J. Nonlinear Funct. Anal. 2016 (2016), Article ID 10

MATHRES
Journal of Nonlinear Functional Analysis
Available online at http://jnfa.mathres.org

EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF (p,q)-KIRCHHOFF TYPE SYSTEMS WITH MULTIPLE PARAMETERS

ANASS OURRAOUI

Laboratoire ANOL, ENSAH-FSO, University of Mohamed I, 60050 Oujda, Morocco

Abstract. In this paper, we establish the existence results of positive solutions for a class of nonlocal (p,q)-Laplacian systems. The main tool used in this article is the sub and supersolution approach.

Keywords. Kirchhoff system; Subsolution; Supersolution; p-Laplacian.

1. Introduction

Consider the following nonlocal elliptic system

$$
(1)\qquad
\begin{aligned}
&-M_1\left(\int_\Omega |\nabla u|^p\,dx\right)|\nabla u|^{p-2}\nabla u = \lambda_1 a(x)f(v)+\mu_1 b(x)h(u) \quad x\in\Omega,\\
&-M_2\left(\int_\Omega |\nabla v|^q\,dx\right)|\nabla v|^{q-2}\nabla v = \lambda_2 c(x)g(u)+\mu_2 d(x)\pi(v) \quad x\in\Omega,\\
&u=v=0 \quad \text{on} \quad \partial\Omega,
\end{aligned}
$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary, $p,q>1$, $\Delta_s u := div(|\nabla u|^{s-2}\nabla u), s>1$, $\lambda_1,\lambda_2,\mu_1$ and μ_2 are nonnegative parameters and f,g,h,π are functions that satisfy conditions, which will be stated later.

Because of the presence of the terms $M_1\left(\int_\Omega |\nabla u|^p dx\right)$ and $M_2\left(\int_\Omega |\nabla v|^q dx\right)$ in the two first equations in (1) so it is called nonlocal problem.

E-mail address: a.ourraoui@gmail.com

Received December 17, 2015

The study of nonlocal problems has received considerable attention in recent years. Such problems arise from models for physical phenomena like the propagation of light in some nonlinear optical materials and biological systems. We refer to the overview papers [3, 8, 17] for the advances and the references in this area.

Problem like (1) is related to a general version of the Kirchhoff equation [16],

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L |\frac{\partial u}{\partial x}|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

where ρ_0, ρ, L and h are constants associated to the effects of the changes in the length of strings during the vibrations. This induced Kirchhoff to propose a nonlinear correction of the classical D'Alembert equation.

The reader may consult, for instance, [1, 3, 4, 7, 10, 11, 12, 16, 18, 19] and the references therein for more information on nonlocal problems.

We make the following assumptions:

(M_0) For $k = 1,2$, $M_k : \ (0,\infty) \to [0,\infty)$ is increasing and continuous function with

$$0 < m_k \leq M_k(t) < \infty, \ for\ all\ t > 0.$$

(H_0) Let $a,b,c,d \in C(\overline{\Omega})$, there exist positive constants a_0, b_0, c_0, d_0 such that

$$a(x) \geq a_0 > 0, b(x) \geq b_0 > 0, c(x) \geq c_0 > 0, d(x) \geq d_0 > 0.$$

(H_1) $f,g,h,\pi \ \in \ C^1(0,\infty) \cap C[0,\infty)$ are increasing functions such that

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = \lim_{t\to\infty} h(t) = \lim_{t\to\infty} \pi(t) = \infty.$$

(H_2) $\lim_{t\to\infty} \frac{f\left(K\left[g(t)\right]^{\frac{1}{q-1}}\right)}{t^{p-1}} = 0, \ \forall\, K > 0.$

(H_3) $\lim_{t\to\infty} \frac{h(t)}{t^{p-1}} = \lim_{t\to\infty} \frac{\pi(t)}{t^{q-1}} = 0.$

Our main result reads as follows:

Theorem 1.1. *Assume that (M_0) and $(H_0) - (H_3)$ hold, then the problem (1) has positive solution for $\lambda_k + \mu_k$ are large, $k = 1,2$.. Moreover, if we assume that f,g,h and π be sufficiently* *smooth functions near to zero and satisfy*

$$f(0)=g(0)=h(0)=\pi(0)=0=f^{(k)}(0)=h^{(k)}(0)=g^{(i)}(0)=\pi^{(i)}(0) \tag{2}$$

for $k=1,2...,\mathbb{E}(p-1)$ *and* $i=1,2...,\mathbb{E}(q-1)$, *with* $\mathbb{E}(s)$ *be the integer part of* s. *Then the problem (1) has at least three weak positive solutions, provided* $\lambda_k+\mu_k$ *are large*, $k=1,2$.

Motivated by the works in [2, 14, 20] and the above mentioned papers, the aim of this paper is to use the sub and supersolution method, due to the loss of the variational structure, to show that the more general system (1) has also at least distinct three positive solutions.

2. Auxiliary results

We recall that $W=W_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|=(\int_\Omega|\nabla u|^p)^{\frac{1}{p}}$. We denote by $|.|_\infty$ the norm of $L^\infty(\Omega)$.

We consider the following classical problem

$$\begin{aligned} -div\Big(|\nabla u|^{s-2}\nabla u\Big)&=\lambda|u|^{s-2}u \quad x\in\Omega,\\ u&=0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3}$$

Let φ_s be the eigenfunction corresponding to the first eigenvalue σ_s of (3) such that $\varphi_s>0$ in Ω with $|\varphi_s|=1$ and $\frac{\partial\varphi_s}{\partial n}<0$ on $\partial\Omega$ (see [6]). Thereby, there exist positive constants m,τ,ρ such that

$$\sigma_s(\varphi_s)^s-|\nabla\varphi_s|^s\leq -m,\ x\in\overline{\Omega}_\rho,$$

$$\varphi_s\geq\tau,\ x\in\Omega\backslash\overline{\Omega}_\rho,$$

where

$$\overline{\Omega}_\rho=\{x\in\Omega\ :\ d(x,\partial\Omega)\leq\rho\}.$$

Taking $l_1>0$ such that

$$a_0f(t),\ b_0h(t),\ c_0g(t),\ d_0\pi(t)>-l_1$$

for all $t \geq 0$. Let e_s be a solution of the following problem

$$
\begin{aligned}
&-div\Big(|\nabla e_s|^{s-2}\nabla u\Big) = 1 \quad x \in \Omega, \\
&e_s = 0 \quad \text{on} \quad \partial\Omega,
\end{aligned} \tag{4}
$$

by Guedda and Véron [13], e_s is bounded and strong principle maximum gives $e_s > 0$ in Ω.

Now, we recall a proposition which plays a crucial role in the arguments. We consider

$$
\begin{aligned}
&-M_1\left(\int_\Omega |\nabla u|^p\,dx\right)|\nabla u|^{p-2}\nabla u = B_1(x,u,v) \quad x \in \Omega, \\
&-M_2\left(\int_\Omega |\nabla v|^q\,dx\right)|\nabla v|^{q-2}\nabla v = B_2(x,u,v) \quad x \in \Omega, \\
&u = v = 0 \quad \text{on} \quad \partial\Omega,
\end{aligned} \tag{5}
$$

where Ω is bounded domain in $\mathbb{R}^N$, and $B_1, B_2 :\ \Omega\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ with the following assumptions:

(C_1) $B_1(x,r,t)$ and $B_2(x,r,t)$ are Carathéodory functions and they are bounded if r,t belong to bounded sets.

(C_2) There exists a function $\chi : \mathbb{R} \to \mathbb{R}$ being continuous, nondecreasing, with $\chi(0) = 0, 0 \leq \chi(s) \leq C(1+|s|^{(min\{p,q\}-1)}$ for some $C > 0$, and applications $s \to B_1(x,s,t)+\chi(s)$ and $t \to B_2(x,s,t)+\chi(t)$ are nondecreasing, for a.e. $x \in \Omega$.

In this work, we use the following sub-supersolution principle, the proof of which is based on the well-known fixed point theorem for the increasing operator on the order interval (see e.g. [5]) and is similar to that given in [15, 9].

Proposition 2.1. Let M_1, *and* $M_2 : (0,\infty) \to [0,\infty)$ be two functions satisfying the condition (C_1). Assume that the functions B_1, B_2 satisfy the conditions (C_1) and (C_2). Assume that $(\underline{u},\overline{v})$, $(\underline{u},\overline{v})$, are respectively, a weak subsolution and a weak supersolution of system (5) with $\underline{u}(x) \leq \overline{u}(x)$ and $\underline{v}(x) \leq \overline{v}(x)$ for a.e. $x \in \Omega$. Then there exists a minimal (u_*, v_*) (and, respectively, a maximal (u^*, v^*)) weak solution for system (5) in the set $[\underline{u},\overline{u}] \times [\underline{v},\overline{v}]$. In particular, every weak solution $(u,v) \in [\underline{u},\overline{u}] \times [\underline{v},\overline{v}]$ of system (5) satisfies $u_*(x) \leq u(x) \leq u^*(x)$ and $v_*(x) \leq v(x) \leq v^*(x)$ for a.e. $x \in \Omega$. Setting

$$B_1(x,u,v) = \lambda_1 a(x) f(v) + \mu_1 b(x) h(u)$$

and

$$B_2(x,u,v) = \lambda_2 c(x) g(u) + \mu_2 d(x) \pi(v).$$

It is clear that the functions B_1 and B_2 satisfy the conditions (C_1) and (C_2) of the previous Proposition 2.1.

3. Main results

In this section, we give the detailed proof of Theorem 1.1.

Proof of Theorem 1.1. We start by constructing a positive weak subsolution $(\psi_1, \psi_2) \in W \times W$ and a supersolution $(z_1, z_2) \in W \times W$ of (1) with $\psi_k \leq z_k,\ k = 1,2$, i.e

$$\begin{aligned}
M_1\left(\int_\Omega |\nabla \psi_1|^p \, dx\right) \int_\Omega |\nabla \psi_1|^{p-2} \nabla \psi_1 \nabla \varphi \, dx &\leq \lambda_1 \int_\Omega a(x) f(\psi_2) \varphi \, dx + \mu_1 \int_\Omega b(x) h(\psi_1) \varphi \, dx, \\
M_2\left(\int_\Omega |\nabla \psi_2|^q \, dx\right) \int_\Omega |\nabla \psi_2|^{q-2} \nabla \psi_2 \nabla \varphi \, dx &\leq \lambda_2 \int_\Omega c(x) g(\psi_1) \varphi \, dx + \mu_2 \int_\Omega d(x) \pi(\psi_2) \varphi \, dx,
\end{aligned}$$

and

$$\begin{aligned}
M_1\left(\int_\Omega |\nabla z_1|^p \, dx\right) \int_\Omega |\nabla z_1|^{p-2} \nabla z_1 \nabla \varphi \, dx &\geq \lambda_1 \int_\Omega a(x) f(z_2) \varphi \, dx + \mu_1 \int_\Omega b(x) h(z_1) \varphi \, dx, \\
M_2\left(\int_\Omega |\nabla z_2|^q \, dx\right) \int_\Omega |\nabla z_2|^{q-2} \nabla z_2 \nabla \varphi \, dx &\geq \lambda_2 \int_\Omega c(x) g(z_1) \varphi \, dx + \mu_2 \int_\Omega d(x) \pi(z_2) \varphi \, dx,
\end{aligned}$$

for all positive $\varphi \in C_0^\infty(\Omega),\ \varphi \geq 0$ in Ω.

We split the proof into four steps.

<u>**Step 1**</u> : We shall verify that

$$(\psi_1, \psi_2) = \left(\left(\frac{(\lambda_1 + \mu_1) l_1}{m m_1} \right)^{\frac{1}{p-1}} \left(\frac{p-1}{p} \right) \varphi_p^{\frac{p}{p-1}}, \left(\frac{(\lambda_2 + \mu_2) l_1}{m m_2} \right)^{\frac{1}{q-1}} \left(\frac{q-1}{q} \right) \varphi_q^{\frac{q}{q-1}} \right)$$

is a subsolution of (1). Thus, from (H_0) we have

$$\begin{aligned}
&M_1\left(\int_\Omega |\nabla\psi_1|^p\,dx\right)\int_\Omega |\nabla\psi_1|^{p-2}\nabla\psi_1\nabla\varphi\,dx\\
&=\frac{(\lambda_1+\mu_1)l_1}{mm_1}M_1\left(\int_\Omega |\nabla\psi_1|^p\,dx\right)\int_\Omega |\nabla\varphi_p|^{p-2}\varphi_p\,\nabla\varphi_p\,\nabla\varphi\,dx\\
&=\frac{(\lambda_1+\mu_1)l_1}{mm_1}M_1\left(\int_\Omega |\nabla\psi_1|^p\,dx\right)\int_\Omega |\nabla\varphi_p|^{p-2}\,\nabla(\varphi_p\;\varphi)\,\nabla\varphi_p-\int_\Omega |\nabla\varphi_p|^p\varphi\,dx\\
&=\frac{(\lambda_1+\mu_1)l_1}{mm_1}M_1\left(\int_\Omega |\nabla\psi_1|^p\,dx\right)\int_\Omega \Big[\sigma_p(\varphi_p)^p-|\nabla\varphi_p|^p\Big]\varphi\,dx. \qquad (6)
\end{aligned}$$

If $(\lambda_1+\mu_1)$ and $(\lambda_2+\mu_2)$ large enough in the definition of ψ_1,ψ_2, so by (H_1),

$$a_0f(\psi_2),\ b_0h(\psi_1),\ c_0g(\psi_1), d_0\psi_2\geq \frac{l_1}{m}\max\left\{\frac{|M_1|_{L^\infty(0,\infty)}}{m_1}\sigma_p,\frac{|M_2|_{L^\infty(0,\infty)}}{m_2}\sigma_q\right\}. \qquad (7)$$

Note that from some results on Auxiliary results section, we have in Ω_ρ,

$$\sigma_p(\varphi_p)^p-|\nabla\varphi_p|^p\leq -m<0.$$

From (7) and for $\lambda_1+\mu_1$ large enough, it yields

$$\begin{aligned}
&\frac{(\lambda_1+\mu_1)l_1}{mm_1}M_1\left(\int_\Omega |\nabla\psi_1|^p\,dx\right)\int_\Omega (\sigma_p(\varphi_p)^p-|\nabla\varphi_p|^p)\,\varphi\,dx\\
&\leq\frac{(\lambda_1+\mu_1)l_1}{mm_1}M_1\left(\int_\Omega |\nabla\psi_1|^p\,dx\right)\int_{\Omega_\rho} (\sigma_p(\varphi_p)^p-|\nabla\varphi_p|^p)\,\varphi\,dx\\
&+\frac{(\lambda_1+\mu_1)l_1}{mm_1}M_1\left(\int_\Omega |\nabla\psi_1|^p\,dx\right)\int_{\Omega\setminus\Omega_\rho} (\sigma_p(\varphi_p)^p-|\nabla\varphi_p|^p)\,\varphi\,dx\\
&\leq -(\lambda_1+\mu_1)l_1\int_{\Omega_\rho}\varphi\,dx+\frac{(\lambda_1+\mu_1)l_1}{mm_1}|M_1|_{L^\infty(0,\infty)}\int_{\Omega\setminus\Omega_\rho}\sigma_p\varphi\,dx\\
&\leq\int_{\Omega_\rho}\Big[\lambda_1a_0f(\psi_2)+\mu_1b_0h(\psi_1)\Big]\varphi+\int_{\Omega\setminus\Omega_\rho}\Big[\lambda_1a_0f(\psi_2)+\mu_1b_0h(\psi_1)\Big]\varphi\,dx\\
&=\int_\Omega\Big[\lambda_1a_0f(\psi_2)\,dx+\mu_1b_0h(\psi_1)\Big]\varphi\,dx\\
&\leq\int_\Omega\Big[\lambda_1a(x)f(\psi_2)\,dx+\mu_1b(x)h(\psi_1)\Big]\varphi\,dx. \qquad (8)
\end{aligned}$$

Similarly, for $(\lambda_2+\mu_2)$ large enough, we obtain

$$\begin{aligned}
&\frac{(\lambda_2+\mu_2)l_1}{mm_2}M_2\left(\int_\Omega |\nabla\psi_2|^q\,dx\right)\int_\Omega \big(\sigma_q(\varphi_q)^q-|\nabla\varphi_q|^q\big)\,\varphi\,dx\\
&\leq\int_\Omega\Big[\lambda_2c(x)g(\psi_1)+\mu_2d(x)\pi(\psi_2)\Big]\varphi\,dx,
\end{aligned}$$

i.e., (ψ_1,ψ_2) is a subsolution of (1).

Step 2: Setting

$$(z_1,z_2)=\left(\left(\frac{1}{m_1}\right)^{\frac{1}{p-1}}Ce_p,\left[\frac{(\lambda_2|c|_\infty+\mu_2|d|_\infty)}{m_2}g\left(\left(\frac{1}{m_1}\right)^{\frac{1}{p-1}}C|e_p|_\infty\right)\right]^{\frac{1}{q-1}}e_q\right),$$

with $e_s, s=p,q$ is defined in the section 2, as solution of (4), so we prove that (z_1,z_2) is a supersolution of (1).

Firstly, for $\varphi\in C_0^\infty(\Omega),\ \varphi\geq 0$, we have

$$\begin{aligned}
& M_1\left(\int_\Omega|\nabla z_1|^p\,dx\right)\int_\Omega|\nabla z_1|^{p-2}\nabla z_1.\nabla\varphi\,dx\\
=\ & \frac{1}{m_1}C^{p-1}M_1\left(\int_\Omega|\nabla z_1|^p\,dx\right)\int_\Omega|\nabla e_p|^{p-2}\nabla e_p.\nabla\varphi\,dx\\
\geq\ & \frac{1}{m_1}C^{p-1}m_1\int_\Omega\varphi\,dx\\
=\ & C^{p-1}\int_\Omega\varphi\,dx. \qquad (9)
\end{aligned}$$

By the condition (H_2) we can choose C large enough so that

$$\begin{aligned}
C^{p-1}\ \geq\ & \lambda_1|a|_\infty f\left(\left[\frac{(\lambda_2|c|_\infty+\mu_2|d|_\infty)}{m_2}g\left(\left(\frac{1}{m_1}\right)^{\frac{1}{p-1}}C|e_p|_\infty\right)\right]^{\frac{1}{q-1}}e_q\right)+\mu_1|b|_\infty h\left(\left(\frac{1}{m_1}\right)^{\frac{1}{p-1}}C|e_p|_\infty\right)\\
(10)\quad & \geq\lambda_1 a(x)f(z_2)+\mu_1 b(x)h(z_1).
\end{aligned}$$

It follows from (9) and (10) that

$$M_1\left(\int_\Omega|\nabla z_1|^p\,dx\right)\int_\Omega|\nabla z_1|^{p-2}\nabla z_1.\nabla\varphi\,dx\geq\lambda_1\int_\Omega a(x)f(z_2)\varphi\,dx+\mu_1\int_\Omega b(x)h(z_1)\varphi\,dx$$

with $\varphi\in C_0^\infty(\Omega),\ \varphi\geq 0$ in Ω.

On the other hand we have

$$\begin{aligned}
M_2\left(\int_\Omega|\nabla z_2|^q\,dx\right)\int_\Omega|\nabla z_2|^{q-2}\nabla z_2.\nabla\varphi\,dx\ =\ & \left(\frac{(\lambda_2|c|_\infty+\mu_2|d|_\infty)}{m_2}g\left(\left(\frac{1}{m_1}\right)^{\frac{1}{p-1}}C|e_p|_\infty\right)\right)\times\\
& M_2\left(\int_\Omega|\nabla|z_1|^p\,dx\right)\int_\Omega|\nabla e_q|^{q-2}\nabla e_q.\nabla\varphi\,dx\\
\geq\ & (\lambda_2|c|_\infty+\mu_2|d|_\infty)g\left(\left(\frac{1}{m_1}\right)^{\frac{1}{p-1}}C|e_p|_\infty\right)\int_\Omega\varphi\,dx.
\end{aligned}$$

Now, in view of (H_3) for C large enough, we have

$$g\left(\left(\frac{1}{m_1}\right)^{\frac{1}{p-1}}C|e_p|_\infty\right)\geq\pi\left(\left[\lambda_2|c|_\infty+\mu_2|d|_\infty)g\left(\left(\frac{1}{m_1}\right)^{\frac{1}{p-1}}C|e_p|_\infty\right)\right]^{\frac{1}{q-1}}|e_q|_\infty\right).$$

It follows that

$$\begin{aligned} M_2\left(\int_\Omega |\nabla z_2|^q\,dx\right)\int_\Omega |\nabla z_2|^{q-2}\nabla z_2.\nabla\varphi\,dx &\geq \int_\Omega (\lambda_2|c|_\infty+\mu_2|d|_\infty)g\Big(\Big(\frac{1}{m_1}\Big)^{\frac{1}{p-1}}C|e_p|_\infty\Big)\varphi\,dx \\ &\geq \int_\Omega \lambda_2|c|_\infty g\Big(\Big(\frac{1}{m_1}\Big)^{\frac{1}{p-1}}C|e_p|_\infty\Big)\varphi\,dx+ \\ &\quad \int_\Omega \mu_2|d|_\infty g\Big(\Big(\frac{1}{m_1}\Big)^{\frac{1}{p-1}}C|e_p|_\infty\Big)\varphi\,dx \\ &\geq \int_\Omega \lambda_2 c(x)g(z_1)\varphi\,dx+\int_\Omega \mu_2 d(x)\pi(z_2)\varphi\,dx, \end{aligned}$$

where $\varphi\in C_0^\infty(\Omega),\ \varphi\geq 0$ in Ω. Thus, (z_1,z_2) is a super-solution of problem (1).

The fact that $\psi_i\leq z_i$ for C large enough, $i=1,2$, according to Proposition 2.1, the problem (1) has a positive solution.

Step 3: Let us consider the following problem

$$\begin{aligned} -M_1\left(\int_\Omega |\nabla \underline{u}|^p\,dx\right)|\nabla \underline{u}|^{p-2}\nabla \underline{u} &= \lambda_1 a(x)f(\underline{v})+\mu_1 b(x)h(\underline{u})-\lambda_1-\mu_1 \quad x\in\Omega, \\ -M_2\left(\int_\Omega |\nabla \underline{v}|^q\,dx\right)|\nabla \underline{v}|^{q-2}\nabla \underline{v} &= \lambda_2 c(x)g(\underline{u})+\mu_2 d(x)\pi(\underline{v})-\lambda_2-\mu_2 \quad x\in\Omega, \\ \underline{u}=\underline{v}&=0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

Taking

$$f_1(t)=f(t)-1, g_1(t)=g(t)-1, h_1(t)=h(t)-1 \text{ and } \pi_1(t)=\pi(t)-1,$$

we have

$$\begin{aligned} -M_1\left(\int_\Omega |\nabla \underline{u}|^p\,dx\right)|\nabla \underline{u}|^{p-2}\nabla \underline{u} &= \lambda_1 a(x)f_1(\underline{v})+\mu_1 b(x)h_1(\underline{u}) \quad x\in\Omega, \\ -M_2\left(\int_\Omega |\nabla \underline{v}|^q\,dx\right)|\nabla \underline{v}|^{q-2}\nabla \underline{v} &= \lambda_2 c(x)g_1(\underline{u})+\mu_2 d(x)\pi_1(\underline{v}) \quad x\in\Omega, \\ \underline{u}=\underline{v}&=0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{11}$$

then by the previous remark in Step 2, problem (11) has a positive solution when $\lambda_1+\mu_1$ and $\lambda_2+\mu_2$ are large. We observe that solution $(\underline{u},\underline{v})$ is a strict subsolution of (1).

In what follows, one takes $(\psi_1,\psi_2)=(0,0)$ as a subsolution of (1) (because it is a solution of (1)).

Step 4: We construct a strict supersolution $(\overline{u},\overline{v})$ of (1).

Let

$$(\overline{u},\overline{v}) = \Big(\Big(\frac{1}{m_1}\Big)^{\frac{1}{p-1}}\varepsilon\varphi_p, \Big(\frac{1}{m_2}\Big)^{\frac{1}{q-1}}\varepsilon\varphi_q\Big),$$

with $\varepsilon > 0$.

Note that there exist two positive constants c_1 and c_2 such that

$$\varphi_p \leq c_1\varphi_q$$

and

$$\varphi_q \leq c_2\varphi_p.$$

We pose

$$\mathscr{A}_p(t) = m_1\sigma_p t^{p-1} - \lambda_1 a(x) f\Big(\Big(\frac{(m_1)^{\frac{1}{p-1}}}{(m_2)^{\frac{1}{q-1}}}c_2 t\Big)\Big) - \mu_1 b(x)h(t)$$

and

$$\mathscr{A}_q(t) = m_2\sigma_q t^{q-1} - \lambda_2 c(x) g\Big(\Big(\frac{(m_2)^{\frac{1}{q-1}}}{(m_1)^{\frac{1}{p-1}}}c_1 t\Big) - \mu_2 d(x)\pi(t).$$

According to assumption (2), it is easy to verify that there exists $\nu > 0$ such that

$$\mathscr{A}_p(t),\ \mathscr{A}_q(t) > 0\ for\ 0 < t \leq \nu.$$

Thereby, for $0 < \varepsilon < \frac{\nu}{|\nu|_\infty}\min\Big\{(m_1)^{\frac{1}{p-1}}, (m_2)^{\frac{1}{q-1}}\Big\}$, we obtain

$$\mathscr{A}_p(\overline{u}) = \sigma_p(\varepsilon\varphi_p)^{p-1} - \lambda_1 a(x) f\left(c_2\Big(\frac{1}{m_2}\Big)^{\frac{1}{q-1}}\varepsilon\varphi_p\right) - \mu_1 b(x)h\left(\Big(\frac{1}{m_1}\Big)^{\frac{1}{p-1}}\varepsilon\varphi_p\right) > 0,$$

(12)

keeping this in mind with the fact that

$$\begin{aligned}
\lambda_1 a(x) f\left(\Big(\frac{1}{m_2}\Big)^{\frac{1}{q-1}}c_2\varepsilon\varphi_p\right) + \mu_1 b(x)h\left(\Big(\frac{1}{m_1}\Big)^{\frac{1}{p-1}}\varepsilon\varphi_p\right) &\geq \lambda_1 a(x) f\left(\Big(\frac{1}{m_2}\Big)^{\frac{1}{q-1}}\varepsilon\varphi_q\right) \\
&\quad + \mu_1 b(x)h\left(\Big(\frac{1}{m_1}\Big)^{\frac{1}{p-1}}\varepsilon\varphi_p\right) \\
&= \lambda_1 a(x) f(\overline{v}) + \mu_1 b(x)h(\overline{u})
\end{aligned}$$

we have

$$\sigma_p(\varepsilon\varphi_p)^{p-1} > \lambda_1 a(x) f(\overline{v}) + \mu_1 b(x)h(\overline{u}).$$

We also have

$$\mathscr{A}_q(\overline{v}) = \sigma_q(\varepsilon\varphi_q)^{q-1} - \lambda_2 c(x) g\left(c_1\left(\frac{1}{m_1}\right)^{\frac{1}{p-1}}\varepsilon\varphi_q\right) - \mu_2 d(x)\pi\left(\left(\frac{1}{m_2}\right)^{\frac{1}{q-1}}\varepsilon\varphi_q\right) > 0.$$

(13)

It follows that

$$\sigma_q(\varepsilon\varphi_q)^{q-1} > \lambda_2 c(x) g(\overline{u}) + \mu_2 d(x)\pi(\overline{v}).$$

Hence,

$$\begin{aligned} M_1\Big(\int_\Omega |\nabla\overline{u}|^p\,dx\Big)\int_\Omega |\nabla\overline{u}|^{p-2}\nabla\overline{u}\nabla\varphi\,dx &\geq m_1\frac{\varepsilon^{p-1}}{m_1}\int_\Omega |\nabla\varphi_p|^{p-2}\nabla\varphi_p.\nabla\varphi\,dx \\ &= \int_\Omega \sigma_p(\varepsilon\varphi_p)^{p-1}\varphi\,dx \\ &> \int_\Omega \lambda_1 a(x) f(\overline{v})\varphi\,dx + \int_\Omega \mu_1 b(x) h(\overline{u})\varphi\,dx. \end{aligned}$$

Similarly,

$$M_2\Big(\int_\Omega |\nabla\overline{v}|^q\,dx\Big)\int_\Omega |\nabla\overline{v}|^{q-2}\nabla\overline{v}\nabla\varphi\,dx > \int_\Omega \lambda_2 c(x) g(\overline{u})\varphi\,dx + \int_\Omega \mu_2 d(x)\pi(\overline{v})\varphi\,dx.$$

Finally, we choose ε small enough in order to have $(\underline{u},\underline{v}) \nleq (\overline{u},\overline{v})$. Therefore, form Proposition 2.1, the problem (1) admits three weak solutions

$$(u_1,v_1) \in [(\psi_1,\psi_2),(\overline{u},\overline{v})],\ (u_2,v_2) \in [(\underline{u},\underline{v}),(z_1,z_2)]$$

and

$$(u_3,v_3) \in [(\psi_1,\psi_2),(z_1,z_2)]\diagdown\Big([(\psi_1,\psi_2),(\overline{u},\overline{v})] \cup [(\underline{u},\underline{v}),(z_1,z_2)]\Big).$$

Thus the proof of Theorem 1.1 is completed.

REFERENCES

[1] M. Allaoui, A Ourraoui, Existence results for a class of p(x)- Kirchhoff problem with a singular weight, Mediterranean J. Math. doi: 10.1007/s00009-015-0518-2.

[2] J. Ali, R. Shivaji, Positive solutions for a class of p-Laplacian systems with multiple parameters, J. Math. Anal. Appl. 335 (2007), 1013-1019.

[3] C.O. Alves, F.J.S.A. Corrêa and T.F. Ma, Positive solutions for a quasilinear elliptic equation of Kirchhoff type, Comput. Math. Appl. 49 (2005), 85-93.

[4] C.O. Alves, D.P. Covei, Existence of solution for a class of nonlocal elliptic problem via sub-supersolution method, Nonlinear Anal. doi: 10.1016/j.nonrwa.2014.11.003.

[5] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. SIAM Rev. 18 (1976), 620-709.

[6] A. Anane, Simplicité et isolation de la premi‘ere valeur propre du p- laplacien avec poids C.R. Acad. Sci. Paris. t. 305, Série I. (1987), 725-728.

[7] S. Aouaoui, On some nonlocal problem involving the N-Laplacian in $\mathbb{R}^N$, Nonlinear Stud. 57-70.

[8] Chipot, B. Lovat, Some remarks on nonlocal elliptic and parabolic problems, Nonlinear Anal., 30 (1997), 4619-4627.

[9] N.T. Chung, An existence result for a class of Kirchhoff type systems via sub and supersolutions method, Appl. Math. Lett. 35 (2014), 95-101.

[10] F. Colasuonno, P. Pucci, Multiplicity of solutions for $p(x)$-polyharmonic elliptic Kirchhoff equations, Nonlinear Anal. 74 (2011), 5962-5974.

[11] M. G. Figueiredo, Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument, J. Math. Anal. Appl. 401 (2013), 706-713.

[12] G.M. Figueiredo, J.A. Santos, On a Φ-Kirchhoff multivalued problem with critical growth in an Orlicz-Sobolev space, Asymptotic Anal. 89 (2014), 151-172.

[13] M. Guedda, L. Vron, Quasilinear elliptic equations involving critical Sobolev exponents, Nonlinear Anal. 13 (1989), 879-902.

[14] D.D. Hai, R. Shivaji, An existence result on positive solutions for a class of p-Laplacian systems, Nonlinear Anal. 56 (2004), 1007-1010.

[15] X. Han and G. Dai, On the sub-supersolution method for $p(x)-$Kirchhoff type equations, J. Inequal. Appl. 2012 (2012), 283.

[16] G. Kirchhoff, Mechanik, Teubner, Leipzig, 1883.

[17] D.L. Mills, Nonlinear Optics, Springer, 1998.

[18] A. Ourraoui, On a class of nonlocal problem involving a critical exponent, Commun. Math. 23 (2015), 47-55.

[19] A. Ourraoui, On a class of nonlocal $p(x)$-Laplacian involving no-flux boundary problem, preprint.

[20] S.H. Rasouli, On the existence of positive solution for class of nonlinear elliptic systems with multiple parameters and singular weights, Commun. Korean Math. Soc. 27 (2012), 557-564.