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SOLVING SYSTEMS OF MONOTONE VARIATIONAL INEQUALITIES ON FIXED POINT SETS OF STRICTLY PSEUDO-CONTRACTIVE MAPPINGS

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Abstract. The aim of this paper is to study strong convergence theorems for fixed point problems of an infinite family of strictly pseudo-contractive mappings and solution problems for a system of monotone variational inequalities. Our results mainly improve and extend the results announced by Yuan [Q. Yuan, On nonexpansive mappings and an inverse-strongly monotone mapping in Hilbert spaces, J. Nonlinear Funct. Anal. 2015 (2015), Article ID 1] and other corresponding results.

Keywords. Fixed point; Variational inequality; Pseudo-contraction mappings; Hilbert space; Nonexpansive mapping.

1. Introduction

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty, closed and convex subset of H. Let $A : C \to H$ be a nonlinear mapping.

(1) A is called strongly positive with constant $\overline{\gamma}$ if there is a constant $\overline{\gamma} > 0$ such that

$$\langle Ax, x \rangle \geq \overline{\gamma} \| x \|^2, \qquad \forall x \in C.$$

(2) A is called monotone if

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$$\langle Ax - Ay, x - y\rangle \geq 0, \qquad \forall x, y \in C.$$

(3) A is called $\eta-$strongly monotone if there exists a positive constant η such that

$$\langle Ax - Ay, x - y\rangle \geq \eta \parallel x - y \parallel^2, \qquad \forall x, y \in C.$$

(4) A is called $k-$Lipschitzian if there exist a positive constant k such that

$$\parallel Ax - Ay \parallel \leq k \parallel x - y \parallel, \qquad \forall x, y \in C.$$

(5) A is called $\alpha-$inverse strongly monotone if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \parallel Ax - Ay \parallel^2, \qquad \forall x, y \in C.$$

It is obvious that any $\alpha-$inverse strongly monotone mapping A is $\frac{1}{\alpha}-$Lipschitzian. The classical variational inequality problem is to fined $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in C. \tag{1}$$

The set of solution of (1) is denoted by $VI(C,A)$, that is,

$$VI(C,A) = \{u \in C : \langle Au, v - u\rangle \geq 0, \quad \forall v \in C\}.$$

Let P_C be the metric projection from H onto the subset C. For a given point $z \in H$, $u \in C$ satisfies the inequality

$$\langle u - z, v - u\rangle \geq 0,\ \forall v \in C \Leftrightarrow \ u = P_C z. \tag{2}$$

It is known that projection operator P_C is nonexpansive. It is also know that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2,\ \forall x, y \in H. \tag{3}$$

One can see that the variational inequality (1) is equivalent to a fixed point problem. The point $u \in C$ is a solution of the variational inequality (1) if and only if $u \in C$ satisfies the relation $u = P_C(u - \lambda Au)$, where $\lambda > 0$ is a constant. A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone map of C into H and let $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v-u, w\rangle \geq 0, \forall u \in C\}$ and define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if $v \in VI(C,A)$; see [1] and the reference therein.

A mapping $T : C \to C$ is called λ-strictly pseudo-contractive of Browder and Petryshyn type [3] if there exists a constant $\lambda \in [0,1)$ such that

$$\| Tx - Ty \|^2 \leq \| x-y \|^2 + \lambda \| (I-T)x - (I-T)y \|^2, \qquad \forall x,y \in C. \tag{4}$$

When $\lambda = 0$, T is said to be nonexpansive, and if $\lambda = 1$, then T is said to a pseudo-contractive mapping, that is,

$$\| Tx - Ty \|^2 \leq \| x-y \|^2 + \| (I-T)x - (I-T)y \|^2, \qquad \forall x,y \in C. \tag{5}$$

This is equivalent to

$$\langle (I-T)x - (I-T)y, x-y \rangle \geq o, \qquad \forall x,y \in C. \tag{6}$$

T is said to be strongly pseudo-contractive if there exists a constant $\delta \in (0,1)$ such that $\langle Tx - Ty, x-y\rangle \leq \delta \| x-y \|^2$ for all $x,y \in C$. Clearly, the class of λ-strict pseudo-contractive mappings falls into the one between classes of nonexpansive mappings and pseudo-contractive mappings. It is well-known that, in a real Hilbert space H, (4) is equivalent to

$$\begin{aligned} \langle Tx - Ty, x-y \rangle \leq & \| x-y \|^2 \\ & - \frac{1-\lambda}{2} \| (I-T)x - (I-T)y \|^2, \ \forall x,y \in C. \end{aligned} \tag{7}$$

Strict pseudo-contractive mappings have more powerful applications than nonexpansive mappings do in solving inverse problems; see Scherzer [2]. Therefore it is interesting to develop the iterative methods for strict pseudo-contractive mappings. As a matter of fact, Browder and Petryshyn [3] show that if a k-strict pseudo-contractive mappings T has a fixed point in C, then starting with an initial $x_0 \in C$, the sequence $\{x_n\}$ generated by the recursive formula:

$$x_{n+1} = \lambda x_n + (1-\lambda) T x_n; \quad n \in \mathbb{N} \cup \{0\},$$

where λ is a constant such that $k < \lambda < 1$, converges weakly to a fixed point of T.

Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of nonexpansive mappings of C into itself and let $\{\lambda_i\}_{i=1}^{\infty}$ be a sequence of nonnegative real numbers in $[0,1]$. For each $n \geq 1$, define a mapping W_n of C into itself as follows:

$$
\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I,\\
&U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I,\\
&\vdots\\
&U_{n,k} = \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I,\\
&U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I,\\
&\vdots\\
&U_{n,2} = \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I,\\
&W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I.
\end{aligned}
\tag{8}
$$

Such a mapping W_n is called the $W-$mapping generated by $T_1, T_2, \cdots, T_n$ and $\lambda_1, \lambda_2, \cdots, \lambda_n$. Recently Yuan [4] proved the following strong convergence theorem.

Theorem 1.1. *Let H be a real Hilbert space and C a nonempty closed convex subset of H such that it is closed for linear operator. Let $A : H \to H$ be an $\alpha-$invers strongly monotone mapping and f a k-contraction on H. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings from C into itself such that $F := \bigcap_{i=1}^{\infty} F(T_i) \cap VI(C,A) \neq \emptyset$. Let B be a strongly positive linear bounded self-adjoint operator of C into itself with the constant $\overline{\gamma} > 0$. Let $\{x_n\}$ be a sequence generated in*

$$
\begin{cases}
x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n,\\
y_n = \beta_n \gamma f(x_n) + (I - \beta_n B) W_n P_C (I - r_n A) x_n,
\end{cases}
$$

where W_n is generated in (8), $\{\alpha_n\}$ and $\{\beta_n\}$ are real number sequences in $(0,1)$. Assume that the control sequence $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy the following restrictions:

(i) $\lim_{n\to\infty} \beta_n = 0, \sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(iii) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;

(iv) $\{r_n\} \subset [a,b]$ *for some a,b with* $0 < a < b < 2\alpha$.

Assume that $0 < \gamma < \overline{\gamma}/k$. *Then* $\{x_n\}$ *strongly converges to some point q, where* $q = P_F(\gamma f + (I - B))q$, *which solve the variational inequality* $\langle \gamma f(q) - Bq, p - q \rangle \leq 0, \forall p \in F$.

Motivated and inspired by the above research work, in this paper, by employing (10) we create a generalized composite viscosity algorithm to find the common element of fixed point set of an infinite family of a strictly pseudo-contractive mapping and the solution set of system of variational inequality problem.

2. Preliminaries

To establish our results, we need the following technical lemmas.

Lemma 2.1. [3] *Let* $T: C \to H$ *be a* λ*-strictly pseudo-contractive mapping. Define* $S: C \to H$ *by* $S(x) = \delta I(x) + (1 - \delta)T(x)$ *for each* $x \in C$. *Then, as* $\delta \in [\lambda, 1)$, *T is nonexpansive mapping such that* $Fix(S) = Fix(T)$.

Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of λ_n-strictly pseudo-contractive mappings of C into itself, we define a mapping W_n of C into itself as follows,

$$
\begin{aligned}
&U_{n,n+1} = I, \\
&U_{n,n} = \gamma_n S_n U_{n,n+1} + (1 - \gamma_n)I, \\
&\vdots \\
&U_{n,k} = \gamma_k S_k U_{n,k+1} + (1 - \gamma_k)I, \\
&U_{n,k-1} = \gamma_{k-1} S_{k-1} U_{n,k} + (1 - \gamma_{k-1})I, \\
&\vdots \\
&W_n = U_{n,1} = \gamma_1 S_1 U_{n,2} + (1 - \gamma_1)I,
\end{aligned}
\tag{9}
$$

where, $0 \leq \gamma_n \leq 1$, $S_n = \delta_n I + (1 - \delta_n)T_n$ and $\gamma_n \leq \delta_n < 1$, for all $n \in \mathbb{N}$. We can obtain S_n is a nonexpansive mapping and $Fix(S_n) = Fix(T_n)$ by Lemma 2.1. Furthermore, we obtain that W_n is a nonexpansive mapping.

Lemma 2.2. [5] *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let* $\{S_n\}$ *be an infinite family of nonexpansive mappings of C into itself and let* $\{\lambda_i\}$ *be a real* *sequence such that* $0 < \lambda_n \leq b < 1$ *for every* $n \in \mathbb{N}$. *Then, for every* $x \in C$ *and* $k \in \mathbb{N}$, *the limit* $\lim_{n\to\infty} U_{n,k}x$ *exists.*

In view of the previous lemma, we will define $Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}$, $\forall x \in C$.

Lemma 2.3. [5] *Let* C *be a nonempty closed convex subset of a strictly convex Banach space. Let* $\{S_n\}$ *be an infinite family of nonexpansive mappings of* C *into itself and let* $\{\lambda_i\}$ *be a real sequence such that* $0 < \lambda_n \leq b < 1$ *for every* $n \in \mathbb{N}$. *Then,* $Fix(W) = \bigcap_{n=1}^{\infty} Fix(S_n) \neq \emptyset$.

The following lemmas follow from Lemmas 2.1, 2.2 and 2.3.

Lemma 2.4. [6] *Let* C *be a nonempty closed convex subset of a strictly convex Banach space. Let* $\{T_n\}_{n=1}^{\infty}$ *be an infinite family of* λ_n*-strictly pseudo-contractive mappings of* C *into itself such that* $\bigcap_{n=1}^{\infty} Fix(T_n) \neq \emptyset$. *Define* $S_n = \delta_n I_n + (1-\delta_n)T_n$ *and* $0 < \lambda_n \leq \delta_n < 1$ *and let* $\{\gamma_n\}$ *be a real sequence such that* $0 < \gamma_n \leq b < 1$ *for every* $n \in \mathbb{N}$. *Then,* $Fix(W) = \bigcap_{n=1}^{\infty} Fix(T_n) = \bigcap_{n=1}^{\infty} Fix(S_n) \neq \emptyset$.

Lemma 2.5. [7] *Let* C *be a nonempty closed convex subset of a Hilbert space. Let* $\{S_n\}_{n=1}^{\infty}$ *be an infinite family of nonexpansive mappings of* C *into itself such that* $\bigcap_{n=1}^{\infty} Fix(S_n) \neq \emptyset$ *and let* $\{\gamma_n\}$ *be a real sequence such that* $0 < \gamma_n \leq b < 1$ *for every* $n \in \mathbb{N}$. *If* K *is any bounded subset of* C, *then* $\lim_{n\to\infty} \sup_{x\in K} \| Wx - W_n x \| = 0$.

Lemma 2.6. [9] *Let* $\{x_n\}$ *and* $\{z_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose* $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then* $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.7. [10] *Assume* B *is a strongly positive linear bounded operator on Hilbert space* H *with coefficient* $\overline{\gamma} > 0$ *and* $0 < \rho \leq \|B\|^{-1}$. *Then* $\|I - \rho B\| \leq 1 - \rho\overline{\gamma}$.

3. Main results

Now, we are in a position to prove our main results.

Theorem 3.1. *Let* H *be a real Hilbert space and* C *a nonempty closed convex subset of* H *such that it is closed for linear operator. Let* $A : H \to H$ *be an* $\alpha-$ *invers strongly monotone mapping and* $\{f_n\}$ *be a sequence of* $k_n - contraction$ *self maps of* C *with* $0 < k_l \leq \liminf_{n\to\infty} k_n \leq$

$\limsup_{n\to\infty} k_n \le k_s < 1$ *and* $\{f_n(x)\}$ *is uniformly convergence for any* $x \in D$*, where* D *is any bounded subset of* C*. Let* $\{T_i\}_{i=1}^{\infty}$ *be an infinite family of* λ_i*-strictly pseudo-contractive mappings of* C *into itself such that* $F := \bigcap_{i=1}^{\infty} F(T_i) \cap VI(C,A) \neq \emptyset$*. Let* B *be a strongly positive bounded linear self-adjoint operator of* C *into itself with the constant* $\overline{\gamma} > 0$ *such that* $\| B \| \le 1$*. Let* $\{x_n\}$ *be a sequence generated in*

$$\begin{cases} x_{n+1} = \alpha_n x_n + (1-\alpha_n)u_n, \\ u_n = \beta_n \gamma f_n(y_n) + (I - \beta_n B)W_n P_C(I - r_n A)y_n, \\ y_n = (1-\delta_n)x_n + \delta_n W_n x_n, \qquad n \ge 1, \end{cases} \tag{10}$$

where W_n *is generated by (9),* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are real number sequence in* $(0,1)$ *and* $\{\delta_n\}$ *be a real number sequence in* $[0,1]$*. Assume that the control sequence* $\{\alpha_n\}$*,* $\{\beta_n\}$ *,* $\{r_n\}$ *and* $\{\delta_n\}$ *satisfy the following restrictions:*

(i) $\lim_{n\to\infty} \beta_n = 0, \sum_{n=1}^{\infty} \beta_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;
(iii) $\lim_{n\to\infty} \delta_n = 0$;
(iv) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;
(v) $\{r_n\} \subset [a,b]$ *for some a,b with* $0 < a < b < 2\alpha$.

Assume that $0 < \gamma < \overline{\gamma}/k_n$*, for all* $n \in \mathbb{N}$*. Then* $\{x_n\}$ *strongly converges to some point* x^**, where* x^* *solve the system of variational inequality*

$$\langle \gamma f_n(x^*) - Bx^*, x - x^* \rangle \le 0, \ \forall x \in F, \forall n \in \mathbb{N}.$$

Proof. First, we show that $\{x_n\}$ is bounded. Indeed, taking an element $p \in F$. From Lemma 2.5 and the nonexpansivity of W_n,we have

$$\begin{aligned} \|y_n - p\| &\le (1-\delta_n)\|x_n - p\| + \delta_n \|W_n x_n - p\| \\ &\le (1-\delta_n)\|x_n - p\| + \delta_n \|x_n - p\| \\ &= \|x_n - p\|, \end{aligned} \tag{11}$$

and

$$
\begin{aligned}
\|x_{n+1}-p\| &= \|\alpha_n x_n + (1-\alpha_n)u_n - p\| \\
&\leq \alpha_n\|x_n-p\| + (1-\alpha_n)\|u_n-p\|. \qquad (12)
\end{aligned}
$$

By taking $\rho_n = P_c(I-r_nA)x_n$ and using (11), we get

$$
\begin{aligned}
\|u_n-p\| &= \|\beta\gamma f_n(y_n) + (I-\beta_n B)W_n\rho_n - p\| \\
&\leq \beta_n\|\gamma f_n(y_n) - \gamma f_n(p)\| + \beta_n\|\gamma f_n(p) - Bp\| + (1-\beta_n\overline{\gamma})\|\rho_n - p\| \\
&\leq \beta_n\gamma k_n\|y_n-p\| + \beta_n\|\gamma f_n(p) - Bp\| + (1-\beta_n\overline{\gamma})\|y_n-p\| \\
&= (1-\beta_n(\overline{\gamma}-k_n\gamma))\|y_n-p\| + \beta_n\|\gamma f_n(p)-Bp\| \\
&\leq (1-\beta_n(\overline{\gamma}-k_n\gamma))\|x_n-p\| + \beta_n\|\gamma f_n(p)-Bp\|. \qquad (13)
\end{aligned}
$$

It follows from (12) and (13) that

$$
\begin{aligned}
\|x_{n+1}-p\| &\leq \alpha_n\|x_n-p\| + (1-\alpha_n)(1-\beta_n(\overline{\gamma}-k_n\gamma))\|x_n-p\| \\
&\quad + \beta_n(1-\alpha_n)\|\gamma f_n(p)-Bp\| \\
&= (1-(1-\alpha_n)\beta_n(\overline{\gamma}-k_n\gamma))\|x_n-p\| \\
&\quad + (1-\alpha_n)\beta_n(\overline{\gamma}-k_n\gamma)\frac{1}{\overline{\gamma}-k_n\gamma}\|\gamma f_n(p)-Bp\| \\
&\leq \max\left\{\|x_n-p\|, \frac{1}{\overline{\gamma}-k_n\gamma}\|\gamma f_n(p)-Bp\|\right\}. \qquad (14)
\end{aligned}
$$

From the uniform convergence of $\{f_n\}_{n=1}^{\infty}$ on bounded subset of C, there exist $M_0 \geq 0$ such that

$$
\|\gamma f_n(p) - Bp\| \leq M_0, \ \forall n \in \mathbb{N}. \qquad (15)
$$

Since $\frac{1}{\overline{\gamma}-k_n\gamma} < \frac{1}{\overline{\gamma}-k_s\gamma}$, then from (14) and (15), we get

$$
\|x_{n+1}-p\| \leq \max\{\|x_n-p\|, \frac{1}{\overline{\gamma}-k_s\gamma}M_0\}.
$$

Which gives by induction $\{x_n\}$ is bounded , so are $\{y_n\}$, $\{W_nx_n\}$ and $\{f_n(y_n)\}$. Put $\tau_n = \alpha_n(1-\delta_n)$ for each $n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$. Then from (ii) and (iii), we have

$$
0 < \liminf_{n\to\infty}\tau_n \leq \limsup_{n\to\infty}\tau_n < 1.
$$

Define z_n by

$$x_{n+1}=\tau_n x_n+(1-\tau_n)z_n,\quad \forall n\in\mathbb{N}_0. \tag{16}$$

Observe that

$$\begin{aligned}
z_{n+1}-z_n &= \frac{x_{n+2}-\tau_{n+1}x_{n+1}}{1-\tau_{n+1}}-\frac{x_{n+1}-\tau_n x_n}{1-\tau_n}\\
&= \frac{\alpha_{n+1}x_{n+1}+(1-\alpha_{n+1})u_{n+1}-\alpha_{n+1}(1-\delta_{n+1})x_{n+1}}{1-\tau_{n+1}}\\
&\quad -\frac{\alpha_n x_n+(1-\alpha_n)u_n-\alpha_n(1-\delta_n)x_n}{1-\tau_n}\\
&= \frac{(1-\alpha_{n+1})u_{n+1}+\alpha_{n+1}\delta_{n+1}x_{n+1}}{1-\tau_{n+1}}-\frac{(1-\alpha_n)u_n+\alpha_n\delta_n x_n}{1-\tau_n}\\
&= \frac{1-\alpha_{n+1}}{1-\tau_{n+1}}(u_{n+1}-u_n)+\Big(\frac{1-\alpha_{n+1}}{1-\alpha_{n+1}(1-\delta_{n+1})}-\frac{1-\alpha_n}{1-\alpha_n(1-\delta_n)}\Big)u_n\\
&\quad +\frac{\alpha_{n+1}\delta_{n+1}}{1-\tau_{n+1}}x_{n+1}-\frac{\alpha_n\delta_n}{1-\tau_n}x_n\\
&= \frac{1-\alpha_{n+1}}{1-\tau_{n+1}}(u_{n+1}-u_n)+\frac{\alpha_n\alpha_{n+1}(\delta_{n+1}-\delta_n)+\alpha_n\delta_n-\alpha_{n+1}\delta_{n+1}}{(1-\alpha_{n+1}(1-\delta_{n+1}))(1-\alpha_n(1-\delta_n))}u_n\\
&\quad +\frac{\alpha_{n+1}\delta_{n+1}}{1-\tau_{n+1}}x_{n+1}-\frac{\alpha_n\delta_n}{1-\tau_n}x_n.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|z_{n+1}-z_n\| &\le \frac{\alpha_{n+1}\delta_{n+1}}{1-\tau_{n+1}}\|x_{n+1}\|+\frac{\alpha_n\delta_n}{1-\tau_n}\|x_n\|+\frac{1-\alpha_{n+1}}{1-\tau_{n+1}}\|u_{n+1}-u_n\|\\
&\quad +\Big|\frac{\alpha_n\alpha_{n+1}(\delta_{n+1}-\delta_n)+\alpha_n\delta_n-\alpha_{n+1}\delta_{n+1}}{(1-\alpha_{n+1}(1-\delta_{n+1}))(1-\alpha_n(1-\delta_n))}\Big|\|u_n\|.
\end{aligned} \tag{17}$$

On the other hand, we have

$$\begin{aligned}
\|u_{n+1}-u_n\| &= \|\beta_{n+1}(\gamma f_{n+1}(y_{n+1})-BW_{n+1}\rho_{n+1})+\beta_n(\gamma f_n(y_n)-BW_n\rho_n)\\
&\quad +W_{n+1}\rho_{n+1}-W_n\rho_n\|\\
&\le \beta_{n+1}\|\gamma f_{n+1}(y_{n+1})-BW_{n+1}\rho_{n+1}\|\\
&\quad +\beta_n\|\gamma f_n(y_n)-BW_n\rho_n\|+\|W_{n+1}\rho_{n+1}-W_n\rho_n\|\\
&\le \beta_{n+1}\big(\|\gamma f_{n+1}(y_{n+1})\|+\|B\|\|W_{n+1}\rho_{n+1}\|\big)\\
&\quad +\beta_n\big(\|\gamma f_n(y_n)\|+\|B\|\|W_n\rho_n\|\big)\\
&\quad +\|W_{n+1}\rho_{n+1}-W_{n+1}\rho_n\|+\|W_{n+1}\rho_n-W_n\rho_n\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \beta_{n+1}\big(\gamma\|f_{n+1}(y_{n+1})-f_{n+1}(p)\|+\gamma\|f_{n+1}(p)\|\\
&\quad+\|B\|\|W_{n+1}\rho_{n+1}-W_{n+1}p\|+\|B\|\|W_{n+1}p\|\big)\\
&\quad\beta_n\big(\gamma\|f_n(y_n)-f_n(p)\|+\gamma\|f_n(p)\|+\|B\|\|W_n\rho_n-W_np\|\\
&\quad+\|B\|\|W_np\|\big)+\|\rho_{n+1}-p\|+\|W_{n+1}\rho_n-W_n\rho_n\|\\
&\leq \beta_{n+1}\big(\gamma k_{n+1}\|y_{n+1}-p\|+\gamma\|f_{n+1}(p)\|+\|B\|\|\rho_{n+1}-p\|\\
&\quad+\|B\|\|W_{n+1}p\|\big)+\beta_n\big(\gamma\|y_n-p\|+\gamma\|f_n(p)\|+\|B\|\|\rho_n-p\|\\
&\quad+\|B\|\|W_np\|\big)\|+\|\rho_{n+1}-p\|+\|W_{n+1}\rho_n-W_n\rho_n\|.
\end{aligned}
$$

By taking $M_n=\gamma k_n\|y_n-p\|+\gamma\|f_n(p)\|+\|B\|\big(\|\rho_n-p\|+\|W_np\|\big)$, we get

$$
\begin{aligned}
\|u_{n+1}-u_n\| &\leq \beta_{n+1}M_{n+1}+\beta_nM_n+\|\rho_{n+1}-\rho_n\|\\
&\quad+\|W_{n+1}\rho_n-W_n\rho_n\|. \qquad (18)
\end{aligned}
$$

Observe that

$$
\|W_{n+1}\rho_n-W_n\rho_n\|\leq M_1\prod_{i=1}^{n}\lambda_i, \qquad (19)
$$

and

$$
\begin{aligned}
\|\rho_{n+1}-\rho_n\| &= \|P_c(I-r_{n+1}A)y_{n+1}-P_c(I-r_nA)y_n\|\\
&\leq \|(I-r_{n+1}A)y_{n+1}-(I-r_nA)y_n\|\\
&= \|(I-r_{n+1}A)y_{n+1}-(I-r_{n+1}A)y_n\|\\
&\quad+\|(I-r_{n+1}A)y_n-(I-r_nA)y_n\|\\
&\leq \|y_{n+1}-y_n\|+|r_{n+1}-r_n|\|Ay_n\|, \qquad (20)
\end{aligned}
$$

and also

$$
\begin{aligned}
\|y_{n+1}-y_n\| &= \|(1-\delta_{n+1})x_{n+1}+\delta_{n+1}W_{n+1}x_{n+1}-(1-\delta_n)x_n-\delta_nW_nx_n\|\\
&\leq \|x_{n+1}-x_n\|+\delta_{n+1}\|W_{n+1}x_{n+1}-x_{n+1}\|+\delta_n\|x_n-W_nx_n\|\\
&\leq \|x_{n+1}-x_n\|+\delta_{n+1}\big[\|x_{n+1}-p\|+\|W_{n+1}p\|+\|x_{n+1}\|\big]\\
&\quad+\delta_n\big[\|x_n\|+\|p-x_n\|+\|W_np\|\big].
\end{aligned}
$$

Put $L_n = \|x_n\| + \|p - x_n\| + \|W_n p\|$. So, we get

$$\|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\| + \delta_{n+1} L_{n+1} + \delta_n L_n. \tag{21}$$

It follows from (17), (18), (20) and (21) that

$$\begin{aligned}\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \leq & \frac{\alpha_{n+1}\delta_{n+1}}{1-\tau_{n+1}}\|x_{n+1}\| + \frac{\alpha_n\delta_n}{1-\tau_n}\|x_n\| \\ & + \beta_{n+1}M_{n+1} + \beta_n M_n + |r_{n+1} - r_n|\|Ay_n\| \\ & + \delta_{n+1}L_{n+1} + \delta_n L_n + M_1\prod_{i=1}^{n}\lambda_i.\end{aligned}$$

Since $\{x_n\}$, $\{M_n\}$, $\{Ay_n\}$ and $\{L_n\}$ are bounded, so from condition we deduce that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0,$$

which immediately from lemma 2.7 implies that

$$\lim_{n\to\infty}\|z_n - x_n\| = 0.$$

It follows that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\tau_n)\|z_n - x_n\| = 0.$$

We have

$$\begin{aligned}\|y_n - x_n\| &\leq \delta_n\left[\|x_n\| + \|p - x_n\| + \|W_n p\|\right] \\ &= \delta_n L_n.\end{aligned}$$

Since $\{L_n\}$ is bounded and $\lim_{n\to\infty}\delta_n = 0$, we get

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{22}$$

We have

$$\begin{aligned}\|u_n - W_n\rho_n\| &= \|\beta_n\gamma f_n(y_n) + (I - \beta_n B)W_n\rho_n - W_n\rho_n\| \\ &\leq \beta_n\left[\gamma k_n\|y_n - p\| + \gamma\|f_n(p)\| + \|B\|\left[\|\rho_n - p\| + \|W_n p\|\right]\right].\end{aligned}$$

Since $\{y_n\}$, $\{f_n(p)\}$, $\{\rho_n\}$ and $\{W_n p\}$ are bounded and $\lim_{n\to\infty}\beta_n = 0$, we find

$$\lim_{n\to\infty}\|u_n - W_n\rho_n\| = 0. \tag{23}$$

Notice that

$$\begin{aligned}
\|\rho_n - p\|^2 &= \|P_c(I - r_nA)y_n - p\|^2 \\
&\le \|(y_n - p) + r_n(Ap - Ay_n)\|^2 \\
&= \langle -p + y_n + r_n(Ap - Ay_n), -p + y_n + r_n(Ap - Ay_n)\rangle \\
&\le \|p - y_n\|^2 - 2r_n\alpha\|Ay_n - Ap\|^2 + r_n^2\|Ap - Ay_n\|^2 \\
&\le \|x_n - p\|^2 + r_n(r_n - 2\alpha)\|Ay_n - Ap\|^2,
\end{aligned}$$

and

$$\begin{aligned}
\|u_n - p\|^2 &= \|\beta_n\gamma f_n(y_n) + (I - \beta_nB)W_n\rho_n - p\|^2 \\
&\le \Big[\beta_n\|\gamma f_n(y_n) - BW_n\rho_n\| + \|\rho_n - p\|\Big]^2 \\
&= \beta_n^2\|\gamma f_n(y_n) - BW_n\rho_n\|^2 + \|\rho_n - p\|^2 \\
&\quad + 2\beta_n\|\gamma f_n(y_n) - BW_n\rho_n\|\|\rho_n - p\|. \qquad (24)
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|u_n - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\Big[\beta_n^2\|\gamma f_n(y_n) - BW_n\rho_n\|^2 \\
&\quad + \|\rho_n - p\|^2 + 2\beta_n\|\gamma f_n(y_n) - BW_n\rho_n\|\|\rho_n - p\|\Big] \\
&\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\Big[\beta_n^2\|\gamma f_n(y_n) - BW_n\rho_n\| \\
&\quad + 2\beta_n\|\gamma f_n(y_n) - BW_n\rho_n\|\|\rho_n - p\|\Big] \\
&\quad + (1 - \alpha_n)\Big[\|x_n - p\|^2 + r_n(r_n - 2\alpha)\|Ay_n - Ap\|^2\Big].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
(1 - \alpha_n)r_n(2\alpha - r_n)\|Ay_n - Ap\|^2 &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad + (1 - \alpha_n)\beta_n\Big[\beta_n\|\gamma f_n(y_n) - BW_n\rho_n\| \\
&\quad + 2\|\gamma f_n(y_n) - BW_n\rho_n\|\|\rho_n - p\|\Big]
\end{aligned}$$

$$\leq \Big[\|x_n - p\| + \|x_{n+1} - p\|\Big]\|x_{n+1} - x_n\|$$
$$+ (1-\alpha_n)\beta_n\Big[\beta_n\|\gamma f_n(y_n) - BW_n\rho_n\|$$
$$+ 2\|\gamma f_n(y_n) - BW_n\rho_n\|\|\rho_n - p\|\Big].$$

From conditions (i) and (iv), we get

$$\lim_{n\to\infty}\|Ay_n - Ap\| = 0. \tag{25}$$

Observe that

$$\begin{aligned}
\|\rho_n - p\|^2 &= \|P_C(I - r_nA)y_n - P_C(I - r_nA)p\|^2\\
&\leq \langle (I - r_nA)y_n - (I - r_nA)p, \rho_n - p\rangle\\
&= \frac{1}{2}\big\{\|(I - r_nA)y_n - (I - r_nA)p\|^2 + \|\rho_n - p\|^2\\
&\quad - \|(I - r_nA)y_n - (I - r_nA)p - (\rho_n - p)\|^2\big\}\\
&\leq \frac{1}{2}\big\{\|y_n - p\|^2 + \|\rho_n - p\|^2 - \|(y_n - \rho_n) - r_n(Ay_n - Ap)\|^2\big\}\\
&\leq \frac{1}{2}\big\{\|x_n - p\|^2 + \|\rho_n - p\|^2 - \|y_n - \rho_n\|^2 - r_n^2\|Ay_n - Ap\|^2\\
&\quad + 2r_n\langle y_n - \rho_n, Ay_n - Ap\rangle\big\},
\end{aligned}$$

which yields that

$$\begin{aligned}
\|\rho_n - p\|^2 &\leq \|x_n - p\|^2 - \|y_n - p\|^2 - r_n^2\|Ay_n - Ap\|^2\\
&\quad + 2r_n\langle y_n - \rho_n, Ay_n - Ap\rangle\\
&\leq \|x_n - p\|^2 - \|y_n - p\|^2 - r_n^2\|Ay_n - Ap\|^2\\
&\quad + 2r_n\|y_n - \rho_n\|\|Ay_n - Ap\|\\
&\leq \|x_n - p\|^2 - \|y_n - p\|^2\\
&\quad + 2r_n\|y_n - \rho_n\|\|Ay_n - Ap\|.
\end{aligned} \tag{26}$$

From (24) and (26), we get

$$\|x_{n+1} - p\|^2 \leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\Big(\beta_n\|\gamma f_n(y_n) - Bp\|^2$$

$$
\begin{aligned}
&+\|\rho_n-p\|^2+2\beta_n\|\gamma f_n(y_n)-Bp\|\|\rho_n-p\|\Big)\\
&\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\Big(\beta_n\|\gamma f_n(y_n)-Bp\|^2\\
&+\big(\|x_n-p\|^2-\|y_n-\rho_n\|^2-r_n^2\|Ay_n-Ap\|^2\big)\\
&+2r_n\|y_n-\rho_n\|\|Ay_n-Ap\|\\
&+2\beta_n\|\gamma f_n(y_n)-Bp\|\|\rho_n-p\|\Big)\\
&\le \|x_n-p\|^2+(1-\alpha_n)\beta_n\|\gamma f_n(y_n)-Bp\|^2\\
&-(1-\alpha_n)\|y_n-\rho_n\|^2\\
&-(1-\alpha_n)r_n^2\|Ay_n-Ap\|^2\\
&+(1-\alpha_n)2r_n\|y_n-\rho_n\|\|Ay_n-Ap\|\\
&+(1-\alpha_n)2\beta_n\|\gamma f_n(y_n)-Bp\|\|\rho_n-p\|.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
(1-\alpha_n)\|y_n-\rho_n\|^2 &\le \|x_n-p\|^2+(1-\alpha_n)\beta_n\|\gamma f_n(y_n)-Bp\|^2\\
&-(1-\alpha_n)r_n^2\|Ay_n-Ap\|^2+(1-\alpha_n)2r_n\|y_n-\rho_n\|\|Ay_n-Ap\|\\
&+(1-\alpha_n)2\beta_n\|\gamma f_n(y_n)-Bp\|\|\rho_n-p\|-\|x_{n+1}-p\|^2\\
&\le \big(\|x_n-p\|+\|x_{n+1}-p\|\big)\|x_n-x_{n+1}\|+(1-\alpha_n)\beta_n\|\gamma f_n(y_n)-Bp\|^2\\
&-(1-\alpha_n)r_n^2\|Ay_n-Ap\|^2+(1-\alpha_n)2r_n\|y_n-\rho_n\|\|Ay_n-Ap\|\\
&+(1-\alpha_n)2\beta_n\|\gamma f_n(y_n)-Bp\|\|\rho_n-p\|.
\end{aligned}
$$

Using the conditions, we obtain that

$$
\lim_{n\to\infty}\|y_n-\rho_n\|=0. \tag{27}
$$

From (17), (18), (19) , (20) and (21), we have

$$
\begin{aligned}
\|u_{n+1}-u_n\|-\|x_{n+1}-x_n\| &\le \beta_{n+1}M_{n+1}+\beta_nM_n+\delta_{n+1}L_{n+1}+\delta_nL_n\\
&+|r_{n+1}-r_n|\|Ay_n\|+M_1\prod_{i=1}^{n}\lambda_i.
\end{aligned}
$$

Then

$$\limsup_{n\to\infty}\left(\|u_{n+1}-u_n\|-\|x_{n+1}-x_n\|\right)\leq 0.$$

By virtue of lemma 2.7, we obtain that

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{28}$$

We have

$$\begin{aligned}\|\rho_n-W_n\rho_n\| &\leq \|y_n-\rho_n\|+\|y_n-u_n\|+\|u_n-W_n\rho_n\|\\ &\leq \|y_n-\rho_n\|+\|y_n-x_n\|+\|u_n-x_n\|+\|u_n-W_n\rho_n\|. \end{aligned}\tag{29}$$

Then, from (22), (23), (27) and (28), we obtain

$$\lim_{n\to\infty}\|W_n\rho_n-\rho_n\|=0. \tag{30}$$

Notice that

$$\|W\rho_n-\rho_n\|\leq \|W_n\rho_n-\rho_n\|+\|W_n\rho_n-W\rho_n\|.$$

From (30) and lemma 2.5, we find that

$$\lim_{n\to\infty}\|W\rho_n-\rho_n\|=0. \tag{31}$$

Next, let us show that, there exist a unique $x^*\in F$ such that for all $m\in\mathbb{N}$

$$\limsup_{n\to\infty}\langle\gamma f_m(x^*)-Bx^*,x_n-x^*\rangle\leq 0. \tag{32}$$

$P_F(\gamma f_m+(I-B))$ is a contraction of H into itself. In fact from Lemma 2.7, we see that

$$\begin{aligned}&\| P_F(\gamma f_m+(I-B))x-P_F(\gamma f_m+(I-B))y \|\\ &\quad\leq\| (\gamma f_m+(I-B))x-(\gamma f_m+(I-B))y \|\\ &\quad\leq\| I-B \|\| x-y \| +\gamma k_m \| x-y \|\\ &\quad=\lim_{p\to\infty} \| I-(1-\frac{1}{p})B \|\| x-y \| +\gamma k_m \| x-y \|\\ &\quad\leq\lim_{p\to\infty}(1-(1-\frac{1}{p})\overline{\gamma}) \| x-y \| +\gamma k_m \| x-y \|\\ &\quad\leq(1-\overline{\gamma}+k_m\gamma) \| x-y \|,\end{aligned}$$

and hence$P_F(\gamma f_m+(I-B))$ is a contraction due to $(1-(\overline{\gamma}-k_m\gamma))\in(0,1)$. Therefore, by Banach's contraction principal, $P_F(I-B+\gamma f_m)$ has a unique fixed point x^*. Then using (2), x^* is the unique solution of the variational inequality:

$$\langle(\gamma f_m-B)x^*,x-x^*\rangle\leq 0,\qquad \forall x\in F,\forall m\in\mathbb{N}. \tag{33}$$

We can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle\gamma f(x^*)-Bx^*,x_n-x^*\rangle=\lim_{i\to\infty}\langle\gamma f(x^*)-Bx^*,x_{n_i}-x^*\rangle. \tag{34}$$

As $\{x_{n_i}\}$ is bounded, there is a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converges weakly to p. We may assume, without loss of generality, that $x_{n_i}\rightharpoonup p$. Now, we show that $p\in F$. Indeed, let us first show that $p\in VI(C,A)$. Let $T:H\to 2^H$ be a set-valued mapping defined by

$$Tx=\begin{cases}Ax+N_cx, & x\in C,\\ \emptyset, & x\notin C.\end{cases} \tag{35}$$

Since A is inverse-strongly monotone, T is maximal monotone (see [1]). Let $(x,y)\in G(T)$. Since $y-Ax\in N_cx$ and $\rho_n\in C$, we have

$$\langle x-\rho_n,y-Ax\rangle\geq 0. \tag{36}$$

Since $\rho_n=P_c(I-r_nA)x_n$, we have

$$\langle x-\rho_n,\rho_n-(I-r_nA)x_n\rangle\geq 0.$$

It follows that

$$\langle x-\rho_n,\frac{\rho_n-x_n}{r_n}+Ax_n\rangle\geq 0. \tag{37}$$

Since $x_{n_i}\rightharpoonup p$ and A is an $\alpha-invers$ strongly monotone, so from (36), (37) and by using similar method as used in the proof of Theorem 3.1 of [11] and Theorem 3.1 of [12], we can prove that $\langle x-p,y\rangle\geq 0$. Since T is maximal monotone, we have $z\in T^{-1}0$, and hence $p\in VI(C,A)$. Since Hilbert spaces are Opial spaces, again by using similar method as used in the proof of Theorem 3.1 of [11] and Theorem 3.1 of [12], we can prove that $z\in Fix(W)$. Thus, from Lemma 2.4, we get $p\in\bigcap_{i=1}^{\infty}F(T_i)$. Therefore $p\in F$ and applying (33) and 34), we have

$$\limsup_{n\to\infty}\langle\gamma f_m(x^*)-Bx^*,x_n-x^*\rangle\leq 0. \tag{38}$$

Finally, by using similar method as used in the proof of Theorem 2.1 of [4] we can show that $x_n\to x^*$ strongly as $n\to\infty$. These completes the proof.

Theorem 3.2. [4] *Let H be a real Hilbert space and C a nonempty closed convex subset of H such that it is closed for linear operator. Let $A : H \to H$ be an $\alpha-$invers strongly monotone mapping and f a $k-contraction$ on H. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of nonexpansive mappings from C into itself such that $F := \bigcap_{i=1}^{\infty} F(T_i) \cap VI(C,A) \neq \emptyset$. Let B be a strongly positive linear bounded self-adjoint operator of C into itself with the constant $\overline{\gamma} > 0$. Let $\{x_n\}$ be a sequence generated in*

$$\begin{cases} x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n, \\ y_n = \beta_n \gamma f(x_n) + (I - \beta_n B) W_n P_c (I - r_n A) x_n, \end{cases}$$

where W_n is generated in (8), $\{\alpha_n\}$ and $\{\beta_n\}$ are real number sequence in $(0,1)$. Assume that the control sequence $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy the following restrictions:

(i) $\lim_{n\to\infty} \beta_n = 0, \sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(iii) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$;

(iv) $\{r_n\} \subset [a,b]$ *for some a,b with* $0 < a < b < 2\alpha$.

Assume that $0 < \gamma < \overline{\gamma}/k$. Then $\{x_n\}$ strongly converges to some point q, where $q = P_F(\gamma f + (I - B))(q)$, which solves the variational inequality

$$\langle \gamma f(q) - Bq, p - q \rangle \leq 0, \ \forall p \in F.$$

Proof. Since every nonexpansive mapping is λ-strictly pseudo-contractive mappings, for $\lambda = 0$, so by taking $\delta_n = 0$ and $f_n = f$ for all $n \in \mathbb{N}$ in Theorem 3.1 the proof is complete.

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