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ON THE PERIODIC LOTKA-VOLTERRA COMPETITIVE SYSTEMS WITH DISTRIBUTED TIME DELAYS AND FEEDBACK CONTROLS

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Abstract. Two classes of periodic n-species Lotka-Volterra competitive systems with distributed time delays and feedback controls are discussed. Based on the continuation theorem of the coincidence degree theory, some new sufficient conditions on the existence of positive periodic solutions are established.

Keywords. Competitive system; Continuation theorem; Positive periodic solution; Distributed time delay; Feedback control.

1. Introduction

In the past decades, a great deal of literature related to the dynamical systems was published, and extensive research results were obtained. In particular, mathematical ecological dynamical systems have become one of the hot topics in modern applied mathematics[1-36]. Its dynamic behaviors often include extinction, permanence, local or global stability, the existence of periodic solution, almost periodic solution, asymptotic periodic solution and the oscillation of solution and so on. Where, the existence of positive periodic solutions already become one of the most interested subjects for scholars[1-19, 21-34]. However, some nonlinear problems arising in many areas of the applied sciences like mathematical ecology can be formulated under a mathematical point of view involving the study of solutions of systems or equations. It is well known that the existence of a solution to systems are, under appropriate conditions, equivalent to the existence of a fixed point for a certain mapping. In this sense, the fixed point

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theorems are very important find the solution of the systems. In the recent years, the application of fixed point theorems on the existence of positive periodic solutions in mathematical ecology have been studied extensively. For example, Mawhins continuation theorem [1-19, 37],Brouwer fixed point theorem[21-24], Schauder fixed point theorem[25-27, 38], Krasnoselskii's fixed point theorem [28-30, 39-41] and Horn's fixed point theorem [31,42] , etc. Where Mawhins continuation theorem is a powerful tool for study the existence of periodic solutions of periodic high-dimensional time-delayed problems.

However, in the real world, the growth rate of a natural species will not often respond immediately to changes in its own population or that of an interacting species but will rather do so after a time lag [20]. Several authors have pointed out that we should introduce time delay into model foundation, which will have more resemblance to the real ecosystem. There has been a lot of literature about the existence of positive periodic solutions for various type delayed population dynamical systems has been extensively studied in [2-15,17-21,25,27-31,36] and the references cited therein. For example, In [8], the authors studied the following n-species Lotka-Volterra cooperative systems with distributed time delays

$$\begin{aligned} \dot{x}_i(t) = \; & x_i(t)[r_i(t) - \sum_{l=1}^{m} a_{iil}(t) \int_{-\tau}^{0} k_{iil}(s) x_i(t+s)ds \\ & + \sum_{j\neq i}^{n} \sum_{l=1}^{m} a_{ijl}(t) \int_{-\tau}^{0} k_{ijl}(s) x_j(t+s)ds], \quad i = 1,2,...n. \end{aligned} \tag{1.1}$$

By means of the methods of Mawhin's continuation theorem, the sufficient conditions for the existence of positive periodic solutions are established for system (1.1).

On the other hand, in some situation, people may wish to change the position of the existing periodic solution but to keep its stability. This is of significance in the control of ecology balance. One of the methods for the realization of it is to alter the system structurally by introducing some feedback control variables so as to get a population stabilizing at another periodic solution. The realization of the feedback control mechanism might be implemented by means of some biological control scheme or by harvesting procedure[15]. In fact, during the last decade, the study of the existence of positive periodic solutions for the population dynamical systems with feedback control and time delays have been studied extensively[11-15,17-19,27,30]. For example, In [19], the authors studied the following non-autonomous n-species Lotka-Volterra cooperative systems with continuous time delays and feedback controls

$$\begin{aligned}\dot{x}_i(t) = x_i(t)[r_i(t) - \sum_{l=1}^{m} a_{iil}(t)x_i(t-\tau_{iil}(t)) + \sum_{j\neq i}^{n}\sum_{l=1}^{m} a_{ijl}(t)x_j(t-\tau_{ijl}(t))]\\ -d_i(t)u_i(t) - e_i(t)u_i(t-\varepsilon_i(t)),\\ \dot{u}_i(t) = -b_i(t)u_i(t) + \beta_i(t)x_i(t) + \gamma_i(t)x_i(t-\sigma_i(t)), \quad i = 1,2,\cdots,n.\end{aligned} \tag{1.2}$$

By using the technique of coincidence degree , the sufficient conditions for the existence of positive periodic solutions are obtained for system (1.2).

It is well known that the focus in theoretical models of population and community dynamics must be not only on how populations depend on their own population densities or the population densities of other organisms, but also on how populations change in response to the physical environment[10]. To consider periodic environmental factor, it is reasonable to study Lotka-Volterra systems with periodic coefficients. Especially, the Lotka-Volterra competition system is of great interest among the scholars and has long been and will continue to be one of the dominant themes in both mathematical ecology and mathematical biology. The traditional two species nonautonomous Lotka-Volterra competition system takes the form

$$\begin{aligned}\frac{dx_1(t)}{dt} = x_1(t)\left[r_1(t) - a_{11}(t)x_1(t) - a_{12}(t)x_2(t)\right],\\ \frac{dx_2(t)}{dt} = x_2(t)\left[r_2(t) - a_{21}(t)x_1(t) - a_{22}(t)x_2(t)\right].\end{aligned} \tag{1.3}$$

There is an extensive literature concerned with the properties of system (1.3) has been discussed recently by many authors [32-35].

Based on the above works, in this paper, we introduce a distributed time delay and feedback control for system (1.3) and investigate the following two classes of n species periodic Lotka-Volterra type competitive systems with distributed time delays and feedback controls

$$\begin{aligned}\dot{x}_i(t) = &\quad x_i(t)[r_i(t) - \sum_{j=1}^{n}\sum_{l=1}^{m} a_{ijl}(t)\int_{-\tau}^{0} k_{ijl}(s)x_j(t+s)ds\\ &\quad -d_i(t)u_i(t) - e_i(t)\int_{-\tau}^{0} k_i(s)u_i(t+s)ds],\\ \dot{u}_i(t) = &\quad -b_i(t)u_i(t) + \beta_i(t)x_i(t) + \gamma_i(t)\int_{-\tau}^{0} k_i(s)x_i(t+s)ds, \quad i = 1,2,\cdots,n,\end{aligned} \tag{1.4}$$

and

$$\begin{aligned} \dot{x}_i(t) = & \quad x_i(t)[r_i(t) - \sum_{j=1}^{n}\sum_{l=1}^{m} a_{ijl}(t)\int_{-\tau}^{0} k_{ijl}(s)x_j(t+s)ds] \\ & -d_i(t)u_i(t) - e_i(t)\int_{-\tau}^{0} k_i(s)u_i(t+s)ds, \\ \dot{u}_i(t) = & \quad -b_i(t)u_i(t) + \beta_i(t)x_i(t) + \gamma_i(t)\int_{-\tau}^{0} k_i(s)x_i(t+s)ds, \quad i = 1,2,\cdots,n. \end{aligned} \tag{1.5}$$

By using the technique of coincidence degree developed by Gains and Mawhin in [37], we will establish some new sufficient conditions which guarantee system (1.4) and system (1.5) have at least one positive periodic solution.

2. Preliminaries

In system (1.4) and system (1.5), we have that $x_i(t)$ $(i = 1,2,...n)$ represent the density of n competitive species x_i $(i = 1,2,...n)$ at time t, respectively; $r_i(t)$ $(i = 1,2,...n)$ represent the intrinsic growth rate of species x_i $(i = 1,2,...n)$ at time t, respectively; $a_{iil}(t)$ $(i = 1,2,\cdots,n, l = 1,2,\cdots,m)$ represent the intrapatch restriction density of species x_i $(i = 1,2,...n)$ at time t, respectively; $a_{ijl}(t)$ $(l = 1,2,\cdots,m, i \neq j, i, j = 1,2,\cdots,n)$ represent the competitive coefficients between n species x_i $(i = 1,2,...n)$ at time t, respectively. $u_i(t)(i = 1,2,...n)$ represent the indirect feedback control variables at time t, respectively. $\beta_i(t), e_i(t), b_i(t), d_i(t), \gamma_i(t)$ $(i = 1,2,...n)$ represent the feedback control coefficients at time t, respectively. $\tau \geq 0$ is a constant and τ may be $+\infty$. In this paper, we always assume that

(H1) $r_i(t)(i = 1,2,...n)$ are continuous ω-periodic functions with $\int_0^{\omega} r_i(t)dt > 0$. $a_{ijl}(t)$ $(i, j = 1,2,\ l = 1,2,\cdots,m), \beta_i(t), e_i(t), b_i(t), d_i(t), \gamma_i(t)$ $(i = 1,2,...n)$ are continuous, positive ω-periodic functions.

From the viewpoint of mathematical biology, in this paper for system (1.4) and system (1.5) we only consider the solution with the following initial conditions

$$\begin{aligned} x_i(t) = \phi_i(t), \quad \text{for all} \quad t \in [-\tau, 0],\ i = 1,2,...n, \\ u_i(t) = \psi_i(t), \quad \text{for all} \quad t \in [-\tau, 0],\ i = 1,2,...n, \end{aligned} \tag{2.1}$$

where $\phi_i(t), \psi_i(t)$ $(i = 1,2,...n)$ are nonnegative continuous functions defined on $[-\tau, 0]$ satisfying $\phi_i(0) > 0, \psi_i(0) > 0$ $(i = 1,2,...n)$.

In this paper, for any ω-periodic continuous function $f(t)$ we denote

$$f^L = \min_{t\in[0,\omega]} f(t), \quad f^M = \max_{t\in[0,\omega]} f(t), \quad \bar{f} = \frac{1}{\omega}\int_0^{\omega} f(t)dt.$$

In order to obtain the existence of positive ω-periodic solutions of system (1.4) and system (1.5), we will use the continuation theorem developed by Gaines and Mawhin in [37]. For the reader's convenience, we will introduce the continuation theorem in the following.

Let X and Z be two normed vetor spaces. Let $L : \mathrm{Dom}L \subset X \to Z$ be a linear operator and $N : X \to Z$ be a continuous operator. The operator L is called a Fredholm operator of index zero, if $\dim\mathrm{Ker}L = \mathrm{codim}\mathrm{Im}L < \infty$ and $\mathrm{Im}L$ is a closed set in Z. If L is a Fredholm operator of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Im}L = \mathrm{Ker}Q = \mathrm{Im}(I-Q)$. It follows that $L|\mathrm{Dom}L\cap\mathrm{Ker}P : \mathrm{Dom}L\cap\mathrm{Ker}P \to \mathrm{Im}L$ is invertible and its inverse is denoted by K_P and denote by $J : ImQ \to KerL$ an isomorphism of ImQ onto $KerL$. Let Ω be a bounded open subset of X, we say that the operator N is L-compact on $\overline{\Omega}$, where $\overline{\Omega}$ denotes the closure of Ω in X, if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.1. [37] *Let L be a Fredholm operator of index zero and let N be L-compact on $\overline{\Omega}$. If*

(a) for each $\lambda \in (0,1)$ and $x \in \partial\Omega\cap DomL$, $Lx \neq \lambda Nx$;

(b) for each $x \in \partial\Omega\cap KerL$, $QNx \neq 0$;

(c) $deg\{JQN, \Omega\cap KerL, 0\} \neq 0$,

then the operator equation $Lx = Nx$ has at least one solution lying in $DomL\cap\overline{\Omega}$.

3. Main results

In order to obtain the existence of positive periodic solutions of system (1.4) and system (1.5), firstly, we introduce the following Lemma.

Lemma 3.1. *Suppose that $(x_1^*(t), x_2^*(t), \cdots, x_n^*(t), u_1^*(t), u_2^*(t), \cdots, u_n^*(t))$ is an ω-periodic solution of (1.4) and (1.5) with initial conditions (2.1), then $(x_1^*(t), x_2^*(t), \cdots, x_n^*(t), u_1^*(t), u_2^*(t), \cdots, u_n^*(t))$*

satisfies system (3.1) and system (3.2). Where

$$\begin{aligned}\dot{x}_i(t) = \ & x_i(t)[r_i(t) - \sum_{j=1}^{n}\sum_{l=1}^{m} a_{ijl}(t)\int_{-\tau}^{0} k_{ijl}(s)x_j(t+s)ds \\ & -d_i(t)u_i(t) - e_i(t)\int_{-\tau}^{0} k_i(s)u_i(t+s)ds], \\ & u_i(t) = \int_t^{t+\omega}[\beta_i(\mu)x_i(\mu) + \gamma_i(\mu)\int_{-\tau}^{0} k_i(s)x_i(\mu+s)ds]G_i(t,\mu)d\mu, \qquad i = 1,2,\cdots,n,\end{aligned} \tag{3.1}$$

$$\begin{aligned}\dot{x}_i(t) = \ & x_i(t)[r_i(t) - \sum_{j=1}^{n}\sum_{l=1}^{m} a_{ijl}(t)\int_{-\tau}^{0} k_{ijl}(s)x_j(t+s)ds] \\ & -d_i(t)u_i(t) - e_i(t)\int_{-\tau}^{0} k_i(s)u_i(t+s)ds, \\ & u_i(t) = \int_t^{t+\omega}[\beta_i(\mu)x_i(\mu) + \gamma_i(\mu)\int_{-\tau}^{0} k_i(s)x_i(\mu+s)ds]G_i(t,\mu)d\mu, \qquad i = 1,2,\cdots,n,\end{aligned} \tag{3.2}$$

and

$$G_i(t,\mu) = \frac{\exp\{\int_t^{\mu} b_i(\theta)d\theta\}}{\exp\{\int_0^{\omega} b_i(\theta)d\theta\} - 1}, \qquad i = 1,2,\cdots,n.$$

The converse is also true.

Proof. Lemma 3.1 can be proved by using the similar method given by Yin and Li in the proof of Lemma 2 in [18], and hence here we omit it.

It is easy to see that system (3.1) and (3.2) are equivalent to the following system (3.3) and system (3.4)

$$\begin{aligned}\dot{x}_i(t) = \ & x_i(t)[r_i(t) - \sum_{j=1}^{n}\sum_{l=1}^{m} a_{ijl}(t)\int_{-\tau}^{0} k_{ijl}(s)x_j(t+s)ds \\ & -d_i(t)u_i(t) - e_i(t)\int_{-\tau}^{0} k_i(s)u_i(t+s)ds], \qquad i = 1,2,\cdots,n,\end{aligned} \tag{3.3}$$

$$\begin{aligned}\dot{x}_i(t) = \ & x_i(t)[r_i(t) - \sum_{j=1}^{n}\sum_{l=1}^{m} a_{ijl}(t)\int_{-\tau}^{0} k_{ijl}(s)x_j(t+s)ds] \\ & -d_i(t)u_i(t) - e_i(t)\int_{-\tau}^{0} k_i(s)u_i(t+s)ds, \qquad i = 1,2,\cdots,n,\end{aligned} \tag{3.4}$$

where

$$u_i(t) = \int_t^{t+\omega} K(x_i)G_i(t,\mu)d\mu, \qquad G_i(t,\mu) = \frac{\exp\{\int_t^{\mu} b_i(\theta)d\theta\}}{\exp\{\int_0^{\omega} b_i(\theta)d\theta\} - 1},$$

and

$$K(x_i) = \beta_i(\mu)x_i(\mu) + \gamma_i(\mu)\int_{-\tau}^{0} k_i(s)x_i(\mu+s)ds, \quad i = 1,2,\cdots,n.$$

It is clear that in order to prove that system (1.4) and (1.5) with initial conditions (2.1) have at least one ω-periodic solution, we only need to prove that system (3.3) and system (3.4) have at least one ω-periodic solution.

Now, for convenience of statements we denote the functions

$$\bar{R}_i = \frac{1}{\omega}\int_0^{\omega} |r_i(t)|dt, \qquad a_{ij}(t) = \sum_{l=1}^{m} a_{ijl}(t), \quad i,j = 1,2,\cdots,n.$$

The following theorem is about the existence of positive periodic solutions of system (1.4).

Theorem 3.2. *Suppose that assumption (**H1**) holds and there exists a constant $\theta_i > 0$, $i = 1,2,\cdots,n$, such that*

$$\bar{r}_i\theta_i - \sum_{j=1, j\neq i}^{n}\sum_{l=1}^{m}\theta_j\bar{a}_{ijl}\frac{\bar{r}_j}{\big(\sum_{l=1}^{m}\bar{a}_{jjl}+\bar{A}_j\big)}\exp\{(\bar{r}_j+\bar{R}_j)\omega\} =: B_i > 0,$$

where

$$\begin{aligned} A_i(t) = \;& d_i(t)\int_t^{t+\omega} G_i(t,\mu)\big(\beta_i(\mu)+\gamma_i(\mu)\big)d\mu \\ & + e_i(t)\int_{-\tau}^{0} k_i(s)\int_{t+s}^{t+s+\omega} G_i(t+s,\mu)\big(\beta_i(\mu)+\gamma_i(\mu)\big)d\mu ds, \quad i = 1,2,\cdots,n. \end{aligned}$$

Then system (1.4) has at least one positive ω-periodic solution.

Proof. For system (3.3) we introduce new variables $y_i(t)$ $(i = 1,2,\cdots,n)$ such that

$$x_i(t) = \exp\{y_i(t)\}, \quad i = 1,2,\cdots,n.$$

Then system (3.3) is rewritten in the following form

$$\begin{aligned} \dot{y}_i(t) = \;& r_i(t) - \sum_{j=1}^{n}\sum_{l=1}^{m} a_{ijl}(t)\int_{-\tau}^{0} k_{ijl}(s)\exp\{y_j(t+s)\}ds \\ & - d_i(t)U_i(t) - e_i(t)\int_{-\tau}^{0} k_i(s)U_i(t+s)ds. \\ U_i(t) = \;& \int_t^{t+\omega} K(e^{y_i})G_i(t,\mu)d\mu, \qquad i = 1,2,\cdots,n, \end{aligned} \tag{3.5}$$

where

$$K(e^{y_i}) = \beta_i(\mu)\exp\{y_i(\mu)\} + \gamma_i(\mu)\int_{-\tau}^{0} k_i(s)\exp\{y_i(\mu+s))\}ds.$$

In order to apply Lemma 2.1 to system (3.5), we introduce the normed vector spaces X and Z as follows. Let $C(R,R^n)$ denote the space of all continuous function $y(t) = (y_1(t), y_2(t),\cdots,y_n(t))$:

$R \to R^n$. We take

$$X = Z = \{y(t) \in C(R, R^n) : y(t) \text{ is an } \omega\text{-periodic function}\}$$

with norm

$$\|y\| = \sum_{i=1}^{n} \max_{t\in[0,\omega]} |y_i(t)|.$$

It is obvious that X and Z are the Banach spaces. We define a linear operator $L : \mathrm{Dom}\, L \subset X \to Z$ and a continuous operator $N : X \to Z$ as follows.

$$Ly(t) = \dot{y}(t)$$

and

$$Ny(t) = (Ny_1(t), Ny_2(t), \cdots, Ny_n(t)),$$

where

$$\begin{aligned} Ny_i(t) = \quad & r_i(t) - \sum_{j=1}^{n}\sum_{l=1}^{m} a_{ijl}(t) \int_{-\tau}^{0} k_{ijl}(s) \exp\{y_j(t+s)\} ds \\ & -d_i(t)U_i(t) - e_i(t) \int_{-\tau}^{0} k_i(s) U_i(t+s) ds. \end{aligned} \tag{3.6}$$

Further, we define continuous projectors $P : X \to X$ and $Q : Z \to Z$ as follows.

$$Py(t) = \frac{1}{\omega}\int_0^{\omega} y(t)dt, \quad Qv(t) = \frac{1}{\omega}\int_0^{\omega} v(t)dt.$$

We easily see $\mathrm{Im}\, L = \{v \in Z : \int_0^{\omega} v(t)dt = 0\}$ and $\mathrm{Ker}\, L = R^n$. It is obvious that $\mathrm{Im}\, L$ is closed in Z and $\mathrm{dimKer}\, L = n$. Since for any $v \in Z$ there are unique $v_1 \in R^n$ and $v_2 \in \mathrm{Im}\, L$ with

$$v_1 = \frac{1}{\omega}\int_0^{\omega} v(t)dt, \quad v_2(t) = v(t) - v_1$$

such that $v(t) = v_1 + v_2(t)$, we have $\mathrm{codimIm}\, L = n$. Therefore, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_p : \mathrm{Im}\, L \to \mathrm{Ker}\, P \cap \mathrm{Dom}\, L$ is given in the following form

$$K_p v(t) = \int_0^t v(s)ds - \frac{1}{\omega}\int_0^{\omega}\int_0^t v(s)dsdt.$$

For convenience, we denote $F(t) = (F_1(t), F_2(t), \cdots, F_n(t))$ as follows

$$\begin{aligned} F_i(t) = \quad & r_i(t) - \sum_{j=1}^{n}\sum_{l=1}^{m} a_{ijl}(t) \int_{-\tau}^{0} k_{ijl}(s) \exp\{y_j(t+s)\} ds \\ & -d_i(t)U_i(t) - e_i(t) \int_{-\tau}^{0} k_i(s) U_i(t+s) ds. \end{aligned} \tag{3.7}$$

Thus, we have

$$QNy(t)=\frac{1}{\omega}\int_0^{\omega}F(t)dt \tag{3.8}$$

and

$$\begin{aligned}K_p(I-Q)Nu(t)= & \ K_pINu(t)-K_pQNu(t)\\ = & \int_0^t F(s)ds-\frac{1}{\omega}\int_0^{\omega}\int_0^t F(s)dsdt\\ & +(\frac{1}{2}-\frac{t}{\omega})\int_0^{\omega}F(s)ds.\end{aligned} \tag{3.9}$$

From formulas (3.8) and (3.9), we easily see that QN and $K_p(I-Q)N$ are continuous operators. Furthermore, it can be verified that $\overline{K_p(I-Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega\subset X$ by using Arzela-Ascoli theorem and $QN(\overline{\Omega})$ is bounded. Therefore, N is L-compact on $\overline{\Omega}$ for any open bounded subset $\Omega\subset X$.

Now, we reach the position to search for an appropriate open bounded subset Ω for the application of the continuation theorem (Lemma 2.1) to system (3.3).

Corresponding to the operator equation $Ly(t)=\lambda Ny(t)$ with parameter $\lambda\in(0,1)$, we have

$$\dot{y}_i(t)=\lambda F_i(t),\quad i=1,2,\cdots,n, \tag{3.10}$$

where $F_i(t)$ $(i=1,2,\cdots,n)$ are given in Eqs.(3.7).

Assume that $y(t)=(y_1(t),y_2(t),\cdots,y_n(t))\in X$ is a solution of system (3.10) for some parameter $\lambda\in(0,1)$. By integrating system (3.10) over the interval $[0,\omega]$, we obtain

$$\begin{aligned}&\int_0^{\omega}\big[r_i(t)-\sum_{j=1}^{n}\sum_{l=1}^{m}a_{ijl}(t)\int_{-\tau}^{0}k_{ijl}(s)\exp\{y_j(t+s)\}ds\\ &-d_i(t)U_i(t)-e_i(t)\int_{-\tau}^{0}k_i(s)U_i(t+s)ds\big]dt=0,\quad i=1,2,\cdots,n.\end{aligned}$$

Consequently, we have

$$\begin{aligned}&\int_0^{\omega}[\sum_{j=1}^{n}\sum_{l=1}^{m}a_{ijl}(t)\int_{-\tau}^{0}k_{ijl}(s)\exp\{y_j(t+s)\}ds\\ &+d_i(t)U_i(t)+e_i(t)\int_{-\tau}^{0}k_i(s)U_i(t+s)ds\big]dt=\bar{r}_i\omega,\quad i=1,2,\cdots,n.\end{aligned} \tag{17}$$

It follows from (3.10) and (3.11) that

$$\begin{aligned}\int_0^\omega |\dot{y}_i(t)|dt &= \lambda \int_0^\omega \Big| r_i(t) - \sum_{j=1}^n \sum_{l=1}^m a_{ijl}(t) \int_{-\tau}^0 k_{ijl}(s) \exp\{y_j(t+s)\} ds \\ &\quad - d_i(t)U_i(t) - e_i(t) \int_{-\tau}^0 k_i(s) U_i(t+s) ds \Big| dt \\ &\le \int_0^\omega |r_i(t)| dt + \sum_{j=1}^n \sum_{l=1}^m \int_0^\omega \big[a_{ijl}(t) \int_{-\tau}^0 k_{ijl}(s) \exp\{y_j(t+s)\} ds \\ &\quad + d_i(t)U_i(t) + e_i(t) \int_{-\tau}^0 k_i(s) U_i(t+s) ds \big] dt \\ &\le (\bar{r}_i + \bar{R}_i)\omega, \quad i = 1,2,\cdots,n,\end{aligned}$$

that is,

$$\int_0^\omega |\dot{y}_i(t)|dt \le (\bar{r}_i + \bar{R}_i)\omega, \quad i = 1,2,\cdots,n. \tag{3.12}$$

From the continuity of $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))$, there exist constants $\xi_i, \eta_i \in [0,\omega]$ $(i = 1,2,\cdots,n)$ such that

$$y_i(\xi_i) = \max_{t\in[0,\omega]} y_i(t), \quad y_i(\eta_i) = \min_{t\in[0,\omega]} y_i(t) \quad i = 1,2,\cdots,n. \tag{3.13}$$

By (3.11) and (3.13) we obtain

$$\begin{aligned}\bar{r}_i\omega \ge &\int_0^\omega a_{ii}(t) \int_{-\tau}^0 k_{ii}(s) \exp\{y_i(t+s)\} ds dt \\ &+ \int_0^\omega \big[d_i(t)U_i(t) + e_i(t) \int_{-\tau}^0 k_i(s) U_i(t+s) ds \big] dt \\ \ge &(\bar{a}_{ii} + \bar{A}_i)\omega \exp\{y_i(\eta_i)\}, \quad i = 1,2,\cdots,n,\end{aligned}$$

where

$$\begin{aligned}A_i(t) = &d_i(t) \int_t^{t+\omega} G_i(t,\mu) \big(\beta_i(\mu) + \gamma_i(\mu)\big) d\mu \\ &+ e_i(t) \int_{-\tau}^0 k_i(s) \int_{t+s}^{t+s+\omega} G_i(t+s,\mu) \big(\beta_i(\mu) + \gamma_i(\mu)\big) d\mu ds, \quad i = 1,2,\cdots,n.\end{aligned}$$

Therefore, we further have

$$y_i(\eta_i) \le \ln\left(\frac{\bar{r}_i}{\bar{a}_{ii} + \bar{A}_i}\right), \quad i = 1,2,\cdots,n. \tag{3.14}$$

From (3.12) and (3.14), one can see that

$$y_i(t) \le y_i(\eta_i) + \int_0^\omega |\dot{y}_i(t)|dt \le \ln\big(\frac{\bar{r}_i}{\bar{a}_{ii} + \bar{A}_i}\big) + (\bar{r}_i + \bar{R}_i)\omega =: M_i, \quad i = 1,2,\cdots,n. \tag{3.15}$$

On the other hand, from (3.11) and (3.13) it follows that

$$\begin{aligned}\bar{r}_i\omega = & \sum_{j=1}^{n}\sum_{l=1}^{m}\int_0^{\omega}\big[a_{ijl}(t)\int_{-\tau}^{0}k_{ijl}(s)\exp\{y_j(t+s)\}ds \\ & +d_i(t)U_i(t)+e_i(t)\int_{-\tau}^{0}k_i(s)U_i(t+s)ds\big]dt \\ \leq & \sum_{j=1}^{n}\sum_{l=1}^{m}\int_0^{\omega}a_{ijl}(t)\exp\{y_j(\xi_j)\}dt+\int_0^{\omega}A_i(t)\exp\{y_i(\xi_i)\}dt \\ \leq & \sum_{j=1}^{n}\sum_{l=1}^{m}\bar{a}_{ijl}\omega\exp\{y_j(\xi_j)\}+\bar{A}_i\omega\exp\{y_i(\xi_i)\}, \quad i=1,2,\cdots,n.\end{aligned} \tag{3.16}$$

In view of (3.15) and (3.16), one obtains that

$$\begin{aligned}\big(\sum_{l=1}^{m}\bar{a}_{iil}+\bar{A}_i\big)\exp\{y_i(\xi_i)\} \geq & \ \bar{r}_i-\sum_{j=1,j\neq i}^{n}\sum_{l=1}^{m}\bar{a}_{ijl}\exp\{y_j(\xi_j)\} \\ \geq & \ \bar{r}_i-\sum_{j=1,j\neq i}^{n}\sum_{l=1}^{m}\bar{a}_{ijl}\frac{\bar{r}_j}{\big(\sum_{l=1}^{m}\bar{a}_{jjl}+\bar{A}_j\big)}\exp\{(\bar{r}_j+\bar{R}_j)\omega\}=:B_i, \\ & \ i=1,2,\cdots,n,\end{aligned} \tag{3.17}$$

which implies

$$y_i(\xi_i)\geq\ln\left\{\frac{B_i}{\sum_{l=1}^{m}\bar{a}_{iil}+\bar{A}_i}\right\}=:C_i, \quad i=1,2,\cdots,n, \tag{3.18}$$

From (3.12) and (3.18), we have

$$y_i(t)\geq y_i(\xi_i)-\int_0^{\omega}|\dot{y}_i(t)|dt\leq C_i-(\bar{r}_i+\bar{R}_i)\omega=:N_i, \quad i=1,2,\cdots,n. \tag{3.19}$$

Therefore, from (3.15) and (3.19), we have

$$\max_{t\in[0,\omega]}|y_i(t)|\leq\max\big\{|M_i|,|N_i|\big\}=:H_i, \quad i=1,2,\cdots,n.$$

It can be seen that the constants $H_i\,(i=1,2,\cdots,n)$ are independent of parameter $\lambda\in(0,1)$. For any $y=(y_1,y_2,\cdots,y_n)\in R^n$, from (12) we can obtain

$$QNy=(QNy_1,QNy_2,\cdots,QNy_n)$$

$$QNy_i=\bar{r}_i-(\bar{A}_i+\bar{a}_{ii})\exp\{y_i\}-\sum_{j\neq i}^{n}\bar{a}_{ij}\exp\{y_j\}, \quad i=1,2,\cdots,n.$$

We consider the following system of algebraic equation

$$\bar{r}_i-(\bar{A}_i+\bar{a}_{ii})v_i-\sum_{j\neq i}^{n}\bar{a}_{ij}v_j=0 \quad i=1,2,\cdots,n.$$

By Lemma 4.1.1 in [36] and the assumption of Theorem 3.2, the system of algebraic equation has a unique positive solution $v^*=(v_1^*,v_2^*,\cdots,,v_n^*)$. Hence, the equation $QNy=0$ has a unique solution $y^*=(y_1^*,y_2^*,\cdots,y_n^*)=(\ln v_1^*,\ln v_2^*,\cdots,\ln v_n^*)\in R^n$.

Choosing constant $H>0$ large enough such that $|y_1^*|+|y_2^*|+\cdots+|y_n^*|<H$ and $H>H_1+H_2+\cdots+H_n$, we define a bounded open set $\Omega\subset X$ as follows

$$\Omega=\{y\in X:\|y\|<H\}.$$

It is clear that Ω satisfies conditions (a) and (b) of Lemma 2.1. On the other hand, by directly calculating we can obtain

$$\deg\{JQN,\Omega\cap \operatorname{Ker}L,(0,0,\cdots,0)\}=\operatorname{sgn}\begin{vmatrix} f_{y_1}^1 & f_{y_2}^1 & \cdots & f_{y_n}^1\\ f_{y_1}^2 & f_{y_2}^2 & \cdots & f_{y_n}^2\\ \cdots & \cdots & \cdots & \cdots\\ f_{y_1}^n & f_{y_2}^n & \cdots & f_{y_n}^n \end{vmatrix},$$

where

$$\begin{cases} f_{y_i}^i=-(\bar{A}_i+\bar{a}_{ii})\exp\{y_i^*\}, & i=j\\ f_{y_j}^i=-\bar{a}_{ij}\exp\{y_j^*\}, & i\neq j \end{cases} \qquad i,j=1,2,\cdots,n.$$

From the assumption of Theorem 3.2, we have

$$\begin{vmatrix} f_{y_1}^1 & f_{y_2}^1 & \cdots & f_{y_n}^1\\ f_{y_1}^2 & f_{y_2}^2 & \cdots & f_{y_n}^2\\ \cdots & \cdots & \cdots & \cdots\\ f_{y_1}^n & f_{y_2}^n & \cdots & f_{y_n}^n \end{vmatrix}\neq 0,$$

from which we finally have

$$\deg\{JQN,\Omega\cap \operatorname{Ker}L,(0,0,\cdots,0)\}\neq 0.$$

This shows that Ω satisfies condition (c) of Lemma 2.1. Therefore, system (3.5) has a ω-periodic solution $y^*(t)=(y_1^*(t),y_2^*(t),\cdots,y_n^*(t))\in\bar{\Omega}$. Finally, we have system (1.4) has a positive ω-periodic solution . This completes the proof .

On the existence of positive periodic solutions of system (1.5), we have the following different result.

Theorem 3.3. *Suppose that assumption (**H1**) holds and there exists a constant $\varepsilon_i > 0$, $W_i > 0$, $i = 1,2,\cdots,n$, such that*

$$\bar{r}_i\varepsilon_i - \sum_{j\neq i}^{n}\sum_{l=1}^{m}\varepsilon_j a_{ijl}^M \frac{\bar{r}_j}{\sum_{l=1}^{m} a_{jjl}^L} := W_i,$$

and the system of algebraic equation

$$\bar{r}_i - \bar{A}_i - \bar{a}_{ii}v_i - \sum_{j\neq i}^{n}\bar{a}_{ij}v_j = 0, \quad i = 1,2,\cdots,n,$$

where

$$\begin{aligned} A_i(t) = & \; d_i(t)\int_t^{t+\omega} G_i(t,\mu)\big(\beta_i(\mu)+\gamma_i(\mu)\big)d\mu \\ & + e_i(t)\int_{-\tau}^{0} k_i(s)\int_{t+s}^{t+s+\omega} G_i(t+s,\mu)\big(\beta_i(\mu)+\gamma_i(\mu)\big)d\mu ds, \quad i = 1,2,\cdots,n, \end{aligned}$$

has a unique positive solution. Then system (1.5) has at least one positive ω-periodic solution.

Proof. For system (3.4) we introduce new variables $y_i(t)$ $(i = 1,2,\cdots,n)$ such that

$$x_i(t) = \exp\{y_i(t)\}, \quad i = 1,2,\cdots,n.$$

Then system (3.4) is rewritten in the following form

$$\begin{aligned} \dot{y}_i(t) = & \; r_i(t) - \sum_{j=1}^{n}\sum_{l=1}^{m} a_{ijl}(t)\int_{-\tau}^{0} k_{ijl}(s)\exp\{y_j(t+s)\}ds \\ & - \exp\{-y_i(t)\}d_i(t)U_i(t) - \exp\{-y_i(t)\}e_i(t)\int_{-\tau}^{0} k_i(s)U_i(t+s)ds. \\ U_i(t) = & \int_t^{t+\omega} K(e^{y_i})G_i(t,\mu)d\mu, \qquad i = 1,2,\cdots,n, \end{aligned} \tag{3.20}$$

where

$$K(e^{y_i}) = \beta_i(\mu)\exp\{y_i(\mu)\} + \gamma_i(\mu)\int_{-\tau}^{0} k_i(s)\exp\{y_i(\mu+s))\}ds.$$

In order to apply Lemma 2.1 to system (3.20), we introduce the normed vector spaces X and Z as follows. Let $C(R,R^n)$ denote the space of all continuous function $y(t) = (y_1(t),y_2(t),\cdots,y_n(t)) : R \to R^n$. We take

$$X = Z = \{y(t) \in C(R,R^n) : y(t) \text{ is an } \omega\text{-periodic function}\}$$

with norm

$$\|y\| = \sum_{i=1}^{n} \max_{t\in[0,\omega]} |y_i(t)|.$$

It is obvious that X and Z are the Banach spaces. We define a linear operator $L: \mathrm{Dom}\, L \subset X \to Z$ and a continuous operator $N: X \to Z$ as follows.

$$Ly(t) = \dot{y}(t)$$

and

$$Ny(t) = (Ny_1(t), Ny_2(t), \cdots, Ny_n(t)),$$

where

$$\begin{aligned} Ny_i(t) = \quad & r_i(t) - \sum_{j=1}^{n}\sum_{l=1}^{m} a_{ijl}(t)\int_{-\tau}^{0} k_{ijl}(s)\exp\{y_j(t+s)\}ds \\ & -\exp\{-y_i(t)\}d_i(t)U_i(t) - \exp\{-y_i(t)\}e_i(t)\int_{-\tau}^{0} k_i(s)U_i(t+s)ds. \end{aligned} \tag{3.21}$$

Further, we define continuous projectors $P: X \to X$ and $Q: Z \to Z$ as follows.

$$Py(t) = \frac{1}{\omega}\int_0^{\omega} y(t)dt, \quad Qv(t) = \frac{1}{\omega}\int_0^{\omega} v(t)dt.$$

We easily see $\mathrm{Im}\, L = \{v \in Z : \int_0^{\omega} v(t)dt = 0\}$ and $\mathrm{Ker}\, L = R^n$. It is obvious that $\mathrm{Im}\, L$ is closed in Z and $\dim\mathrm{Ker}\, L = n$. Since for any $v \in Z$ there are unique $v_1 \in R^n$ and $v_2 \in \mathrm{Im}\, L$ with

$$v_1 = \frac{1}{\omega}\int_0^{\omega} v(t)dt, \quad v_2(t) = v(t) - v_1,$$

such that $v(t) = v_1 + v_2(t)$, we have $\mathrm{codim}\mathrm{Im}\, L = n$. Therefore, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_p : \mathrm{Im}\, L \to \mathrm{Ker}\, P \cap \mathrm{Dom}\, L$ is given in the following form

$$K_p v(t) = \int_0^t v(s)ds - \frac{1}{\omega}\int_0^{\omega}\int_0^t v(s)dsdt.$$

For convenience, we denote $F(t) = (F_1(t), F_2(t), \cdots, F_n(t))$ as follows

$$\begin{aligned} F_i(t) = \quad & r_i(t) - \sum_{j=1}^{n}\sum_{l=1}^{m} a_{ijl}(t)\int_{-\tau}^{0} k_{ijl}(s)\exp\{y_j(t+s)\}ds \\ & -\exp\{-y_i(t)\}d_i(t)U_i(t) - \exp\{-y_i(t)\}e_i(t)\int_{-\tau}^{0} k_i(s)U_i(t+s)ds. \end{aligned} \tag{3.22}$$

Thus, we have

$$QNy(t) = \frac{1}{\omega}\int_0^{\omega} F(t)dt \tag{3.23}$$

and

$$\begin{aligned}K_p(I-Q)Nu(t) = & \ K_pINu(t)-K_pQNu(t)\\ = & \int_0^t F(s)ds-\frac{1}{\omega}\int_0^\omega\int_0^t F(s)dsdt\\ & +\left(\frac{1}{2}-\frac{t}{\omega}\right)\int_0^\omega F(s)ds.\end{aligned}\tag{3.24}$$

From formulas (3.23) and (3.24), we easily see that QN and $K_p(I-Q)N$ are continuous operators. Furthermore, it can be verified that $\overline{K_p(I-Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega\subset X$ by using Arzela-Ascoli theorem and $QN(\overline{\Omega})$ is bounded. Therefore, N is L-compact on $\overline{\Omega}$ for any open bounded subset $\Omega\subset X$.

Now, we reach the position to search for an appropriate open bounded subset Ω for the application of the continuation theorem (Lemma 2.1) to system (3.4).

Corresponding to the operator equation $Ly(t)=\lambda Ny(t)$ with parameter $\lambda\in(0,1)$, we have

$$\dot{y}_i(t)=\lambda F_i(t),\quad i=1,2,\cdots,n,\tag{3.25}$$

where $F_i(t)$ $(i=1,2,\cdots,n)$ are given in Eqs.(3.22).

Assume that $y(t)=(y_1(t),y_2(t),\cdots,y_n(t))\in X$ is a solution of system (3.25) for some parameter $\lambda\in(0,1)$. By integrating system (3.25) over the interval $[0,\omega]$, we obtain

$$\begin{aligned}&\int_0^\omega\Big[r_i(t)-\sum_{j=1}^n\sum_{l=1}^m a_{ijl}(t)\int_{-\tau}^0 k_{ijl}(s)\exp\{y_j(t+s)\}ds\\ &-\exp\{-y_i(t)\}d_i(t)U_i(t)-\exp\{-y_i(t)\}e_i(t)\int_{-\tau}^0 k_i(s)U_i(t+s)ds\Big]dt\\ &=0,\quad i=1,2,\cdots,n.\end{aligned}$$

Consequently, we have

$$\begin{aligned}&\int_0^\omega\Big[\sum_{j=1}^n\sum_{l=1}^m a_{ijl}(t)\int_{-\tau}^0 k_{ijl}(s)\exp\{y_j(t+s)\}ds\\ &+\exp\{-y_i(t)\}d_i(t)U_i(t)+\exp\{-y_i(t)\}e_i(t)\int_{-\tau}^0 k_i(s)U_i(t+s)ds\Big]dt\\ &=\bar{r}_i\omega,\quad i=1,2,\cdots,n.\end{aligned}\tag{3.26}$$

It follows from (3.25) and (3.26) that

$$\begin{aligned}\int_0^{\omega}|\dot{y}_i(t)|dt &= \lambda\int_0^{\omega}\Big|r_i(t)-\sum_{j=1}^{n}\sum_{l=1}^{m}a_{ijl}(t)\int_{-\tau}^{0}k_{ijl}(s)\exp\{y_j(t+s)\}ds\\ &-\exp\{-y_i(t)\}d_i(t)U_i(t)-\exp\{-y_i(t)\}e_i(t)\int_{-\tau}^{0}k_i(s)U_i(t+s)ds\Big|dt\\ &\le\int_0^{\omega}|r_i(t)|dt+\sum_{j=1}^{n}\sum_{l=1}^{m}\int_0^{\omega}\big[a_{ijl}(t)\int_{-\tau}^{0}k_{ijl}(s)\exp\{y_j(t+s)\}ds\\ &+\exp\{-y_i(t)\}d_i(t)U_i(t)+\exp\{-y_i(t)\}e_i(t)\int_{-\tau}^{0}k_i(s)U_i(t+s)ds\big]dt\\ &\le(\bar{r}_i+\bar{R}_i)\omega,\quad i=1,2,\cdots,n,\end{aligned}$$

that is,

$$\int_0^{\omega}|\dot{y}_i(t)|dt\le(\bar{r}_i+\bar{R}_i)\omega,\quad i=1,2,\cdots,n. \tag{3.27}$$

From the continuity of $y(t)=(y_1(t),y_2(t),\cdots,y_n(t))$, there exist constants $\xi_i,\eta_i\in[0,\omega]$ $(i=1,2,\cdots,n)$ such that

$$y_i(\xi_i)=\max_{t\in[0,\omega]}y_i(t),\quad y_i(\eta_i)=\min_{t\in[0,\omega]}y_i(t)\quad i=1,2,\cdots,n. \tag{3.28}$$

By (3.26) and (3.28) we obtain

$$\bar{a}_{ii}\omega\exp\{y_i(\eta_i)\}\le\int_0^{\omega}a_{ii}(t)\int_{-\tau}^{0}k_{ii}(s)\exp\{y_i(t+s)\}dsdt\le\bar{r}_i\omega,\quad i=1,2,\cdots,n.$$

Therefore, we further have

$$y_i(\eta_i)\le\ln\big(\frac{\bar{r}_i}{\bar{a}_{ii}}\big),\quad i=1,2,\cdots,n. \tag{3.29}$$

From (3.27) and (3.29), one see that

$$y_i(t)\le y_i(\eta_i)+\int_0^{\omega}|\dot{y}_i(t)|dt\le\ln\big(\frac{\bar{r}_i}{\bar{a}_{ii}}\big)+(\bar{r}_i+\bar{R}_i)\omega=:M_i,\quad i=1,2,\cdots,n. \tag{3.30}$$

On the other hand, for each $i,j=1,2,\cdots,n$ and $l=1,2,\cdots,m$, we have

$$\begin{aligned}
&\int_0^\omega a_{ijl}(t)\int_{-\tau}^0 k_{ijl}(s)\exp\{y_j(t+s)\}dsdt\\
=&\int_{-\tau}^0\int_0^\omega a_{ijl}(t)k_{ijl}(s)\exp\{y_j(t+s)\}dtds\\
=&\int_{-\tau}^0\int_s^{s+\omega} a_{ijl}(v-s)k_{ijl}(s)\exp\{y_j(v)\}dvds\\
=&\int_{-\tau}^0\int_0^{\omega} a_{ijl}(v-s)k_{ijl}(s)\exp\{y_j(v)\}dvds\\
=&\int_0^{\omega}\int_{-\tau}^0 a_{ijl}(v-s)k_{ijl}(s)\exp\{y_j(v)\}dsdv\\
=&\int_0^{\omega}\big(\int_{-\tau}^0 a_{ijl}(t-s)k_{ijl}(s)ds\big)\exp\{y_j(t)\}dt.
\end{aligned}\tag{3.31}$$

From (3.26) and (3.31) we obtain

$$\begin{aligned}
&\int_0^\omega\big[\sum_{j=1}^n\sum_{l=1}^m\int_{-\tau}^0 a_{ijl}(t-s)k_{ijl}(s)ds\exp\{y_j(t)\}\\
&+\exp\{-y_i(t)\}d_i(t)U_i(t)+\exp\{-y_i(t)\}e_i(t)\int_{-\tau}^0 k_i(s)U_i(t+s)ds\big]dt\\
&=\bar r_i\omega,\quad i=1,2,\cdots,n.
\end{aligned}\tag{3.32}$$

From (3.32), we further obtain

$$\int_0^\omega \exp\{y_i(t)\}dt\le\frac{\bar r_i\omega}{\sum_{l=1}^m a_{iil}^L},\quad i=1,2,\cdots,n.\tag{3.33}$$

It follows from (3.26) and (3.28) that

$$\begin{aligned}
\bar r_i\omega\le\;&\sum_{j\neq i}^n\sum_{l=1}^m a_{ijl}^M\int_0^\omega\exp\{y_j(t)\}dt+\sum_{l=1}^m\int_0^\omega\int_{-\tau}^0 a_{iil}(t-s)k_{iil}(s)ds\exp\{y_i(t)\}dt\\
&+\int_0^\omega d_i(t)U_i(t)dt+\int_0^\omega e_i(t)\int_{-\tau}^0 k_i(s)U_i(t+s)dsdt\\
\le\;&\sum_{j\neq i}^n\sum_{l=1}^m a_{ijl}^M\frac{\bar r_j\omega}{\sum_{l=1}^m a_{jjl}^L}+\big(\bar A_{ii}\omega+\bar A_i\omega\big)\exp\{y_i(\xi_i)\},\quad i=1,2,\cdots,n,
\end{aligned}$$

where

$$A_{ii}(t)=\sum_{l=1}^m\int_0^\omega\int_{-\tau}^0 a_{iil}(t-s)k_{iil}(s)dsdt,$$

and

$$\begin{aligned}
A_i(t)=\;&d_i(t)\int_t^{t+\omega}G_i(t,\mu)\big(\beta_i(\mu)+\gamma_i(\mu)\big)d\mu\\
&+e_i(t)\int_{-\tau}^0 k_i(s)\int_{t+s}^{t+s+\omega}G_i(t+s,\mu)\big(\beta_i(\mu)+\gamma_i(\mu)\big)d\mu ds,\quad i=1,2,\cdots,n,
\end{aligned}$$

that is

$$\bar{r}_i\omega - \sum_{j\neq i}^{n}\sum_{l=1}^{m} a_{ijl}^{M}\frac{\bar{r}_j\omega}{\sum_{l=1}^{m} a_{jjl}^{L}} \leq \left(\bar{A}_{ii}\omega + \bar{A}_i\omega\right)\exp\{y_i(\xi_i)\}, \quad i=1,2,\cdots,n.$$

which implies

$$y_i(\xi_i) \geq \ln\left\{\frac{W_i}{\bar{A}_{ii}+\bar{A}_i}\right\} =: H_i, \quad i=1,2,\cdots,n, \tag{3.34}$$

where

$$W_i = \bar{r}_i - \sum_{j\neq i}^{n}\sum_{l=1}^{m} a_{ijl}^{M}\frac{\bar{r}_j}{\sum_{l=1}^{m} a_{jjl}^{L}}.$$

From (3.27) and (3.34), we have

$$y_i(t) \geq y_i(\xi_i) - \int_0^{\omega}|\dot{y}_i(t)|dt \leq H_i - (\bar{r}_i + \bar{R}_i)\omega =: N_i, \quad i=1,2,\cdots,n. \tag{3.35}$$

Therefore, from (3.30) and (3.35), we have

$$\max_{t\in[0,\omega]}|y_i(t)| \leq \max\left\{|M_i|,|N_i|\right\} =: B_i, \quad i=1,2,\cdots,n.$$

It can be seen that the constants $B_i\,(i=1,2,\cdots,n)$ are independent of parameter $\lambda\in(0,1)$. For any $y=(y_1,y_2,\cdots,y_n)\in R^n$, from (3.21) we can obtain

$$QNy = (QNy_1, QNy_2,\cdots,QNy_n)$$

$$QNy_i = \bar{r}_i - \bar{A}_i - \bar{a}_{ii}\exp\{y_i\} - \sum_{j\neq i}^{n}\bar{a}_{ij}\exp\{y_j\}, \quad i=1,2,\cdots,n.$$

We consider the following algebraic equation

$$\bar{r}_i - \bar{A}_i - \bar{a}_{ii}v_i - \sum_{j\neq i}^{n}\bar{a}_{ij}v_j = 0, \quad i=1,2,\cdots,n.$$

From the assumption of Theorem 3.3, the equation has a unique positive solution $v^*=(v_1^*,v_2^*,\cdots,,v_n^*)$. Hence, the equation $QNy=0$ has a unique solution $y^*=(y_1^*,y_2^*,\cdots,y_n^*)=(\ln v_1^*,\ln v_2^*,\cdots,\ln v_n^*)\in R^n$.

Choosing constant $B>0$ large enough such that $|y_1^*|+|y_2^*|+\cdots+|y_n^*|<B$ and $B>B_1+B_2+\cdots+B_n$, we define a bounded open set $\Omega\subset X$ as follows

$$\Omega = \{y\in X : \|y\| < B\}.$$

It is clear that Ω satisfies conditions (a) and (b) of Lemma 2.1. On the other hand, by directly calculating we can obtain

$$\deg\{JQN,\Omega\cap \operatorname{Ker}L,(0,0,\cdots,0)\} = \operatorname{sgn}\begin{vmatrix} f^1_{y_1} & f^1_{y_2} & \cdots & f^1_{y_n} \\ f^2_{y_1} & f^2_{y_2} & \cdots & f^2_{y_n} \\ \cdots & \cdots & \cdots & \cdots \\ f^n_{y_1} & f^n_{y_2} & \cdots & f^n_{y_n} \end{vmatrix},$$

where

$$\begin{cases} f^i_{y_i} = -\bar{a}_{ii}\exp\{y^*_i\}, & i=j \\ f^i_{y_j} = -\bar{a}_{ij}\exp\{y^*_j\}, & i\neq j \end{cases} \qquad i,j=1,2,\cdots,n.$$

From the assumption of Theorem 3.3, we have

$$\begin{vmatrix} f^1_{y_1} & f^1_{y_2} & \cdots & f^1_{y_n} \\ f^2_{y_1} & f^2_{y_2} & \cdots & f^2_{y_n} \\ \cdots & \cdots & \cdots & \cdots \\ f^n_{y_1} & f^n_{y_2} & \cdots & f^n_{y_n} \end{vmatrix} \neq 0.$$

From this, we finally have

$$\deg\{JQN,\Omega\cap \operatorname{Ker}L,(0,0,\cdots,0)\} \neq 0.$$

This shows that Ω satisfies condition (c) of Lemma 2.1. Therefore, system (3.20) has a ω-periodic solution $y^*(t) = (y^*_1(t),y^*_2(t),\cdots,y^*_n(t)) \in \bar{\Omega}$. Finally, we have system (1.5) has a positive ω-periodic solution . This completes the proof .

From the proof of Theorem 3.2 and Theorem 3.3, on the existence of positive periodic solutions of system (1.4) and system (1.5), we have the following result.

Corollary 3.4. *Suppose that assumption (**H1**) holds and there exists a constant $\rho_i > 0$, $L_i > 0$, $i=1,2,\cdots,n$, such that*

$$\bar{r}_i\rho_i - \sum_{j\neq i}^{n}\sum_{l=1}^{m}\rho_j a^M_{ijl}\frac{\bar{r}_j}{\sum_{l=1}^{m} a^L_{jjl} + \bar{E}_j F^L_j G^L_j} := L_i,$$

where

$$E_i(t) = d_i(t) + \int_{-\tau}^{0} k_i(s)e_i(t-s)ds,\ F_i(t) = \beta_i(t) + \int_{-\tau}^{0} k_i(s)\gamma_i(t-s)ds,$$

$$G_i(t,\mu)=\frac{\exp\{\int_t^{\mu} b_i(\theta)d\theta\}}{\exp\{\int_0^{\omega} b_i(\theta)d\theta\}-1},$$

and the system of algebraic equation

$$\bar{r}_i-(\bar{A}_i+\bar{a}_{ii})v_i-\sum_{j\neq i}^{n}\bar{a}_{ij}v_j=0,\quad i=1,2,\cdots,n,$$

where

$$\begin{aligned}A_i(t)= \;& d_i(t)\int_t^{t+\omega}G_i(t,\mu)\big(\beta_i(\mu)+\gamma_i(\mu)\big)d\mu\\ &+e_i(t)\int_{-\tau}^{0}k_i(s)\int_{t+s}^{t+s+\omega}G_i(t+s,\mu)\big(\beta_i(\mu)+\gamma_i(\mu)\big)d\mu ds,\quad i=1,2,\cdots,n,\end{aligned}$$

has a unique positive solution. Then system (1.4) has at least one positive ω-periodic solution.

Corollary 3.5. *Suppose that assumption (**H1**) holds and there exists a constant $\zeta_i>0,\ D_i>0,\ i=1,2,\cdots,n$, such that*

$$\bar{r}_i\zeta_i-\sum_{j=1,j\neq i}^{n}\sum_{l=1}^{m}\zeta_i\bar{a}_{ijl}\frac{\bar{r}_j}{\sum_{l=1}^{m}\bar{a}_{jjl}}\exp\{(\bar{r}_i+\bar{R}_j)\omega\}=:D_i,$$

and

$$\bar{r}_i-\bar{A}_i>0,$$

where

$$\begin{aligned}A_i(t)= \;& d_i(t)\int_t^{t+\omega}G_i(t,\mu)\big(\beta_i(\mu)+\gamma_i(\mu)\big)d\mu\\ &+e_i(t)\int_{-\tau}^{0}k_i(s)\int_{t+s}^{t+s+\omega}G_i(t+s,\mu)\big(\beta_i(\mu)+\gamma_i(\mu)\big)d\mu ds,\quad i=1,2,\cdots,n.\end{aligned}$$

Then system (1.5) has at least one positive ω-periodic solution.

4. Conclusions

In this paper, two classes of periodic n-species Lotka-Volterra competitive systems with distributed delays and feedback controls are proposed and by coincidence degree theory, analyzed to study the effect of time delays and feedback controls on the existence of positive periodic solutions of system (1.4) and system (1.5).

In the real world, the environments of most natural populations often undergo temporal variation, causing changes in the growth characteristics of these populations. One of the methods of incorporating temporal nonuniformity of the environments in models is to assume that the parameters are periodic with the same period of the time variable [10]. Here, we considered the periodic parameters in our models and have established the sufficient conditions on the existence of positive periodic solutions of system (1.4) and system (1.5).

From Theorem 3.2, Theorem 3.3, Corollary 3.4 and Corollary 3.5 we can see that the time delays and feedback controls have effect on the existence of positive periodic solutions. And also we can see that the results of Theorem 3.2 and Corollary 3.4 as well as the results Theorem 3.3 and Corollary 3.5 are the same, but the conditions are different. Hence, we have a interesting questions on our conditions of the results that is what is the relationship between conditions of Theorem 3.2 and conditions of Corollary 3.4 ? And the same question for the Theorem 3.3 and Corollary 3.5. We leave this for our future work.

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