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MONOTONE PROJECTION METHODS FOR FIXED POINTS OF ASYMPTOTICALLY QUASI-ϕ-NONEXPANSIVE MAPPINGS

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Abstract. In this paper, a monotone projection method is investigated for fixed points of asymptotically quasi-ϕ-nonexpansive mappings. Strong convergence of fixed points is obtained in the framework of Banach spaces.

Keywords. Asymptotically quasi-ϕ-nonexpansive mapping; Relatively nonexpansive mapping; Generalized projection; Fixed point.

1. Introduction-preliminaries

Let E be a real Hilbert space. Let C be a nonempty subset of E and let $T : C \to C$ be a mapping. A point $x \in C$ is a fixed point of T provided $Tx = x$. In this paper, we use $F(T)$ to denote the fixed point set of T T is said to be uniformly asymptotically regular on C if for any bounded subset K of C,

$$\limsup_{n\to\infty} \sup\{\|T^{n+1}x - T^n x\| : x \in K\} = 0.$$

The mapping T is said to be closed if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$.

Recall that the mapping T is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

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T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C, \forall n \geq 1.$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1] in 1972. In uniformly convex Banach spaces, they proved that if C is nonempty bounded closed and convex then every asymptotically nonexpansive self-mapping T on C has a fixed point. Further, the fixed point set of T is closed and convex.

One of classical iterations is the Halpern iteration [2] which generates a sequence in the following manner:

$$\forall x_1 \in C, \quad x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in the interval $(0,1)$ and $u \in C$ is a fixed element.

Since 1967, Halpern iteration has been studied extensively by many authors; see, for example [3-9]. It is well known that the following two restrictions: (C1) $\lim_{n\to\infty} \alpha_n = 0$, (C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$, are necessary in the sense that if the Halbern iterative sequence is strongly convergent for all nonexpansive self-mappings defined on C. Because of condition (C2), Halpern iteration is widely believed to have a slow convergence. To improve the rate of convergence of the Halbern iterative sequence, we can not rely only on the iteration itself. Hybrid projection methods recently have been applied to solve the problem.

Martinez-Yanes and Xu [4] considered the hybrid projection algorithm for a nonexpansive mapping in a Hilbert space. Strong convergence theorems are established under condition (C1) only. To be more precise, they proved the following theorem.

Theorem MYX. *Let H be a real Hilbert space, C a closed convex subset of H and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\} \subset (0,1)$ is such that* $\lim_{n\to\infty} \alpha_n =$

0. *Then the sequence* $\{x_n\}$ *defined by*

$$\begin{cases} x_0 \in C, \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad \forall n \ge 0, \end{cases}$$

converges strongly to $P_{F(T)}x_0$.

Recently, many authors studied the problem of extending Theorem MYX to a Banach space. In this paper, we study, in the framework of Banach spaces, the problem of modifying Halpern iteration via hybrid projection algorithms such that strong convergence is available under assumption (C1) only. The organization of this paper is as follows. In Section 1, we provide some necessary introduction and preliminaries. In Section 2, the main strong convergence theorems are established in the framework of Banach spaces. Some applications are provided to support our main results in this section.

Let E be a Banach space with the dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. $\rightarrow$ and $\rightharpoonup$ denoted by the strong convergence and weak convergence, respectively.

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided $\lim\limits_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$ exists for each $x, y \in U_E$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U_E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. It is also well known that if E is uniformly smooth if and only if E^* is uniformly convex.

Recall that a Banach space E has the Kadec-Klee property if for any sequence $\{x_n\} \subset E$, and $x \in E$ with $x_n \rightharpoonup x$, and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. For more details on Kadec-Klee property, the readers can refer to [10] and the references therein. It is well known that if E is a uniformly convex Banach spaces, then E has the Kadec-Klee property.

As we all know if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [11] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that E is a smooth Banach space. Consider the functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for } x, y \in E.$$

Observe that, in a Hilbert space H, the equality is reduced to $\phi(x,y) = \|x-y\|^2$, $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem $\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x)$. Existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J. In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|y\| + \|x\|)^2, \quad \forall x, y \in E,$$

and

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle, \quad \forall x, y, z \in E.$$

Remark 1.1. If E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x,y) = 0$ if and only if $x = y$.

Let C be a nonempty closed convex subset of E and T a mapping from C into itself. A point p in C is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. A mapping T from C into itself is said to be relatively nonexpansive if $\widetilde{F}(T) = F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The mapping T is said to be relatively asymptotically nonexpansive [12] if $\widetilde{F}(T) = F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(p,Tx) \leq k_n \phi(p,x)$ for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

The mapping T is said to be quasi-ϕ-nonexpansive [13] if $F(T) \neq \emptyset$ and $\phi(p,Tx) \leq \phi(p,x)$ for all $x \in C$ and $p \in F(T)$. T is said to be asymptotically quasi-ϕ-nonexpansive [14,15] if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [0,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(p,Tx) \leq k_n \phi(p,x)$ for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

Remark 1.2. The class of asymptotically quasi-ϕ-nonexpansive mappings is more general than the class of relatively asymptotically nonexpansive mappings which requires the restriction: $F(T) = \widetilde{F}(T)$.

Recently, Qin, Cho, Kang and Zhou [5] studied fixed points of a quasi-ϕ-nonexpansive mapping based on a monotone projection method, which first was introduced by Su and Qin [16]; see also [17]. To be more precise, they proved the following theorem.

Theorem QCKZ. *Let C be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E and $T : C \to C$ a closed and quasi-ϕ-nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}[\alpha_n J x_1 + (1-\alpha_n) J T x_n], \\ C_{n+1} = \{z \in C_n : \phi(z,y_n) \leq \alpha_n \phi(z,x_0) + (1-\alpha_n)\phi(z,x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1. \end{cases}$$

Assume that the control sequence satisfies the restriction: $\lim_{n\to\infty} \alpha_n = 0$. *Then $\{x_n\}$ converges strongly to* $\Pi_{F(T)} x_1$.

In this paper, we investigate a monotone projection algorithm for a pair of asymptotically quasi-ϕ-nonexpansive mappings. Strong convergence of the purposed algorithm is obtained in a uniformly convex and smooth Banach space.

Lemma 1.3. [11] *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if $\langle x_0 - y, Jx - Jx_0 \rangle \geq 0$, $\forall y \in C$.*

Lemma 1.4. [11] *Let E be a reflexive, strictly convex and smooth Banach space, C a nonempty closed convex subset of E and $x \in E$. Then $\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x)$, $\forall y \in C$.*

Lemma 1.5. [18] *Let E be a uniformly convex Banach space and $B_r(0)$ be a closed ball of X. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda \|x\|^2 + \mu \|y\|^2 + \gamma \|z\|^2 - \lambda \mu g(\|x - y\|)$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

Lemma 1.6. [19] *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$.*

2. Main results

Theorem 2.1. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed and convex subset of E. Let $T : C \to C$ be a closed asymptotically quasi-ϕ-nonexpansive mapping and let $S : C \to C$ be a closed asymptotically quasi-ϕ-nonexpansive mapping. Assume that T and S are uniformly asymptotically regular on C their common fixed point set is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ z_n = J^{-1}\big(\beta_n J x_n + \gamma_n J T^n x_n + \delta_n J S^n x_n\big), \\ y_n = J^{-1}\big(\alpha_n J x_1 + (1 - \alpha_n) J z_n\big), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n) + \alpha_n\big(\|x_1\|^2 + 2\langle z, Jx_n - Jx_1 \rangle\big) + (k_n - 1) M_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $M_n = \sup\{\phi(z,x_n) : z \in F(T)\cap F(S)\}$ *for each* $n \geq 1$ *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *and* $\{\delta_n\}$ *are real sequences in* $(0,1)$ *such that* $\beta_n+\gamma_n+\delta_n = 1$, $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$, $\liminf_{n\to\infty}\gamma_n\delta_n > 0$. *Then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_{F(T)\cap F(S)}x_1$.

Proof. First, we show convexness of $F(T)\cap F(S)$. Let $p_1, p_2 \in F(T)$, and $p = tp_1+(1-t)p_2$, where $t \in (0,1)$. We see that $p = Tp$. Indeed, we see from the definition of T that $\phi(p_1,T^n p) \leq k_n\phi(p_1, p)$, and $\phi(p_2,T^n p) \leq k_n\phi(p_2, p)$. Hence, we have $\phi(p_1,T^n p) = \phi(p_1, p)+\phi(p,T^n p)$ $+2\langle p_1 - p,Jp-JT^n p\rangle$, and $\phi(p_1,T^n p) = \phi(p_1, p)+\phi(p,T^n p)+2\langle p_1 - p,Jp-JT^n p\rangle$. This implies that $\phi(p,T^n p) \leq (k_n - 1)\phi(p_1, p) + 2\langle p - p_1,Jp-JT^n p\rangle$, and $\phi(p,T^n p) \leq (k_n - 1)\phi(p_2, p)$ $+2\langle p-p_2,Jp-JT^n p\rangle$. It follows that $\lim_{n\to\infty}\phi(p,T^n p) = 0$. This finds that $T^n p \to p$ as $n \to \infty$. Hence $TT^n p = T^{n+1}p \to p$, as $n \to \infty$. In view of the closedness of T, we can obtain that $p \in F(T)$. This shows that $F(T)$ is convex. Hence, $F(S)$ is also convex. Since T and S are closed, we find that $F(T)$ and $F(S)$ are also closed. This proves that the common fixed point set is closed and convex. From the construction of C_n, we find that C_n is closed and convex. This shows that $\Pi_{C_{n+1}}x_1$ is well defined.

Next, we prove that $F(S)\cap F(T) \subset C_n$ for each $n \geq 1$. $F(S)\cap F(T) \subset C_1 = C$ is obvious. Suppose that $F(S)\cap F(T) \subset C_h$ for some $h \in \mathbb{N}$. Then, for $\forall w \in \mathscr{F} \subset C_h$, we find that

$$\begin{aligned}
\phi(w,z_h) &\leq \|w\|^2 - 2\beta_h\langle w,Jx_h\rangle - 2\gamma_h\langle w,JT^h x_h\rangle - 2\delta_h\langle w,JS^h x_h\rangle \\
&\quad + \beta_h\|x_h\|^2 + \gamma_h\|T^h x_h\|^2 + \delta_h\|S^h x_h\|^2 \\
&\leq \beta_h\phi(w,x_h) + \gamma_h k_h\phi(w,x_h) + \delta_h k_h\phi(w,x_h) \\
&\leq \phi(w,x_h) + (k_h - 1)\phi(w,x_h).
\end{aligned}$$

It follows that

$$\begin{aligned}
\phi(w,y_h) &= \|w\|^2 - 2\langle w,\alpha_h Jx_1 + (1-\alpha_h)JT^h x_h\rangle + \|\alpha_h Jx_1 + (1-\alpha_h)Jz_h\|^2 \\
&\leq \|w\|^2 - 2\alpha_h\langle w,Jx_1\rangle - 2(1-\alpha_h)\langle w,Jz_h\rangle + \alpha_h\|x_1\|^2 + (1-\alpha_h)\|z_h\|^2 \\
&\leq \alpha_h\phi(w,x_1) + (1-\alpha_h)\phi(w,x_h) + (k_h - 1)(1-\alpha_h)\phi(w,x_h) + \xi_h \\
&\leq \phi(w,x_h) + \alpha_h\big(\phi(w,x_1) - \phi(w,x_h)\big) + (k_h - 1)(1-\alpha_h)\phi(w,x_h) \\
&\leq \phi(w,x_h) + \alpha_h\big(\|x_1\|^2 + 2\langle w,Jx_h - Jx_1\rangle\big) + (k_h - 1)M_h.
\end{aligned}$$

This shows that $w \in C_{h+1}$. This implies that $F(S) \cap F(T) \subset C_n$. Note that $x_n = \Pi_{C_n} x_1$, we see that $\langle x_n - z, Jx_1 - Jx_n \rangle \geq 0$, $\forall z \in C_n$. Since $F(S) \cap F(T) \subset C_n$, we find that

$$\langle x_n - w, Jx_1 - Jx_n \rangle \geq 0, \quad \forall w \in \mathscr{F}. \tag{2.1}$$

This implies that

$$\phi(x_n, x_1) = \phi(\Pi_{C_n} x_1, x_1) \leq \phi(w, x_1) - \phi(w, x_n) \leq \phi(w, x_1),$$

for each $w \in F(S) \cap F(T) \subset C_n$. Therefore, the sequence $\phi(x_n, x_1)$ is bounded. This implies that $\{x_n\}$ is bounded. Using the fact $x_n = \Pi_{C_n} x_1$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we have $\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1)$, $\forall n \geq 1$. Therefore, $\{\phi(x_n, x_1)\}$ is nondecreasing. It follows that the limit of $\{\phi(x_n, x_1)\}$ exists.

Using the construction of C_n, we have that $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_1 \in C_n$ for any positive integer $m \geq n$. It follows that $\phi(x_m, x_n) \leq \phi(x_m, x_1) - \phi(x_n, x_1)$. Letting $m, n \to \infty$, we see that $\phi(x_m, x_n) \to 0$. It follows that $x_m - x_n \to 0$ as $m, n \to \infty$. Hence, $\{x_n\}$ is a Cauchy sequence. Since the space is complete and C is closed and convex, we can assume that $\lim_{n\to\infty} x_n = p \in C$. By taking $m = n+1$, we obtain that $\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0$. It follows that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.2}$$

Since $x_{n+1} \in C_{n+1}$, we obtain that

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n) + \alpha_n \big(\|x_1\|^2 + 2\langle z, Jx_n - Jx_1 \rangle \big) + (k_n - 1) M_n.$$

Using the condition imposed on $\{\alpha_n\}$, we find that that $\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0$. This in turn implies that

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0.$$

This further implies that $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Since J is uniformly norm-to-norm continuous on bounded sets, we have $\lim\limits_{n\to\infty} \|Jx_n - Jy_n\| = 0$. Since $Jy_n - Jz_n = \alpha_n (Jx_1 - Jz_n)$, we see that $\lim\limits_{n\to\infty} \|Jy_n - Jz_n\| = 0$. In view of $\|Jx_n - Jz_n\| \leq \|Jx_n - Jy_n\| + \|Jy_n - Jz_n\|$, we find that $\lim\limits_{n\to\infty} \|Jx_n - Jz_n\| = 0$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we obtain that $\lim_{n\to\infty}\|x_n - z_n\| = 0$. Let $r = \sup_{n\geq 1}\{\|x_n\|, \|T^n x_n\|, \|S^n x_n\|\}$. From Lemma 2.8, we have

$$\begin{aligned}
&\phi(w,z_n)\\
&= \|w\|^2 - 2\langle w, \beta_n Jx_n + \gamma_n JT^n x_n + \delta_n JS^n x_n\rangle + \|\beta_n Jx_n + \gamma_n JT^n x_n + \delta_n JS^n x_n\|^2\\
&\leq \|w\|^2 - 2\beta_n\langle w, Jx_n\rangle - 2\gamma_n\langle w, JT^n x_n\rangle - 2\delta_n\langle w, JS^n x_n\rangle\\
&\quad + \beta_n\|x_n\|^2 + \gamma_n\|T^n x_n\|^2 + \delta_n\|S^n x_n\|^2 - \gamma_n\delta_n g(\|JT^n x_n - JS^n x_n\|)\\
&\leq \beta_n\phi(w,x_n) + \gamma_n\phi(w,T^n x_n) + \delta_n\phi(w,S^n x_n) - \gamma_n\delta_n g(\|JT^n x_n - JS^n x_n\|)\\
&\leq \beta_n\phi(w,x_n) + \gamma_n k_n\phi(w,x_n) + \delta_n k_n\phi(w,x_n) - \gamma_n\delta_n g(\|JT^n x_n - JS^n x_n\|)\\
&\leq \phi(w,x_n) + (k_n - 1)\phi(w,x_n) - \gamma_n\delta_n g(\|JT^n x_n - JS^n x_n\|).
\end{aligned}$$

It follows that $\gamma_n\delta_n g(\|JT^n x_n - JS^n x_n\|) \leq \phi(w,x_n) - \phi(w,z_n) + (k_n - 1)\phi(w,x_n)$. On the other hand, we have $\phi(w,x_n) - \phi(w,z_n) \leq \|x_n - z_n\|(\|x_n\| + \|z_n\|) + 2\|w\|\|Jx_n - Jz_n\|$. Hence, we have $\phi(w,x_n) - \phi(w,z_n) \to 0$ as $n \to \infty$. Using $\liminf_{n\to\infty}\gamma_n\delta_n > 0$, we find that $\lim_{n\to\infty} g(\|JT^n x_n - JS^n x_n\|) = 0$. Using the property of g, one sees that $\lim_{n\to\infty}\|JT^n x_n - JS^n x_n\| = 0$. It follows that $\lim_{n\to\infty}\|T^n x_n - S^n x_n\| = 0$. On the other hand, we have

$$\begin{aligned}
\phi(T^n x_n, S^n x_n) &= \|T^n x_n\|^2 - 2\langle T^n x_n, JS^n x_n\rangle + \|S^n x_n\|^2\\
&= \|T^n x_n\|^2 - 2\langle T^n x_n, JT^n x_n\rangle + 2\langle T^n x_n, JT^n x_n - JS^n x_n\rangle + \|S^n x_n\|^2\\
&\leq \|S^n x_n\|^2 - \|T^n x_n\|^2 + 2\|T^n x_n\|\|JT^n x_n - JS^n x_n\|\\
&\leq (\|S^n x_n\| + \|T^n x_n\|)\|S^n x_n - T^n x_n\| + 2\|T^n x_n\|\|JT^n x_n - JS^n x_n\|.
\end{aligned}$$

This implies that $\lim_{n\to\infty}\phi(T^n x_n, S^n x_n) = 0$. On the other hand, we have

$$\begin{aligned}
\phi(T^n x_n, z_n) &= \phi\Big(T^n x_n, J^{-1}\big(\beta_n Jx_n + \gamma_n JT^n x_n + \delta_n JS^n x_n\big)\Big)\\
&= \|T^n x_n\|^2 - 2\langle T^n x_n, \beta_n Jx_n + \gamma_n JT^n x_n + \delta_n JS^n x_n\rangle\\
&\quad + \|\beta_n Jx_n + \gamma_n JT^n x_n + \delta_n JS^n x_n\|^2\\
&\leq \|T^n x_n\|^2 - 2\beta_n\langle T^n x_n, Jx_n\rangle - 2\gamma_n\langle T^n x_n, JT^n x_n\rangle - 2\delta_n\langle T^n x_n, JS^n x_n\rangle\\
&\quad + \beta_n\|x_n\|^2 + \gamma_n\|T^n x_n\|^2 + \delta_n\|S^n x_n\|^2\\
&\leq \beta_n\phi(T^n x_n, x_n) + \delta_n\phi(T^n x_n, S^n x_n).
\end{aligned}$$

Hence, we have $\lim_{n\to\infty}\phi(T^n x_n, z_n) = 0$. Using Lemma 1.6, we find that $\lim_{n\to\infty}\|T^n x_n - z_n\| = 0$. Since $\|T^n x_n - p\| \le \|T^n x_n - z_n\| + \|z_n - x_n\| + \|x_n - p\|$, we find that $\lim_{n\to\infty}\|T^n x_n - p\| = 0$. Using the fact that $\|T^{n+1}x_n - p\| \le \|T^{n+1}x_n - T^n x_n\| + \|T^n x_n - p\|$. Since T is uniformly asymptotically regular, we obtain that $\lim_{n\to\infty}\|T^{n+1}x_n - p\| = 0$. Hence, we have $TT^n x_n \to p$ as $n\to\infty$. It follows that $p \in F(T)$. Furthermore, we also have $p\in F(S)$. This shows that $p\in F(T)\cap F(S)$.

Finally, we show that $p = \Pi_{F(S)\cap F(T)}x_1$. Taking the limit as $n\to\infty$ in (2.1), we obtain that $\langle p - w, Jx_1 - Jp\rangle \ge 0$, $\forall w \in F(T)\cap F(S)$. and hence $p = \Pi_{F(T)\cap F(S)}x_1$. This completes the proof.

Remark 2.2. Theorem 2.1 can be applicable to L^p, where $p \ge 1$.

Corollary 2.3. *Let E be a Hilbert space. Let C be a nonempty closed and convex subset of E. Let $T: C\to C$ be a closed asymptotically quasi-nonexpansive mapping and let $S: C\to C$ be a closed asymptotically quasi-nonexpansive mapping. Assume that T and S are uniformly asymptotically regular on C and $F(T)\cap F(S)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1}x_0, \\ z_n = \beta_n x_n + \gamma_n T^n x_n + \delta_n J S^n x_n, \\ y_n = \alpha_n x_1 + (1-\alpha_n) z_n, \\ C_{n+1} = \{z\in C_n : \|z - y_n\|^2 \le \|z - x_n\|^2 + \alpha_n\big(\|x_1\|^2 + 2\langle z, x_n - x_1\rangle\big) + (k_n - 1)M_n\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \end{cases}$$

where $M_n = \sup\{\|z - x_n\|^2 : z\in F(T)\cap F(S)\}$ for each $n\ge 1$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are real sequences in $(0,1)$ such that: $\beta_n + \gamma_n + \delta_n = 1$, $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$ and $\liminf_{n\to\infty}\gamma_n\delta_n > 0$. Then the sequence $\{x_n\}$ converges strongly to $P_{F(T)\cap F(S)}x_1$.

If both T and S are quasi-ϕ-nonexpansive, we find from Theorem 3.1 the following.

Corollary 2.4. *Let E be a uniformly convex and uniformly smooth Banach space. Let C be a nonempty closed and convex subset of E. Let $T : C \to C$ be a closed quasi-ϕ-nonexpansive mapping, and $S : C \to C$ be a closed quasi-ϕ-nonexpansive mapping with a nonempty common fixed point set. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ z_n = J^{-1}\big(\beta_n J x_n + \gamma_n J T x_n + \delta_n J S x_n\big), \\ y_n = J^{-1}\big(\alpha_n J x_1 + (1-\alpha_n) J z_n\big), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n) + \alpha_n\big(\|x_1\|^2 + 2\langle z, Jx_n - Jx_1\rangle\big)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ are real sequences in $(0,1)$ such that: $\beta_n + \gamma_n + \delta_n = 1$, $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ and $\liminf_{n\to\infty} \gamma_n \delta_n > 0$. Then the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap F(S)} x_1$.

Putting $\beta_n = 0$ and $T = S$, we have the following result.

Corollary 2.5. *Let E be a uniformly convex and uniformly smooth Banach space. Let C be a nonempty closed and convex subset of E. Let $T : C \to C$ be a closed quasi-ϕ-nonexpansive mapping with a nonempty fixed point set. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}\big(\alpha_n J x_1 + (1-\alpha_n) J T x_n\big), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n) + \alpha_n\big(\|x_1\|^2 + 2\langle z, Jx_n - Jx_1\rangle\big)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $\{\alpha_n\}$ *is a real sequence in* $(0,1)$ *such that* $\lim_{n\to\infty}\alpha_n = 0$. *Then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_{F(T)}x_1$.

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