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## A NEW ALGORITHM FOR SOLVING NONLINEAR OPTIMIZATION PROBLEMS

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**Abstract.** The aim of this paper is to study common solutions of two nonlinear problems: variational inequality and fixed point problems. A mean valued algorithm with computational errors is introduced for solving the common solution problem. Convergence analysis of the algorithm is obtained in the framework of Hilbert spaces.

**Keywords.** Monotone operator; Nonexpansive mapping; Variational inequality; Fixed point.

# 1. Introduction-Preliminaries

Throughout this paper, we assume that $H$ is a real Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let $C$ be a convex and closed subset of $H$ and $Proj_C$ be the metric projection from $H$ onto $C$.

Let $T$ be a mapping on $C$. Next, we denote by $F(T)$ the set of fixed points of $T$. Recall that $T$ is said to be contractive iff there exists $\alpha \in (0,1)$ such that

$$\|Tx - Ty\| \le \alpha\|x-y\|, \quad \forall x,y \in C.$$

We also say $T$ is an $\alpha$-contractive mapping. Recall that $T$ is said to be nonexpansive iff

$$\|Tx - Ty\| \le \|x-y\|, \quad \forall x,y \in C.$$

It is known that the fixed point set of nonexpansive mappings is nonempty on bounded closed convex subsets of $H$.

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Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, [1]-[17] and the references therein. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space $H$:

$$\min_{x\in F(T)} \frac{1}{2}\langle Bx,x\rangle - \langle x,y\rangle,$$

where $B$ is a linear bounded operator on $H$, and $y$ is a given point in $H$.

In [16], it is proved that sequence $\{x_n\}$ defined by the iterative algorithm below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = \alpha_n y + (I - \alpha_n B)Tx_n, \quad \forall n \geq 0,$$

converges strongly to the unique solution of the minimization problem provided the sequence $\{\alpha_n\}$ satisfies certain conditions.

Recall that a mapping $A : C \to H$ is said to be inverse-strongly monotone if there exists a positive real number $\mu$ such that

$$\langle Ax - Ay, x - y\rangle \geq \mu\|Ax - Ay\|^2, \quad \forall x,y \in C.$$

For such a case, $A$ is also said to be $\mu$-inverse-strongly monotone.

Recall that a mapping $A : C \to H$ is said to be strongly monotone if there exists a positive real number $\mu$ such that

$$\langle Ax - Ay, x - y\rangle \geq \mu\|x - y\|^2, \quad \forall x,y \in C.$$

For such a case, $A$ is also said to be $\mu$-strongly monotone.

The classical variational inequality problem is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in C.$$

We denoted by $VI(C,A)$ the set of solutions of the variational inequality. For a given $z \in H, u \in C$ satisfies the inequality

$$\langle u - z, v - u\rangle \geq 0, \quad \forall v \in C,$$

if and only if $u = Proj_C z$. It is known that projection operator $P_C$ is firmly nonexpansive. It is also known that $Proj_C x$ is characterized by the property: $Proj_C x \in C$ and $\langle x - Proj_C x, Proj_C x - y\rangle \geq 0$ for all $y \in C$.

One can see that the variational inequality problem is equivalent to a fixed point problem, that is, an element $u \in C$ is a solution of the variational inequality if and only if $u \in C$ is a fixed point of the mapping $Proj_C(I - \lambda A)$, where $\lambda > 0$ is a constant and $I$ is the identity mapping. Recently, variational inequality and fixed point problems have been considered by many authors; see, e.g., [1]-[4], [8], [9], [13]-[15], [18]-[23] and the references therein.

Variational inequalities has emerged as an important tool in studying a wide class of real world problems arising in several branches of pure and applied sciences in a unified and general framework. This field is dynamics and is experiencing an explosive growth in both theory and applications. Recently, several numerical techniques including the Wiener-Hopf equations, resolvents, gradient projections, auxiliary principle, decomposition and descent are being developed for solving various classes of variational inequalities and related optimization problems.

Recently, Marino and Xu [17] introduced a general iterative scheme by the viscosity approximation method:

$$x_0 \in H, \quad x_{n+1} = (I - \alpha_n B)Tx_n + \alpha_n \gamma f(x_n), \quad \forall n \geq 0, \tag{1.1}$$

where $T$ is a nonexpansive mapping on $H$, $f$ is a contraction on $H$ with the coefficient $\alpha$, $B$ is a bounded linear strongly positive operator on $H$ with the coefficient $\bar{\gamma}$ and $\gamma$ is a constant such that $0 < \gamma < \bar{\gamma}/\alpha$. They proved that the sequence $\{x_n\}$ generated by the iterative scheme (1.1) converges strongly to the unique solution of the variational inequality:

$$\langle (B - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(S),$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(T)} \frac{1}{2} \langle Bx, x \rangle - h(x),$$

where $h$ is a potential function for $\gamma f$ (i.e., $h'(x) = \gamma f(x)$ for all $x \in H$.)

Very recently, Chen et al. [19] studied the following iterative process:

$$x_1 \in C, \quad x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) SP_C(x_n - \lambda_n Ax_n), \quad \forall n \geq 1, \tag{1.2}$$

and also obtained a strong convergence theorem by so-called viscosity approximation method discussed by Moudafi [24] in the framework of Hilbert spaces.

**Lemma 1.1.** *In Hilbert spaces, the following inequality holds*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x,y \in H.$$

**Lemma 1.2.** [17] *Assume that $B$ is a strong positive linear bounded operator on a Hilbert space $H$ with coefficient $\bar{\gamma} > 0$ and $\|B\|^{-1} \geq \rho > 0$. Then $\|I - \rho B\| \leq 1 - \rho\bar{\gamma}$.*

**Lemma 1.3.** [17] *Let $H$ be a Hilbert space, $B$ be a strongly positive linear bounded self-adjoint operator on $H$ with the coefficient $\bar{\gamma} > 0$. Assume that $\bar{\gamma}/\alpha > \gamma > 0$. Let $T : H \to H$ be a nonexpansive mapping with a fixed point $x_t$ of the contraction $x \mapsto (I - tB)Tx + t\gamma f(x)$. Then $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point $\bar{x}$ of $T$, which solves the variational inequality:*

$$\langle \bar{x} - z, f(\bar{x}) - \frac{B\bar{x}}{\gamma}\rangle \geq 0, \quad \forall z \in F(T).$$

**Lemma 1.4.** [25] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $X$ and let $\beta_n$ be a sequence in [0,1] with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

*Then $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

**Lemma 1.5.** [26] *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n + e_n,$$

*where $\{\gamma_n\}$ is a sequence in $(0,1)$, $\{e_n\}$ and $\{\delta_n\}$ are sequences such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\sum_{n=1}^{\infty} e_n < \infty$;
(iii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

*Then $\lim_{n\to\infty} \alpha_n = 0$.*

**Lemma 1.6.** [21] *Let $H$ be a Hilbert space, $C$ a closed convex subset of $H$, $f : C \to C$ a contraction with the coefficient $\alpha \in (0,1)$ and $B$ a strongly positive linear bounded operator* *with the coefficient* $\bar{\gamma} > 0$. *Then, for any* $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$,

$$\langle x - y, (B - \gamma f)x - (B - \gamma f)y \rangle \geq (\bar{\gamma} - \gamma\alpha)\|x - y\|^2, \quad \forall x, y \in C.$$

*That is,* $B - \gamma f$ *is strongly monotone with coefficient* $\bar{\gamma} - \alpha\gamma$.

## 2. Main results

**Theorem 2.1.** *Let* $H$ *be a real Hilbert space and let* $C$ *be a nonempty convex closed subset of* $H$. *Let* $T$ *be a nonexpansive self mappings on* $C$. *Let* $f : C \to C$ *be an* $\alpha$*-contraction and Let* $B$ *be a strongly positive linear bounded self-adjoint operator of* $C$ *into itself with the coefficient* $\bar{\gamma} > 0$. *Let* $A : C \to H$ *be a* $\mu$*-inverse-strongly monotone mapping. Assume that* $\bar{\gamma} > \alpha\gamma > 0$ *and* $F = F(T) \cap VI(C,A) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated in the following manner* $x_1 \in C$, $y_n = TProj_C\big(x_n - \lambda_n A x_n + e_n\big)$, $x_{n+1} = Proj_C\Big((1-\alpha_n)\beta_n\gamma f(y_n) + \alpha_n x_n + (1-\alpha_n)(I - \beta_n B)y_n\Big)$, $\forall n \geq 1$, $\{\alpha_n\}$, $\{\beta_n\}$ *are sequences in* $(0,1)$ *and* $\{\lambda_n\}$ *is a sequence in* $(0, 2\mu)$ *and* $\{e_n\}$ *is a sequence in* $H$. *Assume that* $\{\alpha_n\}$, $\{\beta_n\}$, $\{e_n\}$ *and* $\{\lambda_n\}$ *satisfy* $1 > \limsup_{n\to\infty} \alpha_n \geq \liminf_{n\to\infty} \alpha_n > 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$, $\sum_{n=1}^{\infty} \|e_n\| < \infty$, $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$, $\{\lambda_n\} \subset [\lambda, \lambda']$ *for some* $\lambda, \lambda'$ *with* $0 < \lambda \leq \lambda' < 2\mu$. *Then sequence* $\{x_n\}$ *converges strongly to some* $q \in F$, *where* $q = Proj_F(\gamma f + I - B)q$.

**Proof.** First, we show that mappings $I - \lambda_n A$ is nonexpansive. For $\forall x, y \in C$, we have

$$\begin{aligned}
&\|(I - \lambda_n A)x - (I - \lambda_n A)y\|^2 \\
&\leq \|x - y\|^2 - 2\lambda_n \langle Ax - Ay, x - y \rangle + \lambda_n^2 \|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 + \lambda_n(\lambda_n - 2\mu)\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2
\end{aligned}$$

This shows that $I - \lambda_n A$ are nonexpansive. Put $z_n = (I - \beta_n B)y_n + \beta_n \gamma f(y_n)$. With no loss of generality, we may that $\beta_n \leq \|B\|^{-1}$ for all $n \geq 1$. From Lemma 1.2, we know that, if $0 < \beta_n \leq$

$\|B\|^{-1}$ for all $n \geq 1$, then $\|I-\beta_n B\| \leq 1-\beta_n\bar{\gamma}$. Letting $p \in F$, we have

$$\begin{aligned}\|y_n - p\| \leq & \|Proj_C\big(x_n - \lambda_n A x_n + e_n\big) - p\| \\ \leq & \|\big(x_n - \lambda_n A x_n\big) - \big(p - \lambda_n A p\big) + e_n\| \\ \leq & \|x_n - p\| + \|e_n\|.\end{aligned}$$

Putting $z_n = \beta_n \gamma f(y_n) + (I-\beta_n B)y_n$, one has

$$\begin{aligned}\|z_n - p\| \leq & \beta_n \|\gamma f(y_n) - Bp\| + \|I-\beta_n B\| \|y_n - p\| \\ \leq & \beta_n[\gamma\|f(y_n) - f(p)\| + \|\gamma f(p) - Bp\|] + (1-\beta_n\bar{\gamma})\|y_n - p\| \\ \leq & \beta_n[\gamma\|f(y_n) - f(p)\| + \|\gamma f(p) - Bp\|] + (1-\beta_n\bar{\gamma})\|x_n - p\| + e_n \\ \leq & [1-(\bar{\gamma}-\gamma\alpha)\beta_n]\|x_n - p\| + \beta_n\|\gamma f(p) - Bp\| + e_n,\end{aligned}$$

which yields that

$$\begin{aligned}\|x_{n+1} - p\| \leq & (1-\alpha_n)\|z_n - p\| + \alpha_n\|x_n - p\| \\ \leq & (1-\alpha_n)[1-(\bar{\gamma}-\gamma\alpha)\beta_n]\|x_n - p\| + \alpha_n\|x_n - p\| \\ & + (1-\alpha_n)\beta_n\|\gamma f(p) - Bp\| + e_n.\end{aligned}$$

Since $\sum_{n=1}^{\infty}\|e_n\| < \infty$, we find from the mathematical induction that sequence $\{x_n\}$ is bounded.

Next, we show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Letting $\xi_n = Proj_C\big(x_n - \lambda_n A x_n + e_n\big)$, one finds that

$$\begin{aligned}\|\xi_{n+1} - \xi_n\| \leq & \|\big(x_{n+1} - \lambda_{n+1} A x_{n+1} + e_{n+1}\big) - \big(x_n - \lambda_n A x_n + e_n\big)\| \\ \leq & \|\big(x_{n+1} - \lambda_{n+1} A x_{n+1} + e_{n+1}\big) - \big(x_n - \lambda_{n+1} A x_n + e_n\big)\| \\ & + \|\big(x_n - \lambda_{n+1} A x_n + e_n\big) - \big(x_n - \lambda_n A x_n + e_n\big)\| \\ \leq & \|x_{n+1} - x_n\| + \|e_{n+1} - e_n\| + \|Ax_n\||\lambda_{n+1} - \lambda_n|.\end{aligned} \tag{2.1}$$

It follows from (2.1) that

$$\begin{aligned}\|y_{n+1} - y_n\| = & \|T\xi_{n+1} - T\xi_n\| \\ \leq & \|\xi_{n+1} - \xi_n\| \\ \leq & \|x_{n+1} - x_n\| + \|e_{n+1} - e_n\| + \|Ax_n\||\lambda_{n+1} - \lambda_n|.\end{aligned} \tag{2.2}$$

Hence, one has

$$\begin{aligned}\|z_{n+1}-z_n\| &\le \|I-\beta_{n+1}B\|\|y_{n+1}-y_n\|+|\beta_n-\beta_{n+1}|\|By_n\| \\ &\quad +\beta_{n+1}\gamma\|f(y_{n+1})-f(y_n)\|+\gamma|\beta_{n+1}-\beta_n|\|f(y_n)\| \\ &\le \big(1-\beta_{n+1}(\bar{\gamma}-\gamma\alpha)\big)\|y_{n+1}-y_n\|+|\beta_n-\beta_{n+1}|(\|By_n\|+\gamma\|f(y_n)\|).\end{aligned} \tag{2.3}$$

Substituting (2.2) into (2.3), one finds that

$$\begin{aligned}\|z_{n+1}-z_n\| &\le \big(1-\beta_{n+1}(\bar{\gamma}-\gamma\alpha)\big)\|x_{n+1}-x_n\|+\|e_{n+1}-e_n\|+\|Ax_n\||\lambda_{n+1}-\lambda_n| \\ &\quad +|\beta_n-\beta_{n+1}|(\|By_n\|+\gamma\|f(y_n)\|).\end{aligned}$$

This yields that $\limsup_{n\to\infty}(\|z_n-z_{n+1}\|-\|x_{n+1}-x_n\|)\le 0$. By virtue of Lemma 1.4, we obtain that $\lim_{n\to\infty}\|z_n-x_n\|=0$. On the other hand, we have $\|x_{n+1}-x_n\|\le(1-\alpha_n)\|x_n-z_n\|$. This implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{2.4}$$

Since $\beta_n\to 0$ as $n\to\infty$, one finds that

$$\lim_{n\to\infty}\|z_n-y_n\|=0. \tag{2.5}$$

For all $p\in F$, we have

$$\begin{aligned}\|\xi_n-p\|^2 &\le \|(x_n-p)-\lambda_n(Ax_n-Ap)\|^2+\|e_n\|^2+2\|e_n\|\|(x_n-p)-\lambda_n(Ax_n-Ap)\| \\ &\le \|(x_n-p)-\lambda_n(Ax_n-Ap)\|^2+\|e_n\|^2+2\|e_n\|\|x_n-p\| \\ &\le \|x_n-p\|^2-\lambda_n(2\mu-\lambda_n)\|Ax_n-Ap\|^2+\|e_n\|^2+2\|e_n\|\|x_n-p\|.\end{aligned}$$

Hence, one has

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \|\alpha_n(x_n-p)+(1-\alpha_n)(z_n-p)\|^2\\
&\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)(\beta_n\|\gamma f(y_n)-Bp\|+(1-\beta_n\bar{\gamma})\|y_n-p\|)^2\\
&\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\beta_n\|\gamma f(y_n)-Bp\|^2+(1-\alpha_n)(1-\beta_n\bar{\gamma})\|y_n-p\|^2\\
&\quad +2(1-\alpha_n)\beta_n\|\gamma f(y_n)-Bp\|\|y_n-p\|\\
&\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\beta_n\|\gamma f(y_n)-Bp\|^2+(1-\alpha_n)(1-\beta_n\bar{\gamma})\|y_n-p\|^2\\
&\quad +2(1-\alpha_n)\beta_n\|\gamma f(y_n)-Ap\|\|\xi_n-p\|\\
&\le \|x_n-p\|^2-(1-\alpha_n)(1-\beta_n\bar{\gamma})\lambda_n(2\mu-\lambda_n)\|Ax_n-Ap\|^2+\|e_n\|^2+2\|e_n\|\|x_n-p\|\\
&\quad +2\beta_n\|\gamma f(y_n)-Ap\|\|\xi_n-p\|+\beta_n\|\gamma f(y_n)-Bp\|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}&(1-\alpha_n)(1-\beta_n\bar{\gamma})\lambda_n(2\mu-\lambda_n)\|Ax_n-Ap\|^2\\
&\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+\|e_n\|^2+2\|e_n\|\|x_n-p\|\\
&\quad +2\beta_n\|\gamma f(y_n)-Ap\|\|\xi_n-p\|+\beta_n\|\gamma f(y_n)-Bp\|^2.\end{aligned}$$

This yields that

$$\lim_{n\to\infty}\|Ax_n-Ap\|=0. \tag{2.6}$$

On the other hand, we have

$$\begin{aligned}\|\xi_n-p\|^2 &\le \langle (x_n-\lambda_nAx_n+e_n)-(p-\lambda_nAp),\xi_n-p\rangle\\
&\le \frac{1}{2}\Big(\|(I-\lambda_nA)x_n-(I-\lambda_nA)p\|^2+\|e_n\|^2+2\|e_n\|\|(I-\lambda_nA)x_n-(I-\lambda_nA)p\|\\
&\quad +\|\xi_n-p\|^2-\|(x_n-\xi_n)-\lambda_n(Ax_n-Ap)+e_n\|^2\Big)\\
&\le \frac{1}{2}\Big(\|x_n-p\|^2+2\|e_n\|\|x_n-p\|+\|\xi_n-p\|^2\\
&\quad -\|x_n-\xi_n\|^2-\lambda_n^2\|Ax_n-Ap\|^2+2\lambda_n\|x_n-\xi_n\|\|Ax_n-Ap\|\\
&\quad +2\|e_n\|\|(x_n-\xi_n)-\lambda_n(Ax_n-Ap)\|\Big),\end{aligned}$$

which yields that

$$\|\xi_n - p\|^2 \leq \|x_n - p\|^2 + 2\|e_n\|\|x_n - p\| - \|x_n - \xi_n\|^2 + 2\lambda_n\|x_n - \xi_n\|\|Ax_n - Ap\| + 2\|e_n\|\|(x_n - \xi_n) - \lambda_n(Ax_n - Ap)\|.$$

Note that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \|\alpha_n(x_n - p) + (1-\alpha_n)(z_n - p)\|^2\\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)(\beta_n\|\gamma f(y_n) - Bp\| + (1-\beta_n\bar{\gamma})\|y_n - p\|)^2\\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|\gamma f(y_n) - Bp\|^2 + (1-\alpha_n)(1-\beta_n\bar{\gamma})\|y_n - p\|^2\\ &\quad + 2(1-\alpha_n)\beta_n\|\gamma f(y_n) - Bp\|\|y_n - p\|\\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|\gamma f(y_n) - Bp\|^2 + (1-\alpha_n)(1-\beta_n\bar{\gamma})\|\xi_n - p\|^2\\ &\quad + 2(1-\alpha_n)\beta_n\|\gamma f(y_n) - Ap\|\|\xi_n - p\|.\end{aligned}$$

Further, one has

$$\begin{aligned}\|x_{n+1} - p\|^2 \leq{}& \|x_n - p\|^2 + \beta_n\|\gamma f(y_n) - Bp\|^2 + 2\|e_n\|\|x_n - p\|\\ &- (1-\alpha_n)(1-\beta_n\bar{\gamma})\|x_n - \xi_n\|^2 + 2\lambda_n\|x_n - \xi_n\|\|Ax_n - Ap\|\\ &+ 2\|e_n\|\|(x_n - \xi_n) - \lambda_n(Ax_n - Ap)\| + 2\beta_n\|\gamma f(y_n) - Ap\|\|\xi_n - p\|.\end{aligned}$$

Therefore, one has

$$\begin{aligned}&(1-\alpha_n)(1-\beta_n\bar{\gamma})\|x_n - \xi_n\|^2\\ &\leq (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + \beta_n\|\gamma f(y_n) - Bp\|^2 + 2\|e_n\|\|x_n - p\|\\ &\quad + 2\lambda_n\|x_n - \xi_n\|\|Ax_n - Ap\| + 2\|e_n\|\|(x_n - \xi_n) - \lambda_n(Ax_n - Ap)\|\\ &\quad + 2\beta_n\|\gamma f(y_n) - Ap\|\|\xi_n - p\|.\end{aligned}$$

In view of the restrictions, one obtains $\lim_{n\to\infty}\|x_n - \xi_n\| = 0$, which implies $\lim_{n\to\infty}\|T\xi_n - \xi_n\| = 0$. From Lemma 1.6, $P_F(\gamma f + (I - B))$ has a unique fixed point. Next, we use $q$ to denote the unique fixed point $q = P_F(\gamma f + (I - B))(q)$.

Next, we show that $\limsup_{n\to\infty}\langle x_n - q, \gamma f(q) - Bq\rangle \leq 0$. To see this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x_n - q, \gamma f(q) - Bq\rangle = \lim_{i\to\infty}\langle x_{n_i} - q, \gamma f(q) - Bq\rangle.$$

Since $\{\xi_{n_i}\}$ is bounded, there exists a subsequence $\{\xi_{n_{i_j}}\}$ of $\{\xi_{n_i}\}$ which converges weakly to $w$. Without loss of generality, we can assume that $\xi_{n_i} \rightharpoonup w$. Since the space has the Opial's condition, we have $w = Tw$. Put $M\xi = \emptyset$, $\xi \notin C$ and $M\xi = N_C + A\xi$, $\xi \in C$. Then $M$ is also a maximal monotone operator. Let $(\xi, \xi') \in Graph(M)$. Since $\xi' - A\xi \in N_C\xi$ and $\xi_n \in C$, we have $\langle \xi - \xi_n, \xi' - A\xi \rangle \geq 0$. On the other hand, we have from $\xi_n = Proj_C(x_n - \lambda_n Ax_n + e_n)$ that $\langle \xi - \xi_n, \xi_n - (I - \lambda_n A)x_n - e_n \rangle \geq 0$. It follows from the above that

$$\begin{aligned}
\langle \xi - \xi_{n_i}, \xi' \rangle &\geq \langle \xi - \xi_{n_i}, A\xi \rangle \\
&\geq \langle \xi - \xi_{n_i}, A\xi - \frac{\xi_{n_i} - x_{n_i}}{\lambda_{n_i}} - Ax_{n_i} + \frac{e_{n_i}}{\lambda_{n_i}} \rangle \\
&= \langle \xi - \xi_{n_i}, A\xi - A\xi_{n_i} \rangle + \langle \xi - \xi_{n_i}, A\xi_{n_i} - Ax_{n_i} \rangle \\
&\quad - \langle \xi - \xi_{n_i}, \frac{\xi_{n_i} - x_{n_i}}{\lambda_{n_i}} - \frac{e_{n_i}}{\lambda_{n_i}} \rangle \\
&\geq \langle \xi - \xi_{n_i}, A\xi_{n_i} - Ax_{n_i} \rangle - \langle \xi - \xi_{n_i}, \frac{\xi_{n_i} - x_{n_i}}{\lambda_{n_i}} - \frac{e_{n_i}}{\lambda_{n_i}} \rangle,
\end{aligned}$$

which implies $\langle \xi - w, \xi' \rangle \geq 0$. We have $w \in M^{-1}0$ and hence $w \in VI(C, A)$. This completes the proof $w \in F$. Hence, one has $\limsup_{n\to\infty} \langle x_n - q, \gamma f(q) - Bq \rangle \leq 0$. Note that $\|y_n - q\|^2 \leq \|x_n - q\|^2 + \nu_n$, where $\nu_n = \|e_n\|(\|e_n\| + 2\|x_n - q\|)$. It follows that

$$\begin{aligned}
\|z_n - q\|^2 &\leq \|(I - \beta_n B)(y_n - q)\|^2 + 2\beta_n \langle \gamma f(y_n) - Bq, z_n - q \rangle \\
&\leq (1 - \beta_n \bar{\gamma})^2 (\|x_n - q\|^2 + \nu_n) + 2\beta_n \langle \gamma f(y_n) - Bq, z_n - q \rangle \\
&\leq (1 - \beta_n \bar{\gamma})^2 \|x_n - q\|^2 + 2\beta_n \gamma \langle f(y_n) - f(q), z_n - q \rangle \\
&\quad + 2\beta_n \langle \gamma f(q) - Bq, z_n - q \rangle + \nu_n \\
&\leq (1 - \beta_n \bar{\gamma})^2 \|x_n - q\|^2 + \beta_n \gamma \alpha (\|x_n - q\|^2 + \|z_n - q\|^2) \\
&\quad + 2\beta_n \langle \gamma f(q) - Bq, z_n - q \rangle + \nu_n (1 + \beta_n \gamma \alpha),
\end{aligned}$$

which implies that

$$\begin{aligned}
\|z_n - q\|^2 \leq \big(1 - \frac{2\beta_n(\bar{\gamma} - \alpha\gamma)}{1 - \beta_n \gamma \alpha}\big) \|x_n - q\|^2 + \frac{2\beta_n(\bar{\gamma} - \alpha\gamma)}{1 - \beta_n \gamma \alpha} \Big( \frac{1}{\bar{\gamma} - \alpha\gamma} \langle \gamma f(q) - Bq, z_n - q \rangle \\
+ \frac{\beta_n \bar{\gamma}^2}{2(\bar{\gamma} - \alpha\gamma)} K \Big) + \nu_n (1 + \beta_n \gamma \alpha),
\end{aligned}$$

where $M = \sup_{n\geq 1}\{\|x_n - q\|^2\}$. This yields that

$$\|x_{n+1}-p\|^2 \leq \Big(1-(1-\alpha_n)\frac{2\beta_n(\bar{\gamma}-\alpha\gamma)}{1-\beta_n\gamma\kappa}\Big)\|x_n-q\|^2$$
$$+(1-\alpha_n)\frac{2\beta_n(\bar{\gamma}-\alpha\gamma)}{1-\beta_n\gamma\alpha}\Big(\frac{1}{\bar{\gamma}-\alpha\gamma}\langle\gamma f(q)-Bq, z_n-q\rangle+\frac{\beta_n\bar{\gamma}^2}{2(\bar{\gamma}-\alpha\gamma)}K\Big)+\nu_n(1+\beta_n\gamma\alpha).$$

Using 1.5, one obtain the desired conclusion immediately. The proof is completed.

From Theorem 2.1, the following result on the variational inequality is not hard to derive.

**Corollary 2.2.** *Let $H$ be a real Hilbert space and let $C$ be a nonempty convex closed subset of $H$. Let $f: C \to C$ be an $\alpha$-contraction and Let $B$ be a strongly positive linear bounded self-adjoint operator of $C$ into itself with the coefficient $\bar{\gamma} > 0$. Let $A: C \to H$ be a $\mu$-inverse-strongly monotone mapping. Assume that $\bar{\gamma} > \alpha\gamma > 0$ and $VI(C,A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following manner $x_1 \in C$, $y_n = Proj_C\big(x_n - \lambda_n Ax_n + e_n\big)$, $x_{n+1} = Proj_C\Big((1-\alpha_n)\beta_n\gamma f(y_n) + \alpha_n x_n + (1-\alpha_n)(I-\beta_n B)y_n\Big)$, $\forall n \geq 1$, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0,2\mu)$ and $\{e_n\}$ is a sequence in $H$. Assume that $\{\alpha_n\}$, $\{\beta_n\}$, $\{e_n\}$ and $\{\lambda_n\}$ satisfy $1 > \limsup_{n\to\infty}\alpha_n \geq \liminf_{n\to\infty}\alpha_n > 0$, $\sum_{n=1}^{\infty}\beta_n = \infty$, $\sum_{n=1}^{\infty}\|e_n\| < \infty$, $\lim_{n\to\infty}\beta_n = 0$, $\lim_{n\to\infty}|\lambda_{n+1}-\lambda_n| = 0$, $\{\lambda_n\} \subset [\lambda, \lambda']$ for some $\lambda, \lambda'$ with $0 < \lambda \leq \lambda' < 2\mu$. Then sequence $\{x_n\}$ converges strongly to some $q \in VI(C,A)$, where $q = Proj_{VI(C,A)}(\gamma f + I - B)q$.*

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