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IMPROVED CONVERGENCE ANALYSIS OF MIXED SECANT METHODS FOR PERTURBED SUBANALYTIC VARIATIONAL INCLUSIONS

IOANNIS K. ARGYROS[1], DANIEL GONZÁLEZ[2,*]

[1]Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA

[2]Departamento de Matemáticas, Escuela Politécnica Nacional, Quito, Ecuador

Abstract. We present a local convergence analysis of mixed secant method in order to approximate solutions of variational inclusions. The local convergence results are given under weaker conditions than in earlier studies such as [4], resulting to a more precise error analysis and the extension of the applicability of the method. A numerical example illustrating the advantages of our approach is also presented in this study.

Keywords. Mixed secant method; Local convergence; Subanalytic functions; Variational inclusion.

1. Introduction

In this study we are concerned with the problem of approximating a solution x^* of the equation

$$F(x) = 0,$$

$$o \in f(x) + g(x) + F(x), \tag{1}$$

where f is a locally Lipschitz and subanalytic function, g is a Lipschitz function and F is a set–valued function on $\mathbb{R}^n$.

Many problems in Computational Sciences and other disciplines can be brought in form (1) using Mathematical Modelling [1, 2, 5, 6, 7, 11, 13, 14, 15, 16, 17, 18]. In particular, a

*Corresponding author.

E-mail address: ioannisa@cameron.edu (I. K. Argyros), daniel.gonzalezs@epn.edu.ec (D. González)

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large number of problems in Applied Mathematics and also in Engineering such Mathematical Programming and Control problems can be modelled like (1). The unknowns of engineering equations can be functions (difference, differential and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequence converges to an optimal solution of the problem at hand. Except in special cases, the solutions of these equations cannot be found in closed form. That is why most commonly used solution methods for these equations are iterative. In particular, the practice of Numerical Analysis for finding such solutions is essentially connected to variants of Newton's method.

We prove the existence of a sequence $\{x_k\}$ satisfying for each $k = 1, 2, \dots$

$$o \in f(x_k) + g(x_k) + (\Delta f(x_k) + [x_{k+1}, x_k; g])(x_{k+1} - x_k) + F(x_{k+1}), \tag{2}$$

where x_0, x_1 are initial points, $\Delta f(x_k)$ belongs to $\partial^o f(x_k)$, the Clarke Jacobian of f at the point x_k [6] and $[x_{k-1}, x_k; g]$ is a divided difference of order one for g at the points x_{k-1} and x_k (to be precise in Section 2).

The study about convergence matter of iterative methods is usually centered on to types: semi–local and local convergence analysis. The semi–local convergence analysis is based on the information around an initial point, to give convergence conditions guaranteeing the convergence of the iterative process; while the local one is, based on the information around a solution to find estimates of the radii of the convergence balls.

There is a plethora on local as well as semilocal convergence results on specializations of method (2). We refer the reader to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18] and the references therein for this type of results. In the case of local convergence, our convergence ball is larger and the ratio of convergence smaller than before [4, 5, 8, 9, 16, 17]. These advantages are also obtained under weaker hypotheses. This type of improved convergence results are important in Computational Mathematics, since this way we have a wider choice of initial guess and we compute less iterates in order to obtain a desired error tolerance.

The paper is organized as follows. In Section 2, to make the paper as selfcontained as possible, we present some mathematical background on divided differences taken from [1] and some results on subanalytic functions and set valued functions that can be found, e.g., in [1, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. The local convergence analysis of method (2) is given in Section 3. Finally, the numerical examples illustrating the theoretical results are given in Section 4.

2. Preliminaries

First, we shall make use of:

- The distance from a point x to a set A in the metric space (Z,ρ) is defined by:

$$dist(x,A) = \inf\{\rho(x,y), y \in A\}.$$

- The excess e from the set A to the set C is given by:

$$e(A,C) = \sup\{dist(x,A), x \in C\}.$$

- Let $\Lambda :\rightrightarrows Y$ be a set-valued map, we denote by:

$$\text{graph}\Lambda = \{(x,y) \in X \times Y, y \in \Lambda(x)\} \text{ and } \Lambda^{-1}(y) = \{x \in X, y \in \Lambda(x)\}.$$

- $\overline{U}(x,r)$ is the closed ball centered at x with radius r.
- The norm is denoted by $\|\cdot\|$.

We shall use first and second order divided differences in the study of the local convergence, analysis of method (2). For more information, refer the readers to [1].

Definition 2.1. An operator $[x_0, y_0; g] \in \mathscr{L}(X,Y)$ is called a divided difference of the first order of the function $g : X \to Y$ at the points x_0 and y_0 if both following conditions are satisfied:

$a)$ $[x_0, y_0; g](y_0 - x_0) = g(y_0) - g(x_0)$ for $x_0 \neq y_0$.

$b)$ If g is Fréchet differentiable at $x_0 \in X$ then we denote $[x_0, x_0; g] = \nabla g(x_0)$.

Definition 2.2. An operator $[x_0,y_0,z_0;g] \in \mathscr{L}(X,\mathscr{L}(X,Y))$ is called a divided difference of second order of the function $g : X \to Y$ at the points x_0,y_0,z_0 if both following conditions are satisfied:

a) $[x_0,y_0,z_0;g](z_0-x_0) = [y_0,z_0;g]-[x_0,y_0;g]$ for x_0,y_0 and z_0 distincts.

b) If g admits a second order Fréchet derivative at $x_0 \in X$ then denote $[x_0,x_0,x_0;g] = \frac{\nabla^2 g(x_0)}{2}$.

The rest of the section contains wellknown basic concepts and results on semianalytic, subanalytic and pseudo-Lipschitz functions [1, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18].

Definition 2.3. A subset X of $\mathbb{R}^n$ is semianalytic if for each $a \in \mathbb{R}^n$ there is a neighborhood U of a and real analytic functions $f_{i,j}$ on U such that

$$X \cap U = \bigcup_{i=1}^{r} \bigcap_{j=1}^{s_i} \{x \in U | f_{i,j} \varepsilon_{i,j} 0\},$$

where $\varepsilon_{i,j} \in \{>,<,=\}$.

Definition 2.4. A subset X of $\mathbb{R}^n$ is subanalytic if each point $a \in \mathbb{R}^n$ admits a neighborhood U such that $X \cap U$ is a projection of a relatively compact semianalytic set: there is a semianalytic bounded set A in $\mathbb{R}^{n+p}$ such that $X \cap U = \Pi(A)$ where $\Pi : \mathbb{R}^{n+p} \to \mathbb{R}^n$ is the projection.

Definition 2.5. Let X be a subset of $\mathbb{R}^n$. A function $f : X \to \mathbb{R}^m$ is semianalytic (resp. subanalytic) if its graph is semianalytic (resp. subanalytic).

Proposition 2.6. (Canonical approximation). If $f : X \subset \mathbb{R}^n \to \mathbb{R}^n$ is a subanalytic locally Lipschitz mapping, then for all $x \in X$

$$\|f(x+d) - f(x) - f'(x;d)\| = o_x(\|d\|).$$

Remark 2.7. [18] The subanalytic function $t \to o_x(t)$ admits a Puiseux development; so there exist a constant $c > 0$, a real number $\varepsilon > 0$ and a rational number $\gamma > 0$ such that $\|f(x+d) - f(x) - f'(x;d)\| = c\|d\|^\gamma$ whenever $\|d\| \leq \varepsilon$.

Definition 2.8. Let $f:\mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz continuous function. The limiting Jacobian of f at $x \in \mathbb{R}^n$ is defined as

$$\partial f(x) = \{A \in \mathscr{L}(\mathbb{R}^n, \mathbb{R}^m) : \exists u^k \in D; f'(u^k) \to A, k \to +\infty\},$$

where D denotes the points of differentiability of f. The Clarke Jacobian of f at $x \in \mathbb{R}^n$ denoted $\partial^o f(x)$ is the subset of X^* dual of X, defined as the closed convex hull of $\partial f(x)$ (see [6] page 70).

As a result of the Lipschitz property of f, the Clarke Jacobian is a nonempty compact convex set.

Moreover, we have the proposition.

Proposition 2.9. [3] Let $f:\mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitz and subanalytic, there exists a positive rational number γ such that:

$$\|f(y) - f(x) - \Delta(y)(y-x)\| \le C_x \|y-x\|^{1+\gamma}, \tag{3}$$

where y is close to x, $\Delta(y)$ is any element of $\partial^o f(y)$ and C_x is a positive constant.

The following result is a corollary of Proposition 2.9.

Corollary 2.10. *Let x^* be a solution of (1). Let $f:\mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitz and subanalytic. Then, there exists a positive rational number γ such that:*

$$\|f(x) - f(x^*) - \Delta(x)(x-x^*)\| \le C_* \|x-x^*\|^{1+\gamma}, \tag{4}$$

where y is close to x^*, $\Delta(x)$ is any element of $\partial^o f(x)$ and C_* is a positive constant.

Remark 2.11. It is worth noticing that (3) implies (4) but not necessarily viceversa, even if $C_x = C_*$. The proofs of the results in the literature for method (2) or the specialization use (3) instead of the more precise, weaker and really needed (4) [4]. Hence, our results using (4) instead of (3) extend the applicability of method (2) under weaker hypothesis. Moreover, we have that

$$C_* \le C_x^* = \max\{C_x : x \in D\} \tag{5}$$

and

$$\frac{C_x^*}{C_*}$$

can be arbitrarily large [1] (see also the example in Section 4).

Definition 2.12. [13] A set-valued map F is pseudo-Lipschitz around $(x_0, y_0) \in \text{graph}F$ with constant M if there exist constants a and b such that

$$\sup_{z \in F(y') \cap \overline{U}(y_0, a)} \text{dist}(z, F(y'')) \leq M\|y' - y''\|, \text{ for all } y' \text{ and } y'' \text{ in } \overline{U}(x_0, b). \tag{6}$$

Using the excess, the inequality (6) can be replaced by the following

$$e(F(y') \cap \overline{U}(y_0, a), F(y'')) \leq M\|y' - y''\|, \text{ for all } y' \text{ and } y'' \text{ in } \overline{U}(x_0, b). \tag{7}$$

Lemma 2.13. [13] *Let (Z, ρ) be a complete metric space, let ϕ be a set-valued map from Z into the closed subsets of Z, let $\eta_0 \in Z$ and let r and λ be such that $0 \leq \lambda < 1$ and*

(a) *$dist(\eta_0, \phi(\eta_0)) \leq r(1 - \lambda)$,*

(b) *$e(\phi(x_1) \cap \overline{U}(\eta_0, r), \phi(x_2)) \leq \lambda \rho(x_1, x_2), \forall x_1, x_2 \in \overline{U}(\eta_0, r)$,*

then ϕ has a fixed-point in $\overline{U}(\eta_0, r)$. That is, there exists $x \in \overline{U}(\eta_0, r)$ such that $x \in \phi(x)$. If ϕ is single-valued, then x is the unique fixed point of ϕ in $\overline{U}(\eta_0, r)$.

Definition 2.14. [13] A set valued map $F : X \rightrightarrows Y$ is metrically regular at $(x_0, y_0) \in \text{graph}F$ if there exist constants a, b and κ such that

$$\text{dist}(x, F^{-1}(y)) \leq \kappa \text{dist}(y, F(x)), \quad \forall x \in \overline{U}(x_0, a), y \in \overline{U}(y_0, b). \tag{8}$$

The regularity modulus of F denoted by $\text{Reg}F(x_0, y_0)$ is the infimum of all the values of κ for which (8) holds.

Proposition 2.15. [14] Let $F : X \rightrightarrows Y$ be a set-valued map and $(x_0, y_0) \in \text{graph}F$. F is metrically regular around (x_0, y_0) with constant κ if and only if F^{-1} is pseudo-Lipschitz around (y_0, x_0) with the same constant κ.

The following (H) assumptions on a neighborhood $D \subset \mathbb{R}^n$ of x^* shall be used in our local convergence analysis of method (2):

(H_1) $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz and subanalytic,

(H_2) g is differentiable at x^*,

(H_3) $\exists K_2 > 0, \forall x, y$ and $z \in D, \|[x,y,z;g]\| \leq K_2$,

(H_4) $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued map with closed graph,

(H_5) $\forall \Delta f(x^*) \in \partial^o f(x^*)$, the set-valued map $[f(x^*) + g(\cdot) + \Delta f(x^*)(\cdot - x^*) + F(\cdot)]$ is metrically regular around $(x^*, 0)$ with constant L such that $2L(2K_2 + K_1) < 1$.

Remark 2.16.

i) (H_1) implies that there exists $K_1 > 0$ such that for all $x \in D$ and $\Delta f(x) \in \partial^o f(x)$ we have $\|\Delta f(x)\| \leq K_1$.

ii) Using [8], we can show that the metric regularity of the set-valued map $[f(x^*) + g(\cdot) + \Delta f(x^*)(\cdot - x^*) + F(\cdot)]$ is equivalent to the one of $[f(\cdot) + g(\cdot) + F(\cdot)]$, but the constants are not the same.

It is also convenient for the local convergence analysis that follows to define the following functions:

$$P(x) = f(x^*) + g(x) + \Delta f(x^*)(x - x^*) + F(x). \tag{9}$$

For all $k \geq 1$

$$\begin{aligned} Z_k(x) &= f(x^*) + g(x) + \Delta f(x^*)(x - x^*) - f(x_k) - g(x_k) \\ &\quad -(\Delta f(x_k) + [x_{k-1}, x_k; g])(x - x_k), \end{aligned} \tag{10}$$

$$\Phi_k(x) = P^{-1}(Z_k(x)). \tag{11}$$

3. Local convergence analysis

The main result of this study is:

Theorem 3.1. *Let x^* be a solution of (1), and suppose that the (H) conditions are satisfied. Then, there exists a positive constant C_* such that for all $C > C_0 = \frac{L(C_*+K_2)}{1-2L(2K_2+K_1)}$, one can find $\delta > 0$ such that for every starting points $x_0, x_1 \in \overline{U}(x^*, \delta)$ (with $x_0 \neq x^*, x_1 \neq x^*$), there exists a* *sequence* $\{x_k\}_{k\geq 0}$ *defined by (2) which satisfies:*

$$\|x_{k+1}-x^*\| \leq C\|x_k-x^*\| \max\{\|x_k-x^*\|^{\gamma}, \|x_{k-1}-x^*\|\} \tag{12}$$

where γ *is a rational positive number.*

The result is shown in [4] but using (4) instead of (3).

To prove this theorem, we prove the two following propositions:

Proposition 3.2. Under the assumptions of Theorem , the map P^{-1}, inverse of P given by (9) is pseudo-Lipschitz around $(0,x^*)$.

Proof. Using assumption (H_5) and Proposition , since P is metrically regular around $(x^*,0)$ with constant L, then P^{-1} is pseudo-Lipschitz around $(0,x^*)$ with the constant L.

Proposition 3.3. Under the assumptions of Theorem , there exists $\delta > 0$ such that for all $x_0, x_1 \in \overline{U}(x^*,\delta)$ (with $x_0 \neq x^*$ and $x_1 \neq x^*$), the map Φ_1 admits a fixed point $x_2 \in \overline{U}(x^*,\delta)$.

Proof. For the proof of this proposition, we prove that both assertions (a) and (b) of Lemma hold.

Since P^{-1} is pseudo-Lipschitz around $(0,x^*)$, there exist constants a and b such that

$$e(P^{-1}(y')\cap\overline{U}(x^*,a), P^{-1}(y'')) \leq L\|y'-y''\|, \text{ for all } y' \text{ and } y'' \text{ in } \overline{U}(0,b) \tag{13}$$

and choose $\delta > 0$ verifying

$$\delta < \min\left\{a, \sqrt{\frac{b}{8K_2}}, \sqrt[1+\gamma]{\frac{b}{2C_*(1+2^{1+\gamma})}}, \frac{1}{C}, \sqrt[\gamma]{\frac{1}{C}}\right\}. \tag{14}$$

From the definition of the excess e, we have

$$dist(x^*,\Phi_1(x^*)) \leq e(P^{-1}(0)\cap\overline{U}(x^*,\delta), \Phi_1(x^*)). \tag{15}$$

For all x_0, x_1 in $\overline{U}(x^*,\delta)$ (such that $x_0 \neq x^*$ and $x_1 \neq x^*$), we have

$$\begin{aligned} \|Z_1(x^*)\| &= \|f(x^*)+g(x^*)-f(x_1)-g(x_1)-(\Delta f(x_1)+[x_0,x_1;g])(x^*-x_1)\| \\ &\leq \|f(x^*)-f(x_1)-\Delta f(x_1)(x^*-x_1)\| \\ &\quad +\|g(x^*)-g(x_1)-[x_0,x_1;g](x^*-x_1)\|. \end{aligned} \tag{16}$$

Using Definition 2.1, we have

$$\|Z_1(x^*)\| \leq \|f(x_1) - f(x^*) - \Delta f(x_1)(x_1 - x^*)\| + \|([x_1, x^*; g] - [x_0, x_1; g])(x^* - x_1)\|.$$

Therefore, with the help of Definition 2.2, Proposition 2.9 and assumption (H_3), we obtain

$$\|f(x_1) - f(x^*) - \Delta f(x_1)(x_1 - x^*)\| \leq C_* \|x_1 - x^*\|^{1+\gamma} \tag{17}$$

and

$$\|([x_1, x^*; g] - [x_0, x_1; g])(x^* - x_1)\| \leq \|[x_0, x_1, x^*; g]\| \|x_1 - x^*\| \|x_0 - x^*\|. \tag{18}$$

Consequently,

$$\|Z_1(x^*)\| \leq C_* \|x_1 - x^*\|^{1+\gamma} + K_2 \|x_1 - x^*\| \|x_0 - x^*\|$$

which implies, according to (14), $\|Z_1(x^*)\| < b$.

With (13), we have

$$\begin{aligned} e(P^{-1}(0) \cap \overline{U}(x^*, \delta), \Phi_1(x^*)) &= e(P^{-1}(0) \cap \overline{U}(x^*, \delta), P^{-1}[Z_1(x^*)]) \\ &\leq L(C_* \|x_1 - x^*\|^{1+\gamma} + K_2 \|x_1 - x^*\| \|x_0 - x^*\|) \end{aligned}$$

and, with (15), we obtain

$$dist(x^*, \Phi_1(x^*)) \leq L(C_* + K_2) \|x_1 - x^*\| \max\{\|x_1 - x^*\|^\gamma, \|x_0 - x^*\|\}. \tag{19}$$

By setting $\eta = x^*$ and $r = r_1 = C\|x_1 - x^*\| \max\{\|x_1 - x^*\|^\gamma, \|x_0 - x^*\|\}$, since $C > C_0$, one can find $\lambda \in]2L(2K_2 + K_1), 1[$ such that $C(1 - \lambda) > L(C_* + K_2)$ so that the assertion (a) in Lemma 2.13 is satisfied.

Let us notice that the above choice of r_1 implies $r_1 < a$. Let us show that condition (b) is also satisfied. For $x \in \overline{U}(x^*, \delta)$, we have

$$\begin{aligned} \|Z_1(x)\| &\leq \|f(x^*) + g(x) + \Delta f(x^*)(x - x^*) - f(x_1) - g(x_1) \\ &\quad - (\Delta f(x_1) + [x_0, x_1; g])(x - x_1)\| \\ &\leq \|-f(x) + f(x^*) + \Delta f(x^*)(x - x^*)\| + \|g(x) - g(x_1) - [x_0, x_1; g](x - x_1)\| \\ &\quad + \|f(x) - f(x_1) - \Delta f(x_1)(x - x_1)\|. \end{aligned}$$

Thanks to Definition 2.2, Proposition 2.9 and (H_3), it follows

$$\begin{aligned}\|Z_1(x)\| &\leq C_*(\|x-x^*\|^{1+\gamma}+\|x-x_1\|^{1+\gamma}) \\ &\quad +\|[x_0,x_1,x;g]\|\|x-x_0\|\|x-x_1\| \\ &\leq C_*(\|x-x^*\|^{1+\gamma}+\|x-x_1\|^{1+\gamma})+K_2\|x-x_0\|\|x-x_1\|,\end{aligned}$$

which implies $\|Z_1(x)\| \leq C_*(1+2^{1+\gamma})\delta^{1+\gamma}+4K_2\delta^2$. According to (13), $\|Z_1(x)\| < b$. We proved that if $x \in \overline{U}(x^*,\delta)$, then $Z_1(x) \in \overline{U}(0,b)$.

It follows that, for all $x',x'' \in \overline{U}(x^*,r_0)$

$$\begin{aligned}e(\Phi_1(x')\cap\overline{U}(x^*,r_1),\Phi_1(x'')) &\leq e(\Phi_1(x')\cap\overline{U}(x^*,\delta),\Phi_1(x'')) \\ &\leq L\|Z_1(x')-Z_1(x'')\| \\ &\leq L\|(g(x')-g(x'')-[x_0,x_1;g])(x'-x'')\| \\ &\quad +L\|(\Delta f(x^*)-\Delta f(x_1))(x'-x'')\| \\ &\leq L\|([x',x'';g]-[x_0,x_1;g])(x'-x'')\| \\ &\quad +L(\|\Delta f(x^*)\|+\|\Delta f(x_1)\|)\|x'-x''\| \\ &\leq L\|[x_1,x'',x';g](x'-x_1) \\ &\quad +[x_0,x_1,x'';g](x''-x_0)\|\|x'-x''\| \\ &\quad +L(\|\Delta f(x^*)\|+\|\Delta f(x_1)\|)\|x'-x''\| \\ &\leq 2L(2K_2\delta+K_1)\|x'-x''\|\end{aligned} \tag{20}$$

and for δ small enough,

$$e(\Phi_1(x')\cap\overline{U}(x^*,r_1),\Phi_1(x'')) \leq \lambda\|x'-x''\|.$$

Thus, the condition (b) of Lemma 2.13 is satisfied.

We conclude to the existence of $x_2 \in \overline{U}(x^*,r_1)$, a fixed point of Φ_1 which satisfies inequality (12). Proceeding by induction, we suppose that $x_k \in \overline{U}(x^*,r_{k-1})$, keeping $\eta_0 = x^*$ and $\tau_k = C\|x_k-x^*\|\max\{\|x_k-x^*\|^\gamma,\|x_{k-1}-x^*\|\}$ and we obtain the existence of a fixed-point $x_{k+1} \in \overline{U}(x^*,r_k)$ of Φ_k so that x_{k+1} satisfies (12), that achieves the proof of Theorem 3.1.

Remark 3.4.

(a) The first variant consists in replacing x_{k-1} by x_0 in (2). We obtain a *regula-falsi* type method whose convergence is superlinear; however, the convergence of the sequence in this case is slower because the upper bound in (12) involves x_k and x_{k-1}.

For the second variant, we replace x_{k-1} by x_{k+1} in (2) and we obtain the sequence

$$0 \in f(x_k) + \Delta f(x_k)(x_{k+1} - x_k) + g(x_{k+1}) + F(x_{k+1})$$

$$\Delta f(x_k) \in \partial^o f(x_k). \tag{21}$$

This is a Newton type method for solving the variational inclusion (2) where the set-valued map F is replaced by $g+F$. Let us note that in this case, assumption (H_3) is not necessary.

(b) In the corresponding result in [4], where (4) is used instead (3), estimate (12) is satisfied but for

$$C > C_1 = \frac{L(C_x^* + K_2)}{1 - 2L(2K_2 + K_1)}. \tag{22}$$

Then, in view of the Definition of (C_1), (C_2) and (5) we can that

$$C_0 \le C_1. \tag{23}$$

Therefore, our error estimate (12) is tighter when $C_0 < C_1$ (see the example in the next section), since C is smaller in (12) in our case. The above justify the claims made in this study.

(c) Conclusion (H_1) used in [4] and here is only needed to bring in the usage of (3). However, since (4) is only needed in the proof of the main Theorem , (H_1) can be replaced by $(H_1)'$ condition (4) holds in the (H) conditions.

4. A numerical example

We present an example in this section to show that $C_* < C_x^*$.

Example 4.1. Let $X=Y=\mathbb{R}$, $D=U(0,1)$ and define function F on D by

$$F(x)=e^x-1.$$

Then, we have that $x^*=0$. Using (3) we get in turn that

$$\begin{aligned} F'(y)^{-1}(F(y)-F(x^*)-F'(y)(y-x^*)) &= 1-y-\left(1-y+\frac{y^2}{2!}-\frac{y^3}{3!}+\frac{y^4}{4!}-\cdots\right) \\ &= \left(\frac{1}{2!}-\frac{y}{3!}+\frac{y^2}{4!}-\frac{y^3}{5!}+\cdots\right)y^2 \end{aligned}$$

so

$$\begin{aligned} \|F'(y)^{-1}\left(F(y)-F(x^*)-F'(y)(y-x^*)\right)\| &= \left|\frac{1}{2!}-\frac{y}{3!}+\frac{y^2}{4!}-\frac{y^3}{5!}+\cdots\right||y|^2 \\ &\leq \left(\frac{1}{2!}+\frac{|y|}{3!}+\frac{|y|^2}{4!}+\frac{|y|^3}{5!}+\cdots\right)|y|^2 \\ &\leq \left(\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\cdots\right)|y|^2 \\ &= (e-2)|y|^2. \end{aligned}$$

Hence, we can choose f to be $(f')^{-1}f$, $C_*=e-2$ and $\gamma=1$ in (4). Moreover, we have that

$$\begin{aligned} F'(y)^{-1}\left(F(y)-F(x)-F'(y)(y-x)\right) &= e^{-y}\left(e^y-1-e^x+1-e^y(y-x)\right) \\ &= 1-(y-x)-e^{x-y} \\ &= 1+x-y-\Bigg(1+(x-y) \\ &\quad +\frac{(x-y)^2}{2!}+\frac{(x-y)^3}{3}+\cdots\Bigg) \\ &= \left(\frac{1}{2!}+\frac{x-y}{3!}+\frac{(x-y)^2}{4!}+\cdots\right)(x-y)^2. \end{aligned}$$

So,

$$\begin{aligned} \|F'(y)^{-1}\left(F(y)-F(x)-F'(y)(y-x)\right)\| &= \left|\frac{1}{2!}+\frac{x-y}{3!}+\frac{(x-y)^2}{4!}+\cdots\right||y-x|^2 \\ &\leq \left(\frac{1}{2!}+\frac{|x-y|}{3!}+\frac{|x-y|^2}{4!}+\cdots\right)|y-x|^2 \\ &\leq \left(\frac{1}{2!}+\frac{2}{3!}+\frac{2^2}{4!}+\cdots\right)|y-x|^2 \\ &\leq (e^2-3)|y-x|^2. \end{aligned}$$

That is, we can choose $C_x^* = e^2 - 3$ and $\gamma = 1$ in (3). Therefore, we obtain $C^* < C_x^*$.

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