-x||_X\longrightarrow 0,\ as\ n\longrightarrow+\infty.$$

Where $||.||_Z$ denotes stands for a norm in space Z.

Consequently,

$$y\in \lim_{n\longrightarrow\infty}\frac{f(\bar{x}+t_n u+t_n^2 x'_n)-f(\bar{x})}{t_n^2}+Q \quad \forall n\geq 1.$$

Since $y\in -intQ$ and $intQ+Q=intQ$, thus for n large enough

$$\frac{f(\bar{x}+t_n u+t_n^2 x'_n)-f(\bar{x})}{t_n^2}\in -intQ.$$

This yields

$$f(\bar{x}+t_n u+t_n^2 x'_n)\in -intQ \qquad \text{for n large enough.}$$

This conflicts with the fact that $\bar{x}\in K$ is a weakly efficient solution to the VEP. The proof is complete.

Theorem 3.2. *Consider problem VEP with Q has a compact base, $f : X \longrightarrow Y$ is steady at $\bar{x}$ in all the directions $w \in X$ and $(u,v) \in X \times (-Q)$. Assume that $\underline{D}^2 f(\bar{x},u,v)(x)$ exists for all $x \in dom\Big(D_c^2(f+Q)(\bar{x},u,v)\Big)$. Then if $\bar{x} \in K$ is a weakly efficient solution to the VEP then $\forall$ $x \in A^2(K,\bar{x},u) \cap dom\Big(D_c^2(f+Q)(\bar{x},u,v)\Big)$ there exists $\xi \in Y^* \setminus \{0\}$ such that*

$$\xi \in Q^+, \tag{3.3}$$

$$0 \le < \xi, \underline{D}^2 f(\bar{x},u,v)(x) > . \tag{3.4}$$

Proof. Fixed $x \in A^2(K,\bar{x},u) \cap dom\Big(D_c^2(f+Q)(\bar{x},u,v)\Big)$. According to Proposition 3.3 and Theorem 3.1, it yields that

$$\underline{D}^2 f(\bar{x},u,v)(x) \notin -intQ.$$

By the separation theorem of convex sets, there exists $\xi \in Y^* \setminus \{0\}$ such that

$$< \xi, \underline{D}^2 f(\bar{x},u,v)(x) > + < \xi, q >> 0 \quad \forall\, q \in intQ.$$

As intQ is a cone, hence

$$< \xi, \underline{D}^2 f(\bar{x},u,v)(x) > +t < \xi, q >> 0 \quad \forall\, q \in intQ, \forall t > 0. \tag{3.5}$$

By letting $t \longrightarrow 0^+$, one obtains (3.4)

$$< \xi, \underline{D}^2 f(\bar{x},u,v)(x) > \ge 0.$$

By dividing up two sides of (3.5) with $t > 0$,

$$\frac{< \xi, \underline{D}^2 f(\bar{x},u,v)(x) >}{t} + < \xi, q >> 0 \quad \forall\, q \in intQ, \forall t > 0.$$

Letting $t \longrightarrow \infty$,

$$< \xi, q > \ge 0 \quad \forall q \in cl(intQ) = Q,$$

as $Q = cl(Q)$ and ξ is a continuous function on Y. Therefore, $\xi \in Q^+$, that is, (3.3) is satisfied. The proof is complete.

Theorem 3.3. *Consider problem VEP with Q has a compact base B, $f : X \longrightarrow Y$ is steady at $\bar{x}$ in all the directions $w \in X$ and $(u,v) \in X \times (-Q)$. Assume that $\underline{D}^2 f(\bar{x},u,v)(x)$ exists for all $x \in dom\Big(D_c^2(f+Q)(\bar{x},u,v)\Big)$ and $f(K) \subset D_c^2 f(\bar{x},u,v)(x) + Q$. Then $\bar{x} \in K$ is a weakly efficient* *solution to the VEP if and only if* $\forall\ x \in A^2(K,\bar{x},u) \cap dom\Big(D_c^2(f+Q)(\bar{x},u,v)\Big)$ *there exists* $\xi \in Y^* \setminus \{0\}$ *such that*

$$\xi \in Q^+, \tag{3.6}$$

$$0 \leq < \xi, \underline{D}^2 f(\bar{x},u,v)(x) > \leq < \xi, f(y) > \qquad \forall\, y \in K. \tag{3.7}$$

Proof. Firstly, we suppose that $\bar{x} \in K$ is a weakly efficient solution to the VEP. Due to Theorem 3.2, for each $x \in A^2(K,\bar{x},u) \cap dom\Big(D_c^2(f+Q)(\bar{x},u,v)\Big)$ there exists $\xi \in Y^* \setminus \{0\}$ such that the conditions (3.3) and (3.4) are satisfied. The following conclusion can be directly obtained similarly as the proof of [4, Proposition 2.1]

$$D_c^2 f(\bar{x},u,v)(x) + Q \subset D_c^2(f+Q)(\bar{x},u,v)(x).$$

Thus, by hypotheses,

$$f(K) \subset D_c^2(f+Q)(\bar{x},u,v)(x).$$

Let us next consider the following variational system

$$(P_x)\ \begin{cases} \text{Find } v \in Y \text{ such that} \\ < \lambda, v > = inf\{< \lambda, z >: z \in D_c^2(f+Q)(\bar{x},u,v)(x)\} \text{ for any } \lambda \in Q^+. \end{cases}$$

Due to Proposition 3.1, $\underline{D}^2 f(\bar{x},u,v)(x) = IMin\Big(D_c^2(f+Q)(\bar{x},u,v)(x)\Big)$. It is easy to check that $\underline{D}^2 f(\bar{x},u,v)(x)$ is the solution of (P_x), this means that

$$< \lambda, \underline{D}^2 f(\bar{x},u,v)(x) > = inf\{< \lambda, z >: z \in D_c^2(f+Q)(\bar{x},u,v)(x)\}$$

$$\leq < \lambda, F(\bar{x},y) > \quad \forall y \in K,\ \forall\, \lambda \in Q^+.$$

By setting $\lambda = \xi$, this leads to the relations in (3.7) are satisfied. Finally, we suppose that $\bar{x} \in K$ is a feasible point to the VEP such that the converse cases are satisfied. We prove that $\bar{x} \in K$ is a weakly efficient solution to the VEP. In fact, if it were not so, there would exists $y \in K$ such that

$$-q = -F(\bar{x},y) \in intQ.$$

Since $\xi \in Q^+$ and $\xi \neq \emptyset$, hence

$$< \xi, \underline{D} f(\bar{x})(\bar{u}) > - < \xi, q > \geq < \xi, -q > > 0.$$

This conflicts with the origin hypotheses. The proof is complete.

Theorem 3.4. *Under the assumptions of Theorem 3.3. Then $\bar{x} \in K$ is a Henig efficient solution to the VEP if and only if there exists $\xi \in Q^{\Delta}(B)$ such that for any $x \in A^2(K,\bar{x},u) \cap dom\Big(D_c^2(f+Q)(\bar{x},u,v)\Big)$, we have*

$$0 \leq <\xi, \underline{D}^2 f(\bar{x},u,v)(x)> \leq <\xi, f(y)> \qquad \forall\, y \in K.$$

Proof. We firstly suppose that $\bar{x} \in K$ is a Henig efficient solution to the VEP, that is there exists some neighborhood U of the origin with $U \subset V_B$ and such that

$$tF(\bar{x},y) \notin -int[cone(U+B)] \quad \forall\, y \in K,\ \forall\, t > 0,$$

which is equivalent to

$$F(\bar{x},K) \cap -int[cone(U+B)] = \emptyset.$$

Now we fixed $x \in A^2(K,\bar{x},u) \cap dom\Big(D_c^2(f+Q)(\bar{x},u,v)\Big)$. In view of Proposition 3.1, one gets $\underline{D}^2 f(\bar{x},u,v)(x) = IMin\Big(D_c^2(f+Q)(\bar{x},u,v)(x)\Big)$. As $Q \subset cone(U+B)$ and these cones are pointed, thus

$$IMin\Big(D_c^2(f+Q)(\bar{x},u,v)(x)\Big) = IMin\Big(D_c^2(f+Q)(\bar{x},u,v)(x)|cone(U+B)\Big).$$

Consequently,

$$\underline{D}^2 f(\bar{x},u,v)(x) = IMin\Big(D_c^2(f+Q)(\bar{x},u,v)(x)|cone(U+B)\Big).$$

Let us next consider the following variational system

$$(P_{x'}) \quad \begin{cases} \text{Find } v \in Y \text{ such that} \\ <\xi, v> = inf\{<\xi, z>: z \in D_c^2(f+Q)(\bar{x},u,v)(x)\} \\ \text{for any } \xi \in [cone(U+B)]^+. \end{cases}$$

Obviously, $\underline{D}^2 f(\bar{x},u,v)(x)$ is the solution of $(P_{x'})$. This means that,

$$<\xi, \underline{D}^2 f(\bar{x},u,v)(x)> = inf\{<\xi, z>: z \in D_c^2(f+Q)(\bar{x},u,v)(x)\}.$$

$$\forall y \in K,\ \forall\, \xi \in [cone(U+B)]^+.$$

As by Wang and Li [4, Proposition 3.4],

$$f(K) \subset D_c^2(f+Q)(\bar{x},u,v)(x).$$

Therefore, $\forall y \in K$, $\forall\, \xi \in [cone(U+B)]^+$,

$$< \xi, \underline{D}^2 f(\bar{x},u,v)(x) > \leq < \xi, F(\bar{x},y) >$$

Similarly as in the proof of Theorem 3.1, one gets

$$\underline{D}^2 f(\bar{x},u,v)(x) \notin -int[cone(U+B)].$$

By using the separation theorem of convex sets $\{\underline{D}^2 f(\bar{x},u,v)(x)\}$ and $-int[cone(U+B)]$, there exists $\xi \in Y^* \setminus \{0\}$ satisfying

$$\xi \in [cone(U+B)]^+$$

and

$$0 \leq < \xi, \underline{D}^2 f(\bar{x},u,v)(x) > .$$

Therefore,

$$0 \leq < \xi, \underline{D}^2 f(\bar{x},u,v)(x) > \leq < \xi, F(\bar{x},y) > \ \forall\, y \in K.$$

By Xun-Hua-Gong [5, Lemma 2.1],

$$[cone(U+B)]^+ \setminus \{0\} \subset Q^\Delta(B).$$

Consequently, $\xi \in Q^\Delta(B)$. In the converse case, we may assume that there exists $\xi \in Q^\Delta(B)$ such that for any $x \in Dom\Big(D_c^2(f+Q)(\bar{x},u,v)\Big) \cap A^2(K,\bar{x},u)$, one gets

$$0 \leq < \xi, \underline{D}^2 f(\bar{x},u,v)(x) > \leq < \xi, F(\bar{x},y) > \qquad \forall\, y \in K.$$

Since $\xi \in Q^\Delta(B)$, there exists $t > 0$ such that

$$< \xi, b > \geq t \quad \forall b \in B.$$

Setting

$$V = \{y \in Y \,|\, | < \xi, y > | < \frac{t}{2}\}.$$

We have V is a neighborhood of the origin in Y, as $\xi \in Y^*$. Thus we can take an absolutely convex neighborhood U of the origin with $U \subset V$. By an argument similar as in the proof of Xun-Hua-Gong [6, Theorem 3.2], it follows that

$$cone\Big(F(\bar{x},K)\Big) \cap \Big(U - B\Big) = \emptyset.$$

Therefore, $\bar{x} \in K$ is a Henig efficient solution to the VEP. The proof is complete.

Theorem 3.5. *Under the assumptions of Theorem 3.3. Then $\bar{x} \in K$ is a globally efficient solution to the VEP if and only if there exists $\xi \in Q^\sharp$ such that for any $x \in dom\Big(D^2_c(f+Q)(\bar{x},u,v)\Big) \cap A^2(K,\bar{x},u)$, we have*

$$0 \leq < \xi, \underline{D}^2 f(\bar{x},u,v)(x) > \leq < \xi, F(\bar{x},y) > \qquad \forall\, y \in K.$$

Proof. We firstly suppose that $\bar{x} \in K$ is a globally efficient solution to the VEP, that is there exists a pointed convex cone $H \subset Y$ with $Q \setminus \{0\} \subset intH$ such that

$$F(\bar{x},y) \notin -H \setminus \{0\} \quad \forall\, y \in K,$$

which is equivalent to

$$F(\bar{x},y) \notin -intH \quad \forall\, y \in K.$$

We fixed $x \in dom\Big(D^2_c(f+Q)(\bar{x},u,v)\Big) \cap A^2(K,\bar{x},u)$. In view of Proposition 3.1, and by an argument similar as in the proof of Theorem 3.4 above, we also have $\underline{D}^2 f(\bar{x},u,v)(x) = IMin\Big(D^2_c(f+Q)(\bar{x},u,v)(x)|H\Big)$. Let us next consider the following variational system

$$(P_{x''}) \quad \begin{cases} \text{Find } v \in Y \text{ such that} \\ < \xi, v >= inf\{< \xi, z >: z \in D^2_c(f+Q)(\bar{x},u,v)(x)\} \text{ for any } \xi \in H^+. \end{cases}$$

By a direct computation, we are easily see that $\underline{D}^2 f(\bar{x},u,v)(x)$ is the solution of $(P_{x''})$. It means that,

$$< \xi, \underline{D}^2 f(\bar{x},u,v)(x) >= inf\{< \xi, z >: z \in D^2_c(f+Q)(\bar{x},u,v)(x)\} \leq < \xi, F(\bar{x},y) >$$

$$\forall y \in K,\ \forall\, \xi \in H^+.$$

By an argument similar as in the proof of Theorem 3.1, one gets

$$\underline{D}f(\bar{x})(u) \notin -intH.$$

By using the separation theorem of convex sets $\{\underline{D}f(\bar{x})(u)\}$ and $-intH$, there exists $\xi \in Y^* \setminus \{0\}$ satisfying

$$\xi \in H^+ \setminus \{0\}$$

and

$$0 \leq < \xi, \underline{D}^2 f(\bar{x}, u, v)(x) > .$$

Therefore, $0 \leq < \xi, \underline{D}^2 f(\bar{x}, u, v)(x) > \leq < \xi, F(\bar{x}, y) > \ \forall y \in K$. For all $q \in Q \setminus \{0\}$ implies $q \in intH$. As $\xi \neq 0, \xi \in H^+$, one gets $< \xi, q >> 0$. By the definition of quasi-interior of Q^+, $\xi \in Q^\sharp$. Conversely, assume that there exists $\xi \in Q^\sharp$ such that for any $u \in dom\Big(D_c^2(f+Q)(\bar{x}, u, v)\Big) \cap A^2(K, \bar{x}, u)$,

$$0 \leq < \xi, \underline{D}^2 f(\bar{x}, u, v)(x) > \leq < \xi, F(\bar{x}, y) > \qquad \forall y \in K.$$

Setting

$$H_1 = \{y \in Y \mid < \xi, y >< 0\} \cup \{0\}.$$

By Xun-Hua-Gong [5] in the proof of Theorem 3.3, $H := -H_1$ is pointed convex cone and $Q \setminus \{0\} \subset intH$. Since $\xi \in Q^\sharp$, thus for any $y \in Q \setminus \{0\}$, $< \xi, y >> 0$. Now we prove that

$$F(\bar{x}, y) \notin -H \setminus \{0\} \quad \forall y \in K.$$

In the converse case, we may assume that there exists $y \in K$ such that $F(\bar{x}, y) \in -H \setminus \{0\}$. We have

$$< \xi, F(\bar{x}, y) >< 0$$

and this is a contradiction. Therefore, $\bar{x} \in K$ is a globally efficient solution to the VEP. The proof is complete.

Theorem 3.6. *Under the assumptions of Theorem 3.3. Then $\bar{x} \in K$ is a superefficient solution to the VEP if and only if there exists $\xi \in int(Q^+)$ such that for any $u \in dom\Big(D_c^2(f+Q)(\bar{x}, u, v)\Big) \cap A^2(K, \bar{x}, u)$, we have*

$$0 \leq < \xi, \underline{D}^2 f(\bar{x}, u, v)(x) > \leq < \xi, F(\bar{x}, y) > \quad \forall y \in K.$$

Proof. Since B is compact set, hence B is a bounded closed base of Q. In view of Remark 4.2 [5], one gets $int(Q^+) = Q^\Delta(B)$. Furthermore, $\bar{x} \in K$ is superefficient solution to the VEP if and only if it is also Henig efficient solution to the VEP. The proof is complete.

Theorem 3.7. *Under the assumptions of Theorem 3.2. Then, if $\bar{x} \in K$ is a weakly efficient (respectively, Henig efficient, globally efficient, superefficient etc) solution to the VEP then there exists ξ belongs to $Q^+ \setminus \{0\}$ (respectively, $Q^\Delta(B), \quad Q^\sharp, \quad int(Q^+)$) such that for any x belong to the set $dom\Big(D^2_c(f+Q)(\bar{x},u,v)\Big) \cap A^2(K,\bar{x},u)$, we have*

$$0 \leq < \xi, \underline{D}^2 f(\bar{x},u,v)(x) > .$$

Moreover, if $F(\bar{x},K)$ is a subset of $D^2_c f(\bar{x},u,v)(x)+Q$ then

$$< \xi, \underline{D}^2 f(\bar{x},u,v)(x) > \leq < \xi, F(\bar{x},y) > \quad \forall\, y \in K.$$

In this case, the converse case still holds.

Proof. It is a directly consequence from some Theorems 3.2, 3.4, 3.5 and 3.6. We omit it.

To close this part, we provide a condition for the existence of $\underline{D}^2 f(\bar{x},u,v)$ where $\bar{x} \in K$, $(u,v) \in X \times Y$.

Theorem 3.8. *Let $(\bar{x},u,v) \in K \times X \times Y$ and let $f : K \longrightarrow Y$ be steady at $\bar{x}$ in all the directions $v \in X$. Then we have an assertion as follows*
$\underline{D}^2 f(\bar{x},u,v)(x)$ exists if and only if $IMin\Big(D^2_c f(\bar{x},u,v)(x)\Big) \neq \emptyset$ for all $x \in dom(D^2_c f(\bar{x},u,v))$. Moreover,

$$dom(\underline{D}^2 f(\bar{x},u,v)) = dom(D^2_c f(\bar{x},u,v)).$$

Proof. By applying Proposition 3.3. The proof is complete.

4. Frechet differentiable case

In this section, turning back Problem VEP. We fixed some assumptions and notations as in section 2 and in addition the condition $F(\bar{x}, .)$ is Frechet differentiable at $\bar{x}$, its Frechet derivative at $\bar{x}$ (shortly, $\triangledown F(\bar{x},\bar{x})$) is locally Lipchitz. Following [10], if there exists a neighborhood U of

$\overline{x}$ and $L>0$ such that $||F(\overline{x},x)-F(\overline{x},x')||\leq L||x-x'||$ for all $x,x'\in U$ then $F(\overline{x},.)$ is said to be that Lipschitz around $\overline{x}$. If for each $\overline{x}\in X$ there exists a neighborhood U of $\overline{x}$ such that $F(\overline{x},.)$ is Lipschitz around $\overline{x}$, then $F(\overline{x},.)$ is said to be that locally Lipschitz on X. Clearly, if $F(\overline{x},.)$ is locally Lipschitz on X then $F(\overline{x},.)$ is stable at for each $\overline{x}\in K$ (can see [10, page 452]). We denote

$$M_k(u,u)=\lim_{t\longrightarrow 0}\frac{k(\overline{x}+tu)-k(\overline{x})-t<\nabla k(\overline{x}),u>}{t^2}\quad (\forall u\in X)$$

and
$$N(S,\overline{z})=-(T(S,\overline{z}))^{+}.$$

Where the mapping $k:X\longrightarrow Y$ and the contingent cone $T(S,\overline{z})$ of S at $\overline{z}\in cl(S)$ is

$$T(S,\overline{z})=\{y\in Y:\exists t_n\longrightarrow 0^{+},\exists y_n\longrightarrow y\text{ such that }\overline{z}+t_ny_n\in S\ \forall\, n\geq 1\}.$$

Lemma 4.1. *Let $f:X\longrightarrow Y$ be Frechet differentiable at $\overline{x}\in X$ whose Frechet derivative at $\overline{x}$ be locally Lipschitz and let $\{t_n\}_{n\geq 1}$ be a positive real numbers sequence with limit 0 and $\{v_n\}_{n\geq 1}\subset X$ with $\lim_{n\longrightarrow +\infty} v_n=v$. Then, if $\lim_{n\longrightarrow\infty}\frac{f(\overline{x}+t_nu+t_n^2v_n)-f(\overline{x})}{t_n^2}=y$ then for any sequence $\{v'_n\}_{n\geq 1}\subset X$ with $\lim_{n\longrightarrow +\infty} v'_n=v$, we have*

$$\lim_{n\longrightarrow\infty}\frac{f(\overline{x}+t_nu+t_n^2v'_n)-f(\overline{x})}{t_n^2}=y.$$

Proof. By hypotheses, $f:X\longrightarrow Y$ is Frechet differentiable at $\overline{x}\in X$ hence there would exists a locally Lipschitz mapping $\nabla f(\overline{x})$ from X into Y such that

$$f(\overline{x}+ts)=f(\overline{x})+t\nabla f(\overline{x})(s)+o(t)\quad (\forall s\in X),$$

where $||o(t)||/|t|\longrightarrow 0$ as $t\longrightarrow 0$. Taking $t=t_n$ and $s=u+t_nv_n$, it follows that $f(\overline{x}+t_nu+t_n^2v_n)=f(\overline{x})+t_n\nabla f(\overline{x})(u+t_nv_n)+o(t_n)$, where $||o(t_n)||/t_n\longrightarrow 0$ as $n\longrightarrow\infty$. On the other hand, if $s=u+t_nv'_n$, then

$$f(\overline{x}+t_nu+t_n^2v'_n)=f(\overline{x})+t_n\nabla f(\overline{x})(u+t_nv'_n)+o(t_n).$$

We have a representation as follows

$$\begin{aligned}f(\overline{x}+t_nu+t_n^2v_n)-f(\overline{x})&=f(\overline{x}+t_nu+t_n^2v'_n)-f(\overline{x})\\&\quad+t_n\Big(\nabla f(\overline{x})(u+t_nv_n)-\nabla f(\overline{x})(u+t_nv'_n)\Big)\quad(\forall n\geq 1),\end{aligned}$$

which is equivalent to

$$\frac{f(\bar{x}+t_nu+t_n^2v_n)-f(\bar{x})}{t_n^2} = \frac{f(\bar{x}+t_nu+t_n^2v'_n)-f(\bar{x})}{t_n^2} + \frac{\nabla f(\bar{x})(u+t_nv_n)-\nabla f(\bar{x})(u+t_nv'_n)}{t_n} \quad (\forall n \geq 1).$$

By hypotheses, $\nabla f(\bar{x})$ is locally Lipschitz, there would exist a neighborhood U of u and a positive real number $L>0$ such that

$$||\nabla f(\bar{x})(a)-\nabla f(\bar{x})(b)|| \leq L||a-b|| \quad \forall\, a,b \in U.$$

Taking $N>0$ such that $u+t_nv_n, u+t_nv'_n \in U \quad \forall\, n>N$. Then for each $n>N$,

$$\Big|\Big|\frac{\nabla f(\bar{x})(u+t_nv_n)-\nabla f(\bar{x})(u+t_nv'_n)}{t_n}\Big|\Big| \leq L||v_n-v'_n|| \leq L||v_n-v||+L||v'_n-v||.$$

As $||v'_n-v_n|| \leq ||v_n-v||+||v'_n-v|| \longrightarrow 0$ as $n \longrightarrow \infty$. From there we conclude that

$$\lim_{n\longrightarrow+\infty}\Big|\Big|\frac{\nabla f(\bar{x})(u+t_nv_n)-\nabla f(\bar{x})(u+t_nv'_n)}{t_n}\Big|\Big| = 0.$$

Therefore

$$\lim_{n\longrightarrow\infty}\frac{f(\bar{x}+t_nu+t_n^2v_n)-f(\bar{x})}{t_n^2} = \lim_{n\longrightarrow\infty}\frac{f(\bar{x}+t_nu+t_n^2v'_n)-f(\bar{x})}{t_n^2} = y.$$

The proof is complete.

Proposition 4.1. *Let $\bar{x} \in K,\ (u,v) \in X \times (-Q)$. Assume that one of the following two conditions are satisfied*

(i). There exists $x \in A^2(K,\bar{x},u)$ such that $D_c^2F(\bar{x},\bar{x},u,v)(x) \cap -intQ \neq \emptyset$.

(ii). There exists $u \in A(K,\bar{x})$ such that $\nabla F(\bar{x},\bar{x})(u) \in -intQ$.

Then, $\bar{x} \in K$ is not weakly efficient solution to the VEP.

Proof. For case (i) holds: Assume that, there exist $x \in A^2(K,\bar{x},u)$ and $y \in D_c^2F(\bar{x},\bar{x},u,v)(x)$ such that $y \in -intQ$. By the definition of second-order contingent derivative of $F(\bar{x},.)$ at $\bar{x}$ in the direction (u,v), then there would exists $(t_n,(x_n,y_n)) \longrightarrow (0^+,(x,y))$ such that

$$\frac{F(\bar{x},\bar{x}+t_nu+t_n^2x_n)-F(\bar{x},\bar{x})}{t_n^2} = y_n + \frac{1}{t_n}v \quad \forall n \in \mathbb{N},$$

which is implies that

$$\lim_{n\longrightarrow\infty}\frac{F(\bar{x},\bar{x}+t_nu+t_n^2x_n)-F(\bar{x},\bar{x})}{t_n^2} \in y-Q.$$

Since $x \in A^2(K,\bar{x},u)$, there exists $x'_n \longrightarrow x$ such that $\bar{x}+t_nu+t_n^2x'_n \in K$ for all $n \geq 1$. By hypotheses, $F(\bar{x},.)$ Frechet differentiable at $\bar{x} \in X$ whose Frechet derivative at $\bar{x}$ is locally Lipschitz and taking accourt of Lemma 4.1, one obtains

$$\lim_{n\longrightarrow\infty} \frac{F(\bar{x},\bar{x}+t_nu+t_n^2x'_n)-F(\bar{x},\bar{x})}{t_n^2} \in y-Q.$$

Since $y \in -intQ,\ \ intQ+Q = intQ$ and $F(\bar{x},\bar{x}) = 0$ hence for n large enough,

$$\frac{F(\bar{x},\bar{x}+t_nu+t_n^2x'_n)}{t_n^2} \in -intQ,$$

which is equivalent to $F(\bar{x},\bar{x}+t_nu+t_n^2x'_n) \in -intQ$ for sufficiently large n. From there we conclude that $\bar{x} \in K$ is not weakly efficient solution to the VEP.

For case (ii) holds: We take $u \in A(K,\bar{x})$ such that $\triangledown F(\bar{x},\bar{x})(u) \in -intQ$. By hypotheses, $F(\bar{x},.)$ is Frechet differentiable at $\bar{x} \in X$, there exists a locally Lipschitz mapping $\triangledown F(\bar{x},\bar{x}) : X \longrightarrow Y$ such that for every sequence $\{t_n\}_{n\geq 1} \subset \mathbb{R}_+,\ t_n \longrightarrow 0^+$ and for every $n \geq 1$ one gets

$$F(\bar{x},\bar{x}+t_nu)-F(\bar{x},\bar{x}) = \triangledown F(\bar{x},\bar{x})(t_n u)+o(t_n),$$

which is equivalent to

$$\frac{F(\bar{x},\bar{x}+t_nu)-F(\bar{x},\bar{x})}{t_n} = \triangledown F(\bar{x},\bar{x})(u)+\frac{o(t_n)}{t_n} \quad \forall n \geq 1,$$

where $||o(t_n)||/t_n \longrightarrow 0$ as $n \longrightarrow \infty$. By letting $n \longrightarrow +\infty$,

$$\lim_{n\longrightarrow\infty} \frac{F(\bar{x},\bar{x}+t_nu)-F(\bar{x},\bar{x})}{t_n} = \triangledown F(\bar{x},\bar{x})(u).$$

In other words $u \in A(K,\bar{x})$ there exists $w'_n \longrightarrow u$ such that $\bar{x}+t_nw'_n \in K$ for all $n \geq 1$. As $F(\bar{x},.)$ is Frechet differentiable at $\bar{x}$, thus

$$\lim_{n\longrightarrow\infty} \frac{F(\bar{x},\bar{x}+t_nw'_n)}{t_n} = \triangledown F(\bar{x},\bar{x})(u) \in -intQ.$$

Therefore for sufficiently large n, $\frac{F(\bar{x},\bar{x}+t_nw'_n)}{t_n} \in -intQ$, which is equivalent to $F(\bar{x},\bar{x}+t_nw'_n) \in -intQ$ for n large enough. Consequently, $\bar{x} \in K$ is not weakly efficient solution to the VEP, which completes the proof.

Corollary 4.1. *Let $(u,v) \in X \times (-Q)$ and let $\bar{x} \in K$ be a weakly efficient solution to the VEP. Then for any $x \in A^2(K,\bar{x},u)$ and for any $u' \in A(K,\bar{x})$, we have the following assertions are true*
(i). $D_c^2F(\bar{x},\bar{x},u,v)(x) \cap -intQ = \emptyset$,

(ii). $\Big\{\triangledown F(\bar{x},\bar{x})(u')\Big\} \cap -intQ = \emptyset$.
(iii). If $\triangledown F(\bar{x},\bar{x})(u) \in -intQ$ *then* $D_c^2 F(\bar{x},\bar{x},u,v)(x) \cap (-Q + \triangledown F(\bar{x},\bar{x})u) = \emptyset$.

Proof. By Proposition 4.1 and we omit it.

Theorem 4.1. —it Let $\bar{x} \in K,\ (u,v) \in X \times (-Q)$ and assume that $A^2(K,\bar{x},u) \neq \emptyset$. Then, if $\bar{x} \in K$ is a weakly efficient solution to the VEP then

$$\triangledown F(\bar{x},\bar{x})(u) \notin \Big(D_c^2 F(\bar{x},\bar{x},u,v)(x) + Q\Big) \cap \Big(-intQ\Big)\ \ \forall\, x \in A^2(K,\bar{x},u).$$

Proof. If $x \notin dom\Big(D_c^2 F(\bar{x},\bar{x},u,v)\Big)$, nothing to prove. Conversely, $x \in dom\Big(D_c^2 F(\bar{x},\bar{x},u,v)\Big)$. We must prove that $\triangledown F(\bar{x},\bar{x})(u) \notin \Big(D_c^2 F(\bar{x},\bar{x},u,v)(x) + Q\Big) \cap \Big(-intQ\Big)\ \forall\, x \in A^2(K,\bar{x},u)$. In fact, if it were not so, there would exists $x \in A^2(K,\bar{x},u) \cap dom\Big(D_c^2 F(\bar{x},\bar{x},u,v)\Big)$ such that

$$\triangledown F(\bar{x},\bar{x})(u) \in \Big(D_c^2 F(\bar{x},\bar{x},u,v)(x) + Q\Big)$$

and $\triangledown F(\bar{x},\bar{x})(u) \in -intQ$.

Taking $z \in D_c^2 F(\bar{x},\bar{x},u,v)(x)$ such that $z - \triangledown F(\bar{x},\bar{x})(u) \in -Q$. As $\triangledown F(\bar{x},\bar{x})(u) \in -intQ$ and further $-Q - intQ = -intQ$, it follows that $z \in -intQ$, which is equivalent to

$$D_c^2 F(\bar{x},\bar{x},u,v)(x) \cap -intQ \neq \emptyset.$$

According to Proposition 4.1 (i), then $\bar{x} \in K$ is not weakly efficient solution to the VEP. From there we conclude that the proof is complete.

Proposition 4.2. *Let* $\bar{x} \in K,\ u \in X$ *and assume that* $A^2(K,\bar{x},u) \neq \emptyset$*. Then, if* $\bar{x} \in K$ *is a weakly efficient solution to the VEP then*

$$\triangledown F(\bar{x},\bar{x})(x) + M_{F(\bar{x},.)}(u,u) \notin -int\,cone\Big(Q + \{\triangledown F(\bar{x},\bar{x})(u)\}\Big) \quad \forall x \in A^2(K,\bar{x},u).$$

Proof. Posit to the contrary that there exists $x \in A^2(K,\bar{x},u)$ such that

$$\triangledown F(\bar{x},\bar{x})(x) + M_{F(\bar{x},.)}(u,u) \in -int\,cone\Big(Q + \{\triangledown F(\bar{x},\bar{x})(u).\}\Big).$$

Taking the sequence $\{t_n\}_{n\geq 1} \subset \mathbb{R}_+$ such that $t_n \longrightarrow 0$ as $n \longrightarrow +\infty$. By hypotheses $x \in A^2(K,\bar{x},u)$ thus there exists $x'_n \longrightarrow x$ as $n \longrightarrow \infty$ such that $\bar{x}+t_nu+t_n^2x'_n \in K$ for all $n \geq 1$. On the other hand

$$\frac{F(\bar{x},\bar{x}+t_nu+t_n^2x'_n)-F(\bar{x},\bar{x})-t_n\nabla F(\bar{x},\bar{x})(u)}{t_n^2} = \frac{F(\bar{x},\bar{x}+t_nu+t_n^2x'_n)-F(\bar{x},\bar{x}+t_nu+t_n^2x)}{t_n^2}$$

$$+\frac{F(\bar{x},\bar{x}+t_nu+t_n^2x)-F(\bar{x},\bar{x}+t_nu)}{t_n^2}+\frac{F(\bar{x},\bar{x}+t_nu)-F(\bar{x},\bar{x})-t_n\nabla F(\bar{x},\bar{x})(u)}{t_n^2}$$

$$\longrightarrow \nabla F(\bar{x},\bar{x})(x)+M_{F(\bar{x},.)}(u,u) \in -int\,cone\Big(Q+\{\nabla F(\bar{x},\bar{x})(u)\}\Big).$$

Because $F(\bar{x},.):X\longrightarrow Y$ is Frechet differentiable at $\bar{x}$, its Frechet derivative is locally Lipschitz and consequently, there exists $L>0$ such that for n sufficiently large,

$$\Big|\Big|\frac{F(\bar{x},\bar{x}+t_nu+t_n^2x'_n)-F(\bar{x},\bar{x}+t_nu+t_n^2x)}{t_n^2}\Big|\Big|$$

$$=\Big|\Big|\frac{\nabla F(\bar{x},\bar{x})(u+t_nx'_n)-\nabla F(\bar{x},\bar{x})(u+t_nx)}{t_n}\Big|\Big| \leq L||x'_n-x|| \longrightarrow 0 \quad (as\; n\longrightarrow +\infty)$$

and

$$\frac{F(\bar{x},\bar{x}+t_nu+t_n^2x)-F(\bar{x},\bar{x}+t_nu)}{t_n^2} = \nabla F(\bar{x},\bar{x})(x)+\frac{o(t_n^2)}{t_n^2} \longrightarrow \nabla F(\bar{x},\bar{x})(x)$$

as $n\longrightarrow\infty$. On the other hand, we have

$$-int\,cone\Big(Q+\{\nabla F(\bar{x},\bar{x})(u)\}\Big) = IT(-intQ,\nabla F(\bar{x},\bar{x})(u)).$$

Therefore for n large enough,

$$\nabla F(\bar{x},\bar{x})(u)+t_n\frac{F(\bar{x},\bar{x}+t_nu+t_n^2x'_n)-F(\bar{x},\bar{x})-t_n\nabla F(\bar{x},\bar{x})(u)}{t_n^2} \in -intQ,$$

which is equivalent to $F(\bar{x},\bar{x}+t_nu+t_n^2x'_n)-F(\bar{x},\bar{x}) \in -intQ$. As $F(\bar{x},\bar{x})=0$, a consequence is $F(\bar{x},\bar{x}+t_nu+t_n^2x'_n)\in -intQ$ for n large enough. Therefore $\bar{x}\in K$ is not weakly efficient solution to the VEP and the proof is complete.

Theorem 4.2. *Let* $\bar{x}\in K,\ (u,v)\in X\times(-Q)$ *and assume that* $\nabla F(\bar{x},\bar{x})(u)\in -Q$ *and* $A^2(K,\bar{x},u)\neq \emptyset$. *Then, if* $\bar{x}\in K$ *is a weakly efficient solution to the VEP then* $\forall x\in A^2(K,\bar{x},u)\cap dom\Big(D_c^2F(\bar{x},\bar{x},u,v)\Big)$, *there exists* $\xi\in Y^*\setminus\{0\}$ *such that*

$$\xi \in N\Big(-Q,\nabla F(\bar{x},\bar{x})(u)\Big)\setminus\{0\} \tag{4.1}$$

$$and\quad \xi(z)\geq 0\ \forall z\in D_c^2F(\bar{x},\bar{x},u,v)(x). \tag{4.2}$$

Proof. As $\bar{x} \in K$ is a weakly efficient solution to the VEP and according to Proposition 4.2,

$$\nabla F(\bar{x},\bar{x})(x) + M_{F(\bar{x},.)}(u,u) \notin -int\, cone\Big(Q + \{\nabla F(\bar{x},\bar{x})(u)\}\Big)$$

$$\forall x \in \{x \in X \,:\, D_c^2 F(\bar{x},\bar{x},u,v)(x) \neq \emptyset\} \cap A^2(K,\bar{x},u).$$

By the separation theorem of convex sets, there exists $\xi \in Y^* \setminus \{0\}$ such that

$$\xi(\nabla F(\bar{x},\bar{x})(x) + M_{F(\bar{x},.)}(u,u) + q) > 0 \quad \forall\, q \in int\, cone\Big(Q + \{\nabla F(\bar{x},\bar{x})(u)\}\Big).$$

This implies that

$$\xi(\nabla F(\bar{x},\bar{x})(x) + M_{F(\bar{x},.)}(u,u)) + t\xi(q) > 0 \quad \forall\, q \in int\, cone\Big(Q + \{\nabla F(\bar{x},\bar{x})(u)\}\Big), \quad \forall t > 0.$$

Letting $t \longrightarrow 0^+$, one obtains $\xi(\nabla F(\bar{x},\bar{x})(x) + M_{F(\bar{x},.)}(u,u)) \geq 0$. Furthermore, as $t > 0$, this yields that

$$\xi(q) > -\frac{\xi(\nabla F(\bar{x},\bar{x})(x) + M_{F(\bar{x},.)}(u,u))}{t}.$$

By letting $t \longrightarrow +\infty$, one obtains the following result

$$\xi(q) \geq 0 \quad \forall q \in int\, cone\Big(Q + \{\nabla F(\bar{x},\bar{x})(u)\}\Big).$$

As ξ is a continuous function on Y, $\xi(q) \geq 0 \quad \forall q \in cl\, int\, cone\Big(Q + \{\nabla F(\bar{x},\bar{x})(u)\}\Big)$. In other words, $Q + \{\nabla F(\bar{x},\bar{x})(u)\} \subset cone\Big(Q + \{\nabla F(\bar{x},\bar{x})(u)\}\Big) \subset cl\, int\, cone\Big(Q + \{\nabla F(\bar{x},\bar{x})(u)\}\Big)$. Therefore

$$\xi(q) \geq 0 \quad \forall\, q \in Q + \{\nabla F(\bar{x},\bar{x})(u)\}.$$

As $0 \in Q$, it follows that $\nabla F(\bar{x},\bar{x})(u) \in Q + \nabla F(\bar{x},\bar{x})(u)$ and thus

$$\xi(\nabla F(\bar{x},\bar{x})(u)) \geq 0.$$

By hypotheses, $-\nabla F(\bar{x},\bar{x})(u) \in Q$ and this yields that $-2\nabla F(\bar{x},\bar{x})(u) \in Q$. Consequently, $-\nabla F(\bar{x},\bar{x})(u) \in Q + \nabla F(\bar{x},\bar{x})(u)$. Therefore,

$$\xi(\nabla F(\bar{x},\bar{x})(u)) \leq 0.$$

This implies as follows $\xi(\nabla F(\bar{x},\bar{x})(u)) = 0$. Furthermore, for all $q \in Q$,

$$q + \nabla F(\bar{x},\bar{x})u \in Q + \nabla F(\bar{x},\bar{x})u,\ \xi(q) = \xi(q + \nabla F(\bar{x},\bar{x})u) \geq 0.$$

By the definition of the dual cone of Q, $\xi \in Q^+ = -(-Q)^+$. Therefore,

$$\xi \in N(-Q, \triangledown F(\overline{x},\overline{x})u) \setminus \{0\}.$$

From there we conclude that the condition (4.1) is proven complete.

To prove for case (4.2), by an argument similar as in the proof of Proposition 4.1 and Proposition 4.2, for all $z \in D_c^2 F(\overline{x},\overline{x},u,v)(x)$ one gets

$$z - \lim_{n \longrightarrow \infty} \frac{F(\overline{x},\overline{x}+t_n u + t_n^2 x_n') - F(\overline{x},\overline{x})}{t_n^2} \in Q$$

and

$$\xi_0 \Big(\lim_{n \longrightarrow \infty} \frac{F(\overline{x},\overline{x}+t_n u + t_n^2 x_n') - F(\overline{x},\overline{x}) - t_n \triangledown F(\overline{x},\overline{x})(u)}{t_n^2} \Big) = \xi \big(\triangledown F(\overline{x},\overline{x})(x) + M_{F(\overline{x},.)}(u,u) \big) \geq 0.$$

As $\xi(\triangledown F(\overline{x},\overline{x})(u)) = 0$, and consequently,

$$\xi_0 \Big(\lim_{n \longrightarrow \infty} \frac{F(\overline{x},\overline{x}+t_n u + t_n^2 x_n') - F(\overline{x},\overline{x})}{t_n^2} \Big) \geq 0.$$

Obviously,

$$< \xi, z > \geq \xi_0 \Big(\lim_{n \longrightarrow \infty} \frac{F(\overline{x},\overline{x}+t_n u + t_n^2 x_n') - F(\overline{x},\overline{x})}{t_n^2} \Big) \geq 0.$$

From there we conclude that (4.2) is proven. The proof is complete.

To close this part, we introduce some properties involving to the second-order contingent derivatives.

Proposition 4.3. *Let $\overline{x} \in K$ and $(u,v) \in X \times (-Q)$. Then the following assertions are true*
i). If $\triangledown F(\overline{x},\overline{x})(u) \in intQ$ then $D_c^2 F(\overline{x},\overline{x},u,v)(x) \subset Q$ for all $x \in X$.
ii). If there exists $x \in A^2(K,\overline{x},u) \cap dom\Big(D_c^2 F(\overline{x},\overline{x},u,v)\Big)$ such that $D_c^2 F(\overline{x},\overline{x},u,v)(x) \subset -int\,Q$ then $\triangledown F(\overline{x},\overline{x})(u) \in -Q$.

Proof. Now we prove for case i): If $D_c^2 F(\overline{x},\overline{x},u,v)(x) = \emptyset$, nothing to prove. Conversely, we take $y \in D_c^2 F(\overline{x},\overline{x},u,v)(x)$. By the definition of second-order contingent derivative of $F(\overline{x},.)$ at $\overline{x}$ in the direction (u,v), there would exists $(t_n,(x_n,y_n)) \longrightarrow (0^+,(x,y))$ such that

$$\lim_{n \longrightarrow +\infty} \frac{F(\overline{x},\overline{x}+t_n u + t_n^2 x_n) - F(\overline{x},\overline{x})}{t_n^2} \in \lim_{n \longrightarrow +\infty} y_n - Q = y - Q.$$

On the other hand,

$$\frac{F(\overline{x},\overline{x}+t_n u + t_n^2 x_n) - F(\overline{x},\overline{x})}{t_n} = \triangledown F(\overline{x},\overline{x})(u + t_n x_n) + \frac{o(t_n)}{t_n},$$

where $\underset{n\longrightarrow\infty}{lim} \frac{o(t_n)}{t_n} = 0$. Consequently,

$$\lim_{n\longrightarrow\infty} \frac{F(\bar{x},\bar{x}+t_nu+t_n^2x_n)-F(\bar{x},\bar{x})}{t_n} = \nabla F(\bar{x},\bar{x})(u).$$

In other words, $\nabla F(\bar{x},\bar{x})(u) \in intQ$, and therefore for n large enough,

$$\frac{F(\bar{x},\bar{x}+t_nu+t_n^2x_n)-F(\bar{x},\bar{x})}{t_n^2} \in \frac{intQ}{t_n} = intQ.$$

By letting $n \longrightarrow \infty$, it follows that $y \in Q+Q \subset Q$. Consequently, $D_c^2F(\bar{x},\bar{x},u,v)(x) \subset Q$ for all $x \in X$.

For case (ii): If there exists $x \in A^2(K,\bar{x},u) \cap dom\Big(D_c^2F(\bar{x},\bar{x},u,v)\Big)$ such that $D_c^2F(\bar{x},\bar{x},u,v)(x) \subset -intQ$. By an argument similar as above, for n large enough

$$\frac{F(\bar{x},\bar{x}+t_nu+t_n^2x_n)-F(\bar{x},\bar{x})}{t_n^2} \in y-Q \subset -inQ-Q = -intQ,$$

which is equivalent to

$$\nabla F(\bar{x},\bar{x})(u+t_nx_n)+\frac{o(t_n)}{t_n} \in -t_n\, intQ \subset -intQ.$$

By letting $n \longrightarrow \infty$ it follows that

$$\nabla F(\bar{x},\bar{x})u \in -Q$$

and the proof is complete.

5. Second-order optimality conditions to the VEPC

Throughout this section, we always assume that X, Y, Z and W are Banach spaces with $dim(Y) < \infty, dim(Z) < \infty$, $dim(W) < \infty$ and Q is a pointed closed convex cone in Y with its interior be nonempty, S is convex cone in Z with its interior nonempty, and take $C \subset X$ is a nonempty convex subset in X. Let $F : X \times X \longrightarrow Y$ be a vector bifunction, $g : X \longrightarrow Z$, $h : X \longrightarrow W$ be vector functions. Let us denote by K the feasible set or the constraint set

$$K = \Big\{y \in C \mid -g(y) \in S,\ h(y) = 0\Big\}.$$

The vector equilibrium problem with constraints corresponding to K and F is denoted by (VEPC) or VEPC and written as follows: Find $\bar{x} \in K$ such that

$$F(\bar{x},y) \notin -intQ, \text{ for all } y \in K.$$

Then $\overline{x} \in K$ is called a weakly efficient solution of VEPC. Some other definitions related to Henig efficient, globally efficient and superefficient solutions can see in Section 2. Next, let us denotes

$$h^{-1}(0) = Ker h := \{x \in X \mid h(x) = 0\},$$

$$g^{-1}(-S) = \{z \in Z : -g(z) \in S\} \quad \text{and} \quad G_0 = \{x \in X : -g(x) \in intS\}.$$

Therefore, $K = g^{-1}(-S) \cap h^{-1}(0)$. Let $\overline{x} \in K$. We denote by $N(-S, g(\overline{x})) = -T(-S, g(\overline{x}))^+$ is the normal cone to $-S$ at $g(\overline{x})$. The second-order interior tangent set to S at $\overline{x}$ in the direction u is defined as

$$IT^2(S,\overline{x},u) = \{v \in X \mid \exists \delta > 0 \; s.t \; \overline{x} + tu + t^2u' \in S \; \forall t \in (0;\delta], \forall u' \in B(v,\delta)\}.$$

The second-order sequential interior tangent set to S at $\overline{x}$ in the direction u is defined as

$$IT_s^2(S,\overline{x},u) = \{v \in X \mid \exists \delta > 0 \, \exists t_n \longrightarrow 0^+ \; s.t \; \overline{x} + t_n u + t_n^2 u' \in S \; \forall n \geq 1, \forall u' \in B(v,\delta)\}.$$

Obviously, we have relations as follows

$$IT^2(S,\overline{x},u) = IT^2(intS,\overline{x},u) \subset A^2(S,\overline{x},u) \subset T^2(S,\overline{x},u) \subset cl\, cone\,[cone(S-\overline{x}) - u].$$

To simply some states, in all the results are stated in this section, we always assume that $(\overline{x},(u,v)) \in K \times (X \times (-Q))$, $F(\overline{x},\overline{x}) = 0$, the single-valued functions $F(\overline{x},.), g, h$ are Frechet differentiable at $\overline{x}$ and its Frechet derivatives are locally Lipschitz.

In this section, we provide necessary and sufficient second-order optimality conditions for the weakly efficient solution to the VEPC.

Theorem 5.1. *If $\overline{x} \in K$ is weakly efficient solution to the VEPC then*

$$\Big(\nabla F(\overline{x},\overline{x})(x) + M_{F(\overline{x},.)}(u,u); \nabla g(\overline{x})(x) + M_g(u,u); \nabla h(\overline{x})(x) + M_h(u,u)\Big) \notin$$

$$-int\, cone\Big(Q + \{\nabla F(\overline{x},\overline{x})(u)\}\Big) \times IT^2(-S, g(\overline{x}), \nabla g(\overline{x})(u)) \times IT^2(\{0\}, h(\overline{x}), \nabla h(\overline{x})(u))$$

$$\forall x \in dom\Big(D_c^2 F(\overline{x},\overline{x},u,v)\Big) \cap A^2(C,\overline{x},u).$$

Proof. Posit to the contrary that there exists $x \in A^2(C,\overline{x},u)$ such that $D_c^2 F(\overline{x},\overline{x},u,v)(x) \neq \emptyset$ and

$$\Big(\nabla F(\overline{x},\overline{x})(x) + M_{F(\overline{x},.)}(u,u); \nabla g(\overline{x})(x) + M_g(u,u); \nabla h(\overline{x})(x) + M_h(u,u)\Big) \in$$

$$-int\, cone\Big(Q+\{\nabla F(\bar{x},\bar{x})(u)\}\Big)\times IT^2(-S,g(\bar{x}),\nabla g(\bar{x})(u))\times IT^2(\{0\},h(\bar{x}),\nabla h(\bar{x})u).$$

Taking $y\in D^2_cF(\bar{x},\bar{x},u,v)(x)$. By the second-order contingent derivative of $F(\bar{x},.)$ at $\bar{x}$ in the direction (u,v), there exists $(t_n,x_n)\longrightarrow(0^+,x)$ such that

$$y\in\lim_{n\longrightarrow\infty}\frac{F(\bar{x},\bar{x}+t_nu+t_n^2x_n)-F(\bar{x},\bar{x})}{t_n^2}+Q.$$

As $x\in A^2(C,\bar{x},u)$, there exists $x'_n\longrightarrow x$ as $n\longrightarrow\infty$ such that $\bar{x}+t_nu+t_n^2x'_n\in C$ for all $n\geq 1$. Since $F(\bar{x},.)$ is Frechet differentiable at $\bar{x}\in C$ hence by Proposition 4.1 above, one obtains as follows

$$y\in\lim_{n\longrightarrow\infty}\frac{F(\bar{x},\bar{x}+t_nu+t_n^2x'_n)-F(\bar{x},\bar{x})}{t_n^2}+Q.$$

By a similar argument as in the proof of Theorem 4.2, it follows that

$$F(\bar{x},\bar{x}+t_nu+t_n^2x'_n)\in -intQ \qquad \text{for sufficiently large n.}$$

Now we have

$$\begin{aligned}
&\frac{g(\bar{x}+t_nu+t_n^2x'_n)-g(\bar{x})-t_n\nabla g(\bar{x})(u)}{t_n^2}\\
&=\frac{g(\bar{x}+t_nu+t_n^2x'_n)-g(\bar{x}+t_nu+t_n^2x)}{t_n^2}\\
&\quad+\frac{g(\bar{x}+t_nu+t_n^2x)-g(\bar{x})-t_n\nabla g(\bar{x})(u)}{t_n^2}\\
&=\frac{g(\bar{x}+t_nu+t_n^2x'_n)-g(\bar{x}+t_nu+t_n^2x)}{t_n^2}\\
&\quad+\frac{g(\bar{x}+t_nu+t_n^2x)-g(\bar{x}+t_nu)}{t_n^2}\\
&\quad+\frac{g(\bar{x}+t_nu)-g(\bar{x})-t_n\nabla g(\bar{x})u}{t_n^2}\\
&\longrightarrow\nabla g(\bar{x})(x)+M_g(u,u)\in IT^2(-S,g(\bar{x}),\nabla g(\bar{x})(u)).\}
\end{aligned}$$

Because $g:X\longrightarrow Y$ is Frechet differentiable at $\bar{x}$, $\nabla g(\bar{x}):X\longrightarrow Z$ is locally Lipschitz on X, hence there exist a neighborhood U of $u\in X$ and $L>0$ such that

$$||\nabla g(\bar{x})(k)-\nabla g(\bar{x})(h)||\leq L||k-h|| \quad \forall\, k,h\in U.$$

Taking a interger number $N>0$ such that

$$||\nabla g(\bar{x})(u+t_nx'_n)-\nabla g(\bar{x})(u+t_nx)||\leq Lt_n||x'_n-x|| \quad \forall\, n>N.$$

This together with the fact

$$\Big|\Big|\frac{g(\bar{x}+t_nu+t_n^2x_n')-g(\bar{x}+t_nu+t_n^2x)}{t_n^2}\Big|\Big| = \Big|\Big|\frac{\nabla g(\bar{x})(u+t_nx_n')-\nabla g(\bar{x})(u+t_nx)||}{t_n}\Big|\Big|$$

and we obtain as follows

$$\Big|\Big|\frac{g(\bar{x}+t_nu+t_n^2x_n')-g(\bar{x}+t_nu+t_n^2x)}{t_n^2}\Big|\Big| \longrightarrow 0,$$

as $n \longrightarrow \infty$ and moreover

$$\frac{g(\bar{x}+t_nu+t_n^2x)-g(\bar{x}+t_nu)}{t_n^2} = \nabla g(\bar{x})(x)+\frac{o(t_n^2)}{t_n^2} \longrightarrow \nabla g(\bar{x})(x)$$

as $n \longrightarrow \infty$. By the definition of IT^2, for sufficiently large n, we have

$$g(\bar{x})+t_n\nabla g(\bar{x})(u)+t_n^2\frac{g(\bar{x}+t_nu+t_n^2x_n')-g(\bar{x})-t_n\nabla g(\bar{x})(u)}{t_n^2} \in -S$$

and thus $g(\bar{x}+t_nu+t_n^2x_n') \in -S$ for n large enough. Similarly, we obtain

$$h(\bar{x})+t_n\nabla h(\bar{x})(u)+t_n^2\frac{h(\bar{x}+t_nu+t_n^2x_n')-h(\bar{x})-t_n\nabla h(\bar{x})(u)}{t_n^2} \in \{0\}$$

for n large enough and therefore $h(\bar{x}+t_nu+t_n^2x_n') \in \{0\}$ for n large enough. So, we conclude that $\bar{x}+t_nu+t_n^2x_n' \in K$ for n large enough. Therefore, $\bar{x}$ is not weakly efficient solution to the VEPC. The proof is completed.

Proposition 5.1. *Let $\bar{x}, u \in X$. Then the following are well known*

(i). $IT^2(S,\bar{x},u) \subset A^2(S,\bar{x},u) \subset T^2(S,\bar{x},u) \subset cl\,cone[cone(S-\bar{x})-u]$.

(ii). $IT^2(S,\bar{x},u) = IT^2(intS,\bar{x},u)$.

(iii). If $u \in cone(S-\bar{x})$ then $IT^2(S,\bar{x},u) = int\,cone\,[cone(S-\bar{x})-u]$ and $A^2(S,\bar{x},u) = cl\,cone\,[cone(S-\bar{x})-u]$.

(iv). If $u \notin T(S,\bar{x})$ then $T^2(S,\bar{x},u) = \emptyset$.

(v). If $A^2(S,\bar{x},u) \neq \emptyset$ then $IT^2(S,\bar{x},u) = int\,A^2(S,\bar{x},u)$ and $cl\,IT^2(S,\bar{x},u) = A^2(S,\bar{x},u)$.

Theorem 5.2. *Assume that $u \in A(K,\bar{x})$ and $\nabla F(\bar{x},\bar{x})(u) \in -Q \setminus intQ$. If $\bar{x} \in K$ is a weakly efficient solution to the VEPC then $\forall\, x \in dom\Big(D_c^2F(\bar{x},\bar{x},u,v)\Big) \cap A^2(C,\bar{x},u)$ and $\forall\,(x',y,z) \in D_c^2F(\bar{x},\bar{x},u,v)(x) \times D_c^2g(\bar{x},u,v)(x) \times D_c^2h(\bar{x},u,v)(x)$ there exist $(\xi,\lambda,\gamma) \in Y^* \times Z^* \times W^* \setminus \{(0,0,0)\}$ such that*

$$\xi \in N(-Q,\nabla F(\bar{x},\bar{x})(u)), \tag{5.1}$$

$$\lambda \in N(-S,\nabla g(\bar{x})(u)), \tag{5.2}$$

$$\xi(x') + \lambda(y) + \gamma(z) \geq 0. \tag{5.2}$$

Proof. We fixed $x \in dom\Big(D_c^2 F(\bar{x},\bar{x},u,v)\Big) \cap A^2(C,\bar{x},u)$ and

$$(x',y,z) \in D_c^2 F(\bar{x},\bar{x},u,v)(x) \times D_c^2 g(\bar{x},u,v)(x) \times D_c^2 h(\bar{x},u,v)(x).$$

According to Theorem 5.1, we obtain

$$\Big(\triangledown F(\bar{x},\bar{x})(x) + M_{F(\bar{x}.)}(u,u); \triangledown g(\bar{x})(x) + M_g(u,u); \triangledown h(\bar{x})(x) + M_h(u,u)\Big) \notin$$
$$-int\, cone\Big(Q + \{\triangledown F(\bar{x},\bar{x})(u)\}\Big) \times IT^2(-S, g(\bar{x}), \triangledown g(\bar{x})(u)) \times \{0\}.$$

By Proposition 5.1,

$$IT^2(-S, g(\bar{x}), \triangledown g(\bar{x})(u)) = int\, cone[cone(-S - g(\bar{x})) - \triangledown g(\bar{x})(u))]$$

because $\triangledown g(\bar{x})(u) \in cone(-S - g(\bar{x}))$. By the separation theorem of convex sets, there exist $(\xi,\lambda,\gamma) \in Y^* \times Z^* \times W^* \setminus \{(0,0,0)\}$ such that

$$\xi\Big(\triangledown F(\bar{x},\bar{x})(x) + M_{F(\bar{x}.)}(u,u)\Big) + \lambda\Big(\triangledown g(\bar{x})(x) + M_g(u,u)\Big) + \gamma\Big(\triangledown h(\bar{x})(x) + M_h(u,u)\Big)$$
$$+\xi(p) + \lambda(q) > 0, \forall (p,q) \in int\, cone\Big(Q + \{\triangledown F(\bar{x},\bar{x})(u)\}\Big) \times -IT^2(-S, g(\bar{x}), \triangledown g(\bar{x})(u)).$$

We fixed $q \in -IT^2(-S, g(\bar{x}), \triangledown g(\bar{x})(u))$ and as $int\, cone\Big(Q + \{\triangledown F(\bar{x},\bar{x})(u)\}\Big)$ is a cone, this yields that

$$\xi\Big(\triangledown F(\bar{x},\bar{x})(x) + M_{F(\bar{x}.)}(u,u)\Big) + \lambda\Big(\triangledown g(\bar{x})(x) + M_g(u,u)\Big) + \gamma\Big(\triangledown h(\bar{x})(x) + M_h(u,u)\Big)$$
$$+t\xi(p) + \lambda(q) > 0$$

for all $t > 0$ and $p \in int\, cone\Big(Q + \{\triangledown F(\bar{x},\bar{x})(u)\}\Big)$. Consequently, for all t, p as above,

$$\xi(p) > -\frac{\xi\Big(\triangledown F(\bar{x},\bar{x})(x) + M_{F(\bar{x}.)}(u,u)\Big) + \lambda\Big(\triangledown g(\bar{x})(x) + M_g(u,u)\Big) + \gamma\Big(\triangledown h(\bar{x})(x) + M_h(u,u)\Big)}{t}.$$

Letting $t \longrightarrow \infty$, we have

$$\xi(p) \geq 0 \quad \forall p \in int\, cone\Big(Q + \{\triangledown F(\bar{x},\bar{x})(u)\}\Big).$$

As ξ is a continuous function on Y and Q closed cone, it follows that

$$\xi(p) \geq 0,\ \forall p \in cl\, int\, cone\Big(Q + \{\triangledown F(\bar{x},\bar{x})(u)\}\Big) = cone\Big(Q + \{\triangledown F(\bar{x},\bar{x})(u)\}\Big).$$

On the other hand,

$$Q+\{\nabla F(\bar{x},\bar{x})(u)\}\subset cone\Big(Q+\{\nabla F(\bar{x},\bar{x})(u)\}\Big),$$

therefore

$$\xi(p)\geq 0 \qquad \forall p\in Q+\{\nabla F(\bar{x},\bar{x})(u)\}.$$

As $0\in Q$, $\nabla F(\bar{x},\bar{x})(u)\in Q+\{\nabla F(\bar{x},\bar{x})(u)\}$ and by hypotheses, $-\nabla F(\bar{x},\bar{x})(u)\in Q+\{\nabla F(\bar{x},\bar{x})(u)\}$. From there we infer that $\xi(\nabla F(\bar{x},\bar{x})(u))=0$. Therefore $\xi(q)\geq 0 \qquad \forall q\in Q$. By the definition of the dual cone Q^+ of Q, we have $\xi\in Q^+$. A consequence is

$$\xi\in N(-Q,\nabla F(\bar{x},\bar{x})(u))$$

and (5.1) is proven complete.

Next, we fixed p as above. Similarly, we also have

$$\lambda(q)\geq 0 \quad \forall q\in -int\,cone[cone(-S-g(\bar{x}))-\nabla g(\bar{x})(u))].$$

As λ is a continuous function on Z, consequently,

$$\lambda(q)\geq 0 \quad \forall q\in -cone[cone(-S-g(\bar{x}))-\nabla g(\bar{x})(u))].$$

This implies that $\lambda(q)\geq 0 \quad \forall q\in -cone(-S-g(\bar{x}))+\nabla g(\bar{x})(u)$. Thus $\lambda(q)-\lambda(\nabla g(\bar{x})(u)))\geq 0 \quad \forall q\in -cone(-S-g(\bar{x}))$. Consequently,

$$t\lambda(q)-\lambda(\nabla g(\bar{x})(u)))\geq 0 \quad \forall q\in -cone(-S-g(\bar{x})),\forall\, t>0.$$

As $t>0$,

$$\lambda(q)>\frac{\lambda(\nabla g(\bar{x})u))}{t} \quad \forall q\in -cone(-S-g(\bar{x})),\forall\, t>0.$$

Letting $t\longrightarrow\infty$, we obtain as follows $\lambda(q)\geq 0 \qquad \forall q\in S+g(\bar{x})$. As $0\in S$ implies $g(\bar{x})\in S+g(\bar{x})$ and $\bar{x}\in K$ implies $-g(\bar{x})\in S+g(\bar{x})$. Consequently, $\lambda(g(\bar{x}))=0$ and this yields that $\lambda(q)\geq 0 \qquad \forall\, q\in S$. So that $\lambda\in N(-S,g(\bar{x}))$ and (5.2) is proven complete. Moreover, we also have

$$\xi\Big(\nabla F(\bar{x},\bar{x})(x)+M_{F(\bar{x}.)}(u,u)\Big)+\lambda\Big(\nabla g(\bar{x})(x)+M_g(u,u)\Big)+\gamma\Big(\nabla h(\bar{x})(x)+M_h(u,u)\Big)\geq 0.$$

Obviously,

$$\xi\Big(\nabla F(\bar{x},\bar{x})(x)+M_{F(\bar{x}.)}(u,u)\Big)=\xi(x');$$

$$\lambda\Big(\nabla g(\overline{x})(x)+M_g(u,u)\Big)=\lambda(y) \text{ and } \gamma\Big(\nabla h(\overline{x})(x)+M_h(u,u)\Big)=\gamma(z).$$

Therefore, (5.3) is proven and the proof is complete.

Remark 5.1. In Theorem 5.2, let us easy to check that $\gamma(z)=0$ for every $z\in D_c^2h(\overline{x},u,v)(x)$.

Theorem 5.3. *Under the assumptions of Theorem 5.2 with $h=0$ and X is finite dimensional, $F(\overline{x},.):X\longrightarrow Y, g:X\longrightarrow Z$ be stable functions at $\overline{x}\in K$. Then, if $\forall\, x\in T^2(K,\overline{x},u)\setminus\{0\}$ and $\forall\,(x',y)\in D_c^2F(\overline{x},\overline{x},u,v)(x)\times D_c^2g(\overline{x},u,v)(x)$, there exist $(\xi,\lambda)\in Y^*\times Z^*$ such that*

$$\xi\in Q^+, \tag{5.4}$$

$$\lambda\in N(-S,g(\overline{x})), \tag{5.5}$$

$$\xi(x')+\lambda(y)>0. \tag{5.6}$$

Then $\overline{x}\in K$ is a locally weakly efficient solution to the VEPC, that is there exists a neighborhood U of $\overline{x}$ in K such that

$$F(\overline{x},y)\notin -intQ\quad\forall\, y\in K\cap U.$$

Proof. Posit to the contrary that there exist sequences $x_n\in K\cap B(\overline{x},\frac{1}{n})\setminus\{\overline{x}\}$ and $q_n\in Q$ such that

$$F(\overline{x},x_n)-F(\overline{x},\overline{x})+q_n=b_n\in B(0,\frac{||x_n-\overline{x}||^2}{n}), \tag{5.7}$$

where $t_n:=||x_n-\overline{x}||\longrightarrow 0^+$ as $n\longrightarrow\infty$. It follows that

$$\frac{||F(\overline{x},x_n)-F(\overline{x},\overline{x})+q_n||}{t_n^2}<\frac{1}{n},\quad\forall\, n\geq 1. \tag{5.8}$$

It is clear that, taking a new subsequence if necessary, there exists $x\in T^2(K,\overline{x},u)$ with $||x||=1$ such that

$$\lim_{n\longrightarrow\infty}v_n:=\lim_{n\longrightarrow\infty}\frac{x_n-\overline{x}-t_nu}{t_n^2}=x.$$

As $x_n=x_0+t_nu+t_n^2v_n\in K$, from (5.8) it follows that

$$\frac{F(\overline{x},\overline{x}+t_nu+t_n^2v_n)-F(\overline{x},\overline{x})}{t_n^2}=\frac{-q_n}{t_n^2}+\frac{b_n}{t_n^2}. \tag{5.9}$$

As $(F(\overline{x},.),g)$ is stable at $\overline{x}\in K$, choosing a subsequence if necessary, we may assume that

$$\frac{(F(\overline{x},.),g)(\overline{x}+t_nu+t_n^2v_n)-(F(\overline{x},.),g)(\overline{x})}{t_n^2}\longrightarrow(x',y)\in D_c^2\Big(F(\overline{x},.),g\Big)(\overline{x},u,v)(x). \tag{5.10}$$

On the other hand, by hypothesis there exist $(\xi,\lambda)\in Y^*\times Z^*$ such that the conditions (5.4), (5.5) and (5.6) are satisfied. From (5.10) it follows that

$$\frac{F(\bar{x},\bar{x}+t_nu+t^2v_n)-F(\bar{x},\bar{x})}{t_n^2}\longrightarrow x',\quad \text{as } n\longrightarrow\infty.$$

We have

$$\frac{||b_n||}{||x_n-\bar{x}||^2}<\frac{1}{n}\quad \forall\, n\geq 1.$$

Consequently, $\frac{b_n}{t_n^2}\longrightarrow 0$ as $n\longrightarrow\infty$. Thus $x'=\lim_{n\longrightarrow\infty}\frac{-q_n}{t_n^2}\in -cl\,Q=-Q$. As $\xi\in Q^+$, it follows that $\xi(x')\leq 0$. (*) Moreover, from (5.10) it also follows that

$$\frac{g(\bar{x}+t_nu+t_n^2v_n)-g(\bar{x})}{t_n^2}\longrightarrow y\quad \text{as } n\longrightarrow\infty.$$

On the other hand, $g(\bar{x}+t_nu+t_n^2v_n)\in -S$, it implies that $y\in cl\,cone(-S-g(\bar{x}))=T(-S,g(\bar{x}))$. Since $\lambda\in N(-S,g(\bar{x}))=-T(-S,g(\bar{x}))^+$, by the definition of the dual cone of $T(-S,g(\bar{x}))$, one has $\lambda(y)\leq 0$. (**) Together (*) with (**), we obtain

$$\xi(x')+\lambda(y)\leq 0,$$

this is a contradiction. From there we conclude that $\bar{x}$ is locally weakly efficient solution to the VEPC. The proof is complete.

Remark 5.2. If in addition, $F(\bar{x},.)$ is positively homogeneous, then locally weakly efficient solution $\bar{x}\in K$ also is a weakly efficient solution to the VEPC.

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