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SYMMETRIC PROPERTY OF POSITIVE SOLUTIONS TO SYSTEMS INVOLVING NON-LOCAL OPERATORS

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Abstract. In this paper, we study symmetry results of positive solutions for the following system involving nonlocal operators

$$\begin{cases} (-\Delta+id)^{\alpha_1}u = f_1(v) \quad \text{in } \mathbb{R}^N, \\ (-\Delta+id)^{\alpha_2}v = f_2(u) \quad \text{in } \mathbb{R}^N, \\ \lim_{|x|\to+\infty} u(x) = \lim_{|x|\to+\infty} v(x) = 0, \end{cases}$$

where $N \geq 2$ and $0 < \alpha_1, \alpha_2 < 1$. We use the method of moving planes to obtain the radial symmetry of positive solutions to the above system.

Keywords. System; Nonlocal operator; Radial symmetry; Moving plane.

1. Introduction

The purpose of this paper is to consider symmetry results of positive solutions for system involving non-local operators

$$\begin{cases} (-\Delta+id)^{\alpha_1}u = f_1(v) \quad \text{in } \mathbb{R}^N, \\ (-\Delta+id)^{\alpha_2}v = f_2(u) \quad \text{in } \mathbb{R}^N, \\ \lim_{|x|\to+\infty} u(x) = \lim_{|x|\to+\infty} v(x) = 0, \end{cases} \tag{1}$$

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where $N\geq 2$ and $\alpha_i\in(0,1)$ for $i=1,2$. The non-local operator $(-\Delta+id)^{\alpha_i}$ can be characterized as $(-\Delta+id)^{\alpha_i}\phi(x)=\mathscr{F}^{-1}[(1+|\xi|^2)^{\alpha_i}\mathscr{F}(\phi)](x)$, where $\mathscr{F}$ is the Fourier transform and $\mathscr{F}^{-1}$ is the inverse of Fourier transform. Taking Fourier transform to differential system (1), it is closely related to the following integral system

$$
\begin{cases}
u(x)=\int_{\mathbb{R}^N}K_1(x-\xi)f_1(v(\xi))d\xi, & x\in\mathbb{R}^N,\\
v(x)=\int_{\mathbb{R}^N}K_2(x-\xi)f_2(u(\xi))d\xi, & x\in\mathbb{R}^N,\\
\lim_{|x|\to+\infty}u(x)=\lim_{|x|\to+\infty}v(x)=0,
\end{cases}
\tag{2}
$$

where K_i is the Bessel kernel, $i=1,2$, which is given by

$$
K_i(x)=\mathscr{F}^{-1}\Big(\frac{1}{(1+|y|^2)^{\alpha_i}}\Big)(x).
\tag{3}
$$

When $N\geq 2$ and $\alpha_i\in(0,1)$, the kernel $K_i\in L^1(\mathbb{R}^N)$ is positive, radially symmetric and smooth, moreover, it is decreasing as a function of $r=|x|$, refer to the work of Ma and Chen [13], Pólya [15], Stein [16], Ziemer [18]. These properties of the kernel K_i plays a key role when we use the moving planes method introduced by Li [11], Chen, Li and Ou [6, 7], Felmer et al [8, 9], Ma and Chen [13, 14]. These ideas provide an approach suitable for integral equations, which is different from the usual moving planes method originated in [10, 1] for Laplacian case.

During these years there has been a renewed and increasing interest in the study of linear and nonlinear integral operators, refer to [2, 3, 4] and so on. Especially, the authors in [8] studied the existence and symmetry properties of positive solutions for nonlinear schrödinger equation with the fractional laplacian

$$
\begin{cases}
(-\Delta)^{\alpha}u+u=f(u) \quad \text{in } \mathbb{R}^N,\\
\lim_{|x|\to+\infty}u(x)=0,
\end{cases}
\tag{4}
$$

where $N\geq 2$, $\alpha\in(0,1)$ and the function $f:\mathbb{R}\to\mathbb{R}$ is superlinear and has subcritical growth with respect to u. They applied the moving planes technique to prove that the solutions of (4) are radially symmetric.

For single equation related to $(-\Delta+id)^{\alpha}$, Tan, Wang and Yang [17] have studied the existence of ground state solutions and bounded state solutions. More precisely, they investigated the following nonlinear fractional field equation

$$(-\Delta + id)^{\frac{1}{2}} u = u^p \quad \text{in } \mathbb{R}^N, \qquad u \in H^{\frac{1}{2}}(\mathbb{R}^N). \tag{5}$$

It was shown that if $1 < p < \frac{N+1}{N-1}$, then there exist at least one C^2 positive ground state solution and this solution decays exponentially at infinity. In particular, the solution of (5) is radial symmetry by the Theorem 1 in [13].

In what follows, symmetry result for positive solutions of system (1) is considered. We assume the following hypotheses on the functions $f_i : \mathbb{R} \to \mathbb{R}$

(F) $\quad f_i \in C^1(\mathbb{R})$, increasing and there exists $\tau_i > 0$ such that

$$\lim_{\xi \to 0^+} \frac{f_i'(\xi)}{\xi^{\tau_i}} = 0.$$

The symmetry result is established as follows.

Theorem 1.1. *Assume that the function f_i satisfies (F) for $i = 1, 2$. Let the pair $(u_1, u_2) \in C^2(\mathbb{R}^N) \times C^2(\mathbb{R}^N)$ be a positive solution of system (1) with $u(x) = O(\frac{1}{|x|^{\beta_1}})$ and $v(x) = O(\frac{1}{|x|^{\beta_2}})$ at infinity for some $\beta_1, \beta_2 > 0$. Then both u and v are radially symmetric.*

2. Proof of Theorem 1.1

The radial symmetry of positive solutions for elliptic equations in the unit ball or the whole space have been studied by Gidas, Ni and Nirenberg [10] using the method of moving planes based on the maximum principle. However, our system (1) involving nonlocal operator absences the maximum principle. Here we use a new type of moving plane method as developed recently by Li [11], Chen, Li and Ou [5, 7], to obtain symmetry result. In order to apply this moving plane method, we need to introduce a Hardy-Littlewood-Sobolev type inequality.

Proposition 2.1. *Let K_i be defined in (3) and $m_i > \max\{\tau_i + 1, \frac{N\tau_i}{\alpha_i}\}$ for $i = 1, 2$. If $g \in L^{\frac{m_i}{\tau_i+1}}(\mathbb{R}^N)$, then $\int_{\mathbb{R}^N} K_i(\cdot - y) g(y) dy \in L^{m_i}(\mathbb{R}^N)$. Moreover, we have the estimate*

$$\| \int_{\mathbb{R}^N} K_i(\cdot - y) g(y) dy \|_{L^{m_i}(\mathbb{R}^N)} \le C \|g\|_{L^{\frac{m_i}{\tau_i+1}}(\mathbb{R}^N)}, \tag{6}$$

for some constant $C > 0$.

Proof. See the Theorem 9 in [13], also see [12]. $\square$

Now we use the method of moving planes to prove symmetry result in Theorem 1.1. To this end, we denote

$$\Sigma_\lambda = \{x = (x_1, x') \in \mathbb{R}^N \mid x_1 > \lambda\},$$

$$T_\lambda = \{x = (x_1, x') \in \mathbb{R}^N \mid x_1 = \lambda\},$$

$$u_\lambda(x) = u(x_\lambda) \ \ and \ \ v_\lambda(x) = v(x_\lambda),$$

where $\lambda \in \mathbb{R}$, $x_\lambda = (2\lambda - x_1, x')$ for $x = (x_1, x') \in \mathbb{R}^N$.

Proposition 2.2. *For any $x \in \mathbb{R}^N$, we have*

$$u_1^\lambda(x) - u_1(x) = \int_{\Sigma_\lambda} (K_1(x-y) - K_1(x_\lambda - y))(h_1(u_2^\lambda(y)) - h_1(u_2(y)))dy \tag{7}$$

and

$$v_\lambda(x) - v(x) = \int_{\Sigma_\lambda} (K_2(x-\xi) - K_2(x_\lambda - \xi))(f_2(u_\lambda(\xi)) - f_2(u(\xi)))d\xi. \tag{8}$$

Proof. Since K_1 is radially symmetric, we have

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^N} K_1(x-\xi) f_1(v(\xi))d\xi \\ &= \int_{\Sigma_\lambda} K_1(x-\xi) f_1(v(\xi))d\xi + \int_{\mathbb{R}^N \setminus \Sigma_\lambda} K_1(x-\xi) f_1(v(\xi))d\xi \\ &= \int_{\Sigma_\lambda} K_1(x-\xi) f_1(v(\xi))d\xi + \int_{\Sigma_\lambda} K_1(x_\lambda-\xi) f_1(v_\lambda(\xi))d\xi, \end{aligned}$$

and

$$u_\lambda(x) = \int_{\Sigma_\lambda} K_1(x_\lambda - \xi) f_1(v(\xi))d\xi + \int_{\Sigma_\lambda} K_1(x-\xi) f_1(v_\lambda(\xi))d\xi.$$

Subtracting above equalities, we obtain that

$$u_\lambda(x) - u(x) = \int_{\Sigma_\lambda} (K_1(x-\xi) - K_1(x_\lambda - \xi))(f_1(v_\lambda(\xi)) - f_1(v(\xi)))d\xi,$$

i.e. (7) is true. Similarly, we have (8). □

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. We divide the proof into several steps.

Step 1: We prove that $u_\lambda \leq u$ and $v_\lambda \leq v$ in Σ_λ, for λ large (negative).

To this end, define

$$\Sigma_{\lambda,u} = \{x \in \Sigma_\lambda \mid u_\lambda(x) > u(x)\}$$

and

$$\Sigma_{\lambda,v} = \{x \in \Sigma_\lambda \mid v_\lambda(x) > v(x)\},$$

we just need to show that $\Sigma_{\lambda,u}$ and $\Sigma_{\lambda,v}$ are empty for λ large (negative). By contradiction, we assume that $\Sigma_{\lambda,u}$ is not empty. For any $x \in \Sigma_{\lambda,u}$ and $\xi \in \Sigma_\lambda$, we have $|x_\lambda - \xi| \geq |x - \xi|$. Since K_1 is decreasing, then

$$K_1(x-\xi) \geq K_1(x_\lambda - \xi).$$

Using the condiction that f_1 is increasing and Proposition 2.2, we obtain

$$\begin{aligned}
u_\lambda(x) - u(x) &= \int_{\Sigma_\lambda} (K_1(x-\xi) - K_1(x_\lambda - \xi))(f_1(v_\lambda(\xi)) - f_1(v(\xi)))d\xi \\
&= \int_{\Sigma_{\lambda,v}} (K_1(x-\xi) - K_1(x_\lambda - \xi))(f_1(v_\lambda(\xi)) - f_1(v(\xi)))d\xi \\
&\quad + \int_{\Sigma_\lambda \setminus \Sigma_{\lambda,v}} (K_1(x-\xi) - K_1(x_\lambda - \xi))(f_1(v_\lambda(\xi)) - f_1(v(\xi)))d\xi \\
&\leq \int_{\Sigma_{\lambda,v}} (K_1(x-\xi) - K_1(x_\lambda - \xi))(f_1(v_\lambda(\xi)) - f_1(v(\xi)))d\xi \\
&\leq \int_{\Sigma_{\lambda,v}} K_1(x-\xi)(f_1(v_\lambda(\xi)) - f_1(v(\xi)))d\xi.
\end{aligned}$$

By (F), denoting $M = \sup_{\mathbb{R}^N} v$, there exists $C > 0$ such that $f_1'(t) \leq C|t|^{\tau_1}$ for all $0 \leq t \leq M$, therefore by the positivity of K_1 and the mean value theorem we get, for any $x \in \Sigma_{\lambda,u}$,

$$0 < u_\lambda(x) - u(x) \leq C \int_{\Sigma_{\lambda,v}} K_1(x-\xi) v_\lambda^{\tau_1}(\xi)(v_\lambda(\xi) - v(\xi))d\xi.$$

Choosing $m_1 > 0$ large such that $m_1 > \max\{\tau_1 + 1, \frac{N\tau_1}{\alpha_1}\}$ and $m_1\beta_2 > N$, by Proposition 2.1, we obtain

$$\begin{aligned}
\|u_\lambda - u\|_{L^{m_1}(\Sigma_{\lambda,u})} &\leq C \Big\| \int_{\Sigma_{\lambda,v}} K_1(\cdot - \xi) v_\lambda^{\tau_1}(\xi)(v_\lambda(\xi) - v(\xi))d\xi \Big\|_{L^{m_1}(\Sigma_{\lambda,u})} \\
&\leq C \|v_\lambda^{\tau_1}(v_\lambda - v)\|_{L^{\frac{m_1}{\tau_1+1}}(\Sigma_{\lambda,v})} \\
&\leq C \|v_\lambda\|^{\tau_1}_{L^{m_1}(\Sigma_{\lambda,v})} \|v_\lambda - v\|_{L^{m_1}(\Sigma_{\lambda,v})},
\end{aligned}$$

the last inequality holds by Hölder inequality.

Similarly, choosing $m_2 > 0$ large such that $m_2 > \max\{\tau_2 + 1, \frac{N\tau_2}{\alpha_2}\}$ and $m_2\beta_1 > N$, we can get

$$\|v_\lambda - v\|_{L^{m_2}(\Sigma_{\lambda,v})} \leq C\|u_\lambda\|^{\tau_2}_{L^{m_2}(\Sigma_{\lambda,u})}\|u_\lambda - u\|_{L^{m_2}(\Sigma_{\lambda,u})}.$$

Let $m = \max\{m_1, m_2\}$. Then

$$\|u_\lambda - u\|_{L^{m}(\Sigma_{\lambda,u})} \leq C\|v_\lambda\|^{\tau_1}_{L^{m}(\Sigma_{\lambda,v})}\|v_\lambda - v\|_{L^{m}(\Sigma_{\lambda,v})} \tag{9}$$

and

$$\|v_\lambda - v\|_{L^{m}(\Sigma_{\lambda,v})} \leq C\|u_\lambda\|^{\tau_2}_{L^{m}(\Sigma_{\lambda,u})}\|u_\lambda - u\|_{L^{m}(\Sigma_{\lambda,u})}. \tag{10}$$

Combining (9) and (10), we have

$$\|u_\lambda - u\|_{L^{m}(\Sigma_{\lambda,u})} \leq C\|v_\lambda\|^{\tau_1}_{L^{m}(\Sigma_{\lambda,v})}\|u_\lambda\|^{\tau_2}_{L^{m}(\Sigma_{\lambda,u})}\|u_\lambda - u\|_{L^{m}(\Sigma_{\lambda,u})}. \tag{11}$$

Since $m\beta_2, m\beta_1 > N$, $u(x) = O(\frac{1}{|x|^{\beta_1}})$ and $v(x) = O(\frac{1}{|x|^{\beta_2}})$ at infinity, by choosing λ large (negative), we have

$$C\|v_\lambda\|^{\tau_1}_{L^{m}(\Sigma_{\lambda,v})}\|u_\lambda\|^{\tau_2}_{L^{m}(\Sigma_{\lambda,u})} \leq \frac{1}{2},$$

which combines with (11) yielding that

$$\|u_\lambda - u\|_{L^{m}(\Sigma_{\lambda,u})} = 0,$$

and then $|\Sigma_{\lambda,u}| = 0$. Hence, $\Sigma_{\lambda,u}$ is empty for λ large (negative). Similarly, we can get $\Sigma_{\lambda,v}$ is empty for λ large (negative). That is, there exists λ_L such that for any $\lambda < \lambda_L$, we have

$$u_\lambda \leq u \quad and \quad v_\lambda \leq v \quad in \;\; \Sigma_\lambda.$$

Step 2: We need to show that λ_0 is finite, where

$$\lambda_0 = \sup\{\lambda \mid u_\lambda \leq u \;\; and \;\; v_\lambda \leq v \;\; in \; \Sigma_\lambda\}.$$

Indeed, since u decays at infinity, it is easy to know that there exist λ_s so that $u(x) < u_{\lambda_s}(x)$ for some $x \in \Sigma_{\lambda_s}$. Therefore, $\lambda_0 < \lambda_s$. And by Step 1, we know $\lambda_0 \geq \lambda_L$. As a consequence, λ_0 is finite.

Step 3: Prove that $u \equiv u_{\lambda_0}$ and $v \equiv v_{\lambda_0}$ in Σ_{λ_0}.

By contradiction, we can suppose that $u \neq u_{\lambda_0}$ in Σ_{λ_0}. By continuity of u with respect to λ, $u \geq u_{\lambda_0}$ in Σ_{λ_0}. By (8) in Proposition 2.2 and the monotonicity of f_2 and K_2, for any $x \in \Sigma_{\lambda_0}$, we have

$$v_{\lambda_0}(x) - v(x) = \int_{\Sigma_{\lambda_0}} (K_2(x-\xi) - K_2(x_{\lambda_0}-\xi))(f_2(u_{\lambda_0}(\xi)) - f_2(u(\xi)))d\xi < 0,$$

which implies that

$$v > v_{\lambda_0} \quad in \ \Sigma_{\lambda_0}.$$

Similarly, by (7) and the monotonicity of f_1 and K_1, we get

$$u > u_{\lambda_0} \quad in \ \Sigma_{\lambda_0}.$$

Claim. $u \geq u_\lambda$ and $v \geq v_\lambda$ in Σ_λ continues to hold when $\lambda_0 < \lambda < \lambda_0 + \varepsilon$, where $\varepsilon > 0$ small.

Now we assume that Claim is true, then this contradicts the definition of λ_0. Hence $u \equiv u_{\lambda_0}$ in Σ_{λ_0}, by the similar way, we can get $v \equiv v_{\lambda_0}$ in Σ_{λ_0}.

We only need to prove Claim to complete Step 3.

Proof of Claim. Assume $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$, where $\varepsilon > 0$ will be chosen later. If $u(x) < u_\lambda(x)$ in $\Sigma_{\lambda,u}$, using the similar approach in Step 1, we have

$$\|u_\lambda - u\|_{L^m(\Sigma_{\lambda,u})} \leq C\|v_\lambda\|^{\tau_1}_{L^m(\Sigma_{\lambda,v})}\|u_\lambda\|^{\tau_2}_{L^m(\Sigma_{\lambda,u})}\|u_\lambda - u\|_{L^m(\Sigma_{\lambda,u})}. \tag{12}$$

Since $m\beta_2, m\beta_1 > N$, $u(x) = O(\frac{1}{|x|^{\beta_1}})$ and $v(x) = O(\frac{1}{|x|^{\beta_2}})$ at infinity, then there exist $\varepsilon_1 > 0$ and R large such that for all $\lambda \in (\lambda_0, \lambda_0 + \varepsilon_1)$, we have

$$C\|v_\lambda\|^{\tau_1}_{L^m(B^c_R(0))}\|u_\lambda\|^{\tau_2}_{L^m(B^c_R(0))} \leq \frac{1}{4}. \tag{13}$$

By continuity of u with respect to λ and $u > u_{\lambda_0}$ in Σ_{λ_0}, we know that $|\Sigma_{\lambda,u} \cap B_R(0)|$ is small for $\varepsilon_2 > 0$ small enough, where $\lambda \in (\lambda_0, \lambda_0 + \varepsilon_2)$. Then

$$C\|v_\lambda\|^{\tau_1}_{L^m(\Sigma_{\lambda,u}\cap B_R(0))}\|u_\lambda\|^{\tau_2}_{L^m(\Sigma_{\lambda,u}\cap B_R(0))} \leq \frac{1}{4}. \tag{14}$$

Let $\bar{\varepsilon} = \min\{\varepsilon_1, \varepsilon_2\}$, for $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$, by (12)-(14), we obtain

$$\|u_\lambda - u\|_{L^m(\Sigma_{\lambda,u})} = 0, \tag{15}$$

and then $|\Sigma_{\lambda,u}| = 0$. Hence, $\Sigma_{\lambda,u}$ is empty for $\lambda \in (\lambda_0, \lambda_0 + \bar{\varepsilon})$. Similarly, there exists $\tilde{\varepsilon} > 0$ such that $\Sigma_{\lambda,v}$ is empty for $\lambda \in (\lambda_0, \lambda_0 + \tilde{\varepsilon})$. Taking $\varepsilon = \min\{\bar{\varepsilon}, \tilde{\varepsilon}\}$, we get $u \geq u_\lambda$ and $v \geq v_\lambda$ in Σ_λ for any $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$.

Finally, by translation, we may say that $\lambda_0 = 0$. Thus we have that u is symmetric about x_1-axis, using the same approach in any arbitrary direction, implies that u is radially symmetric. We finish the proof. □

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