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INCLUSION PROPERTIES ON A LINEAR OPERATOR ASSOCIATED WITH GAUSSIAN HYPERGEOMETRIC FUNCTIONS

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Abstract. In this paper, we introduce and investigate various inclusion relationships and convolution properties of a certain class of meromorphically univalent functions $f(z)$ defined by the linear operator $L(\alpha,\beta)f(z)$. The aim of the present paper is to prove some properties for the class $\Sigma(\alpha,\beta,k,\lambda;h)$ to satisfy the certain subordination.

Keywords. Hypergeometric function; Meromorphic function; Hadamard product; Subordination; Linear operator.

1. Introduction

Let Σ denote the class of meromorphic functions $f(z)$ normalized by

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the punctured unit disk

$$\mathbb{U}^* = \{z : z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\} = \mathbb{U} \setminus \{0\},$$

$\mathbb{C}$ being (as usual) the set of complex numbers. We denote by $\Sigma S^*(\beta)$ and $\Sigma K(\beta)$ $(\beta \geqq 0)$ the subclasses of Σ consisting of all meromorphic functions which are, respectively, starlike of order β and convex of order β in $\mathbb{U}^*$ (see also the recent works [1] and [2]).

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For functions $f_j(z)$ $(j=1,2)$ defined by

$$f_j(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_{n,j}z^n \qquad (j=1,2), \tag{1.2}$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1*f_2)(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_{n,1}a_{n,2}z^n. \tag{1.3}$$

Let us consider the function $\widetilde{\phi}(\alpha,\beta;z)$ defined by

$$\widetilde{\phi}(\alpha,\beta;z)=\frac{1}{z}+\sum_{n=0}^{\infty}\frac{(\alpha)_{n+1}}{(\beta)_{n+1}}a_nz^n \tag{1.4}$$

$$\left(\beta\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ \alpha\in\mathbb{C}\right),$$

where

$$\mathbb{Z}_0^-=\{0,-1,-2,\cdots\}=\mathbb{Z}^-\cup\{0\}.$$

Here, and in the remainder of this paper, $(\lambda)_\kappa$ denotes the general Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_\kappa:=\frac{\Gamma(\lambda+\kappa)}{\Gamma(\lambda)}=\begin{cases}\lambda(\lambda+1)\cdots(\lambda+n-1) & (\kappa=n\in\mathbb{N};\ \lambda\in\mathbb{C}),\\ 1 & (\kappa=0;\ \lambda\in\mathbb{C}\setminus\{0\}),\end{cases} \tag{1.5}$$

it being understood *conventionally* that $(0)_0:=1$ and assumed *tacitly* that the Γ-quotient exists (see, for details, [3, p. 21 *et seq.*]), $\mathbb{N}$ being the set of positive integers.

It is easy to see that, in the case when $a_n=1$ $(k=0,1,2,\cdots)$, the following relationship holds true between the function $\widetilde{\phi}(\alpha,\beta;z)$ and the Gaussian hypergeometric function [4]:

$$\widetilde{\phi}(\alpha,\beta;z)=\frac{1}{z}\,{}_2F_1(1,\alpha;\beta;z), \tag{1.6}$$

where

$${}_2F_1(b,\alpha,\beta;z)=\sum_{n=0}^{\infty}\frac{(b)_n(\alpha)_n}{(\beta)_n}\frac{z^n}{n!}$$

is the well-known Gaussian hypergeometric function. Corresponding to the function $\tilde{\phi}(\alpha,\beta;z)$, using the Hadamard product for $f(z)\in\Sigma$, we define a new linear operator $L(\alpha,\beta)$ on Σ by

$$L(\alpha,\beta)f(z)=\tilde{\phi}(\alpha,\beta;z)*f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}\left|\frac{(\alpha)_{n+1}}{(\beta)_{n+1}}\right|a_nz^n. \tag{1.7}$$

The meromorphic functions with the generalized and Gaussian hypergeometric functions were considered recently by Dziok and Srivastava [5], [6], Liu [7], Liu and Srivastava [8], [9],[10], Cho and Kim [11] .

For a function $f \in L(\alpha,\beta) f(z)$ we define

$$I^0 (L(\alpha,\beta) f(z)) = L(\alpha,\beta) f(z),$$

and for $n = 1,2,3,...,$

$$I^k (L(\alpha,\beta) f(z)) = z\left(I^{k-1} L(\alpha,\beta) f(z)\right)' + \frac{2}{z} = \frac{1}{z} + \sum_{n=1}^{\infty} n^k \left|\frac{(\alpha)_{n+1}}{(\beta)_{n+1}}\right| a_n z^n. \tag{1.8}$$

We note that I^k in (1.6) was studied by Ghanim and Darus [12], [13], [14] and [15]. It follows from (1.7) that

$$z(L(\alpha,\beta)f(z))' = \alpha L(\alpha+1,\beta)f(z) - (\alpha+1)L(\alpha,\beta)f(z), \tag{1.9}$$

which implies that

$$z\left(I^k L(\alpha,\beta)f(z)\right)' = \alpha I^k L(\alpha+1,\beta)f(z) - (\alpha+1)\, I^k L(\alpha,\beta)f(z). \tag{1.10}$$

Let Ω be the class of all analytic, convex and univalent functions $h(z)$ in the open unit disk satisfying $h(0) = 1$ and

$$\Re\{h(z)\} > 0,\ |z| < 1 \tag{1.11}$$

for two functions $f,g \in \Omega$, we say that f is *subordinate* to g or g is *superordinate* to f in $\mathbb{U}$ and write $f \prec g$, $z \in \mathbb{U}$, if there exist a Schwarz function ω, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|w(z)| \leq 1$ when $z \in \mathbb{U}$ such that $f(z) = g(\omega(z))$, $z \in \mathbb{U}$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have following equivalence:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \textit{and} \quad f(U) \subset g(U), \qquad (z \in \mathbb{U}).$$

Definition 1.1. A function $f \in \Sigma$ is said to be in the class $\Sigma(\alpha, \beta, k, \lambda; h)$, if it satisfies the subordination condition

$$(1+\lambda)\, z\left(I^k L(\alpha,\beta) f(z)\right) + \lambda z^2 \left(I^k L(\alpha,\beta) f(z)\right)' \prec h(z), \tag{1.12}$$

where λ is a complex number and $h(z) \in \Omega$. Let A be class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.13}$$

which are analytic in $\mathbb{U}$. A function $h(z) \in A$ is said to be in the class $S^*(\alpha)$, if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in \mathbb{U}).$$

For some $\alpha(\alpha < 1)$. When $0 < \alpha < 1$, $S^*(\alpha)$ is the class of starlike functions of order α in $\mathbb{U}$. A function $h(z) \in A$ is said to be prestarlike of order α in $\mathbb{U}$, if

$$\frac{z}{(1-z)^{2(1-\alpha)}} * f(z) \in S^*(\alpha) \quad (\alpha < 1),$$

where the symbol $*$ means the familiar Hadamard product (or convolution) of two analytic functions in $\mathbb{U}$. We denote this class by $R(\alpha)$ (see [16] and [17]). A function $f(z) \in A$ is in the class $R(0)$, if and only if $f(z)$ is convex univalent in $\mathbb{U}$ and $R\left(\frac{1}{2}\right) = S^*\left(\frac{1}{2}\right)$.

In this paper, we introduce and investigate various inclusion relationships and convolution properties of a certain class of meromorphically univalent functions, which are defined in this paper by means of a linear operator.

2. Preliminaries

In order to prove our main results, we need the following lemmas.

Lemma 2.1. [18] *Let $g(z)$ be analytic in $\mathbb{U}$, and $h(z)$ be analytic and convex univalent in $\mathbb{U}$ with $h(0) = g(0)$. If,*

$$g(z) + \frac{1}{\mu} z g'(z) \prec h(z), \tag{2.1}$$

where $\Re\mu \geq 0$ and $\mu \neq 0$, then

$$g(z) \prec \widetilde{h}(z) = \mu z^{-\mu} \int_0^z t^{\mu-1} h(t)\, dt \prec h(z)$$

and $\widetilde{h}(z)$ is the best dominant of (2.1).

Lemma 2.2. [17] *Let $a < 1$, $f(z) \in S^*(a)$ and $g(z) \in R(\alpha)$. For any analytic function $F(z)$ in $\mathbb{U}$, then*

$$\frac{g * (fF)}{g * f}(\mathbb{U}) \subset \overline{co}(F(\mathbb{U})),$$

where $\overline{co}(F(\mathbb{U}))$ *denotes the convex hull of* $F(\mathbb{U})$.

3. Main results

Theorem 3.1. *Let* $0 \leq \lambda_1 < \lambda_2$. *Then* $\Sigma(\alpha, \beta, k, \lambda_2; h) \subset \Sigma(\alpha, \beta, k, \lambda_1; h)$

Proof. Let $0 \leq \lambda_1 < \lambda_2$ and suppose that

$$g(z) = z\left(I^k L(\alpha,\beta) f(z)\right) \tag{3.1}$$

for $f(z) \in \Sigma(\alpha, \beta, k, \lambda_2; h)$. Then the function $g(z)$ is analytic in $\mathbb{U}$ with $g(0) = 1$. Differentiating both sides of (3.1) with respect to z and using (1.10), we have

$$(1+\lambda_2)z\left(I^k L(\alpha,\beta) f(z)\right) + \lambda_2 z^2 \left(I^k L(\alpha,\beta) f(z)\right)' = g(z) + \lambda_2 z g'(z) \prec h(z). \tag{3.2}$$

Hence an application of Lemma 2.1 with $\mu = \frac{1}{\lambda_2} > 0$ yields that

$$g(z) \prec h(z). \tag{3.3}$$

Noting that $0 \leq \frac{\lambda_1}{\lambda_2} < 1$ and that $h(z)$ is convex univalent in $\mathbb{U}$, it follows from (3.1), (3.2) and (3.3) that

$$(1+\lambda_1)z\left(I^k L(\alpha,\beta) f(z)\right) + \lambda_1 z^2 \left(I^k L(\alpha,\beta) f(z)\right)'$$

$$= \frac{\lambda_1}{\lambda_2}\left[(1+\lambda_2)z\left(I^k L(\alpha,\beta) f(z)\right) + \lambda_2 z^2 \left(I^k L(\alpha,\beta) f(z)\right)'\right] + \left(1 - \frac{\lambda_1}{\lambda_2}\right) g(z) \prec h(z).$$

Thus, $f(z) \in \Sigma(\alpha, \beta, k, \lambda_1; h)$ and the proof of Theorem 3.1 is completed.

Theorem 3.2. *Let*

$$\Re\left\{z\widetilde{\phi}(\alpha_1, \alpha_2; z)\right\} > \frac{1}{2} \qquad (z \in \mathbb{U};\ \alpha_2 \notin \{0, -1, -2, ...\}), \tag{3.4}$$

where $\widetilde{\phi}(\alpha_1, \alpha_2; z)$ *is defined as in (1.6). Then,*

$$\Sigma(\alpha_2, \beta, k, \lambda; h) \subset \Sigma(\alpha_1, \beta, k, \lambda; h).$$

Proof. For $f(z)\in\Sigma$, it is easy to verify that

$$z\left(I^kL(\alpha_1,\beta)f(z)\right)=\left(z\widetilde{\phi}(\alpha_1,\alpha_2;z)*\left(zI^kL(\alpha_2,\beta)f(z)\right)\right) \tag{3.5}$$

and

$$z^2\left(I^kL(\alpha_1,\beta)f(z)\right)'=\left(z\widetilde{\phi}(\alpha_1,\alpha_2;z)*z^2\left(I^kL(\alpha_2,\beta)f(z)\right)'\right). \tag{3.6}$$

Let $f(z)\in\Sigma(\alpha_2,\beta,k,\lambda;h)$. Then from (3.5) and (3.6), we deduce that

$$(1+\lambda)z\left(I^kL(\alpha_1,\beta)f(z)\right)+\lambda z^2\left(I^kL(\alpha_1,\beta)f(z)\right)'=\left(z\widetilde{\phi}(\alpha_1,\alpha_2;z)\right)*\Psi(z) \tag{3.7}$$

and

$$\Psi(z)=(1+\lambda)z\left(I^kL(\alpha_2,\beta)f(z)\right)+\lambda z^2\left(I^kL(\alpha_2,\beta)f(z)\right)'\prec h(z) \tag{3.8}$$

In view of (3.4), the function $z\widetilde{\phi}(\alpha_1,\alpha_2;z)$ has the Herglotz representation

$$z\widetilde{\phi}(\alpha_1,\alpha_2;z)=\int_{|x|=1}\frac{d\mu(x)}{1-xz}\quad(z\in\mathbb{U}), \tag{3.9}$$

where $\mu(x)$ is a probability measure defined on the unit circle $|x|=1$ and

$$\int_{|x|=1}d\mu(x)=1.$$

Since $h(z)$ is convex univalent in $\mathbb{U}$, it follows from (3.7), (3.8) and (3.9) that:

$$(1+\lambda)z\left(I^kL(\alpha_1,\beta)f(z)\right)+\lambda z^2\left(I^kL(\alpha_1,\beta)f(z)\right)'=\int_{|x|=1}\Psi(xz)\,d\mu(x)\prec h(z).$$

This shows that $f(z)\in\Sigma(\alpha_1,\beta,k,\lambda;h)$ and the theorem is proved.

Theorem 3.3. *Let* $0<\alpha_1<\alpha_2$. *Then* $\Sigma(\alpha_2,\beta,k,\lambda;h)\subset\Sigma(\alpha_1,\beta,k,\lambda;h)$.

Proof. Define

$$g(z)=z+\sum_{n=1}^{\infty}\left|\frac{(\alpha_1)_{n+1}}{(\alpha_2)_{n+1}}\right|z^{n+1}\qquad(z\in\mathbb{U};\ 0<\alpha_1<\alpha_2).$$

Then,

$$z^2\widetilde{\phi}(\alpha_1,\alpha_2;z)=g(z)\in A, \tag{3.10}$$

where $\widetilde{\phi}(\alpha_1,\alpha_2;z)$ is defined as in (1.6), and

$$\frac{z}{(1-z)^{\alpha_2}}*g(z)=\frac{z}{(1-z)^{\alpha_1}}. \tag{3.11}$$

By (3.11), we see that

$$\frac{z}{(1-z)^{\alpha_2}} * g(z) \in S^*\left(1-\frac{\alpha_1}{2}\right) \subset S^*\left(1-\frac{\alpha_2}{2}\right)$$

for $0 < \alpha_1 < \alpha_2$, which implies that

$$g(z) \in R\left(1-\frac{\alpha_2}{2}\right) \tag{3.12}$$

Let $f(z) \in \Sigma(\alpha_2, \beta, k, \lambda; h)$. Then we deduce from (3.7) and (3.8) (used in the proof of Theorem 3.2 and (3.10) that

$$\begin{aligned}(1+\lambda)z\left(I^k L(\alpha_1,\beta) f(z)\right) + \lambda z^2 \left(I^k L(\alpha_1,\beta) f(z)\right)' \\ = \frac{g(z)}{z} * \Psi(z) = \frac{g(z) * (z\Psi(z))}{g(z) * z},\end{aligned} \tag{3.13}$$

where

$$\Psi(z) = (1+\lambda)z\left(I^k L(\alpha_2,\beta) f(z)\right) + \lambda z^2 \left(I^k L(\alpha_2,\beta) f(z)\right)' \prec h(z). \tag{3.14}$$

Since z belongs to $S^*\left(1-\frac{\alpha_2}{2}\right)$ and $h(z)$ is convex univalent in $\mathbb{U}$. it follows from (3.12), (3.13), (3.14) and Lemma 2.2 that

$$(1+\lambda)z\left(I^k L(\alpha_1,\beta) f(z)\right) + \lambda z^2 \left(I^k L(\alpha_1,\beta) f(z)\right)' \prec h(z).$$

Thus $f(z) \in \Sigma(\alpha_1, \beta, k, \lambda; h)$ and the proof is completed.

As a special case of Theorem 3.3, we have:

$$\Sigma(\alpha+1, \beta, k, \lambda; h) \subset \Sigma(\alpha, \beta, k, \lambda; h) \quad (\alpha > 0).$$

In Theorem 3.4 below we give a generalization of the above result.

Theorem 3.4. *Let* $\Re\alpha \geq 0$ *and* $\alpha \neq 0$*. Then,*

$$\Sigma(\alpha+1, \beta, k, \lambda; h) \subset \Sigma\left(\alpha, \beta, k, \lambda; \widetilde{h}\right),$$

where

$$\widetilde{h}(z) = \alpha z^{-\alpha} \int_0^z t^{\alpha-1} h(t)dt \prec h(z).$$

Proof. Let us define

$$g(z) = (1+\lambda) z \left(I^k L(\alpha, \beta) f(z) \right) + \lambda z^2 \left(I^k L(\alpha, \beta) f(z) \right)' \tag{3.15}$$

for $f(z) \in \Sigma$. Then (1.10) and (3.15) lead to

$$\frac{g(z)}{z} = \alpha\lambda \left(I^k L(\alpha+1, \beta) f(z) \right) + (1-\alpha\lambda) \left(I^k L(\alpha, \beta) f(z) \right) \tag{3.16}$$

Differentiating both sides of (3.16) and using (1.10), we arrive at

$$\begin{aligned} g'(z) - \frac{g(z)}{z} &= \alpha\lambda z \left(I^k L(\alpha+1, \beta) f(z) \right)' \\ &+ (1-\alpha\lambda) \left[\alpha \left(I^k L(\alpha+1, \beta) f(z) \right) - (1+\alpha) \left(I^k L(\alpha, \beta) f(z) \right) \right]. \end{aligned} \tag{3.17}$$

By (3.16) and (3.17), we get

$$g'(z) - \frac{\alpha g(z)}{z} = \alpha\lambda z \left(I^k L(\alpha+1, \beta) f(z) \right)' + \alpha(1+\lambda) \left(I^k L(\alpha+1, \beta) f(z) \right),$$

that is,

$$g(z) + \frac{z g'(z)}{\alpha} = (1+\lambda) z \left(I^k L(\alpha+1, \beta) f(z) \right) + \lambda z^2 \left(I^k L(\alpha+1, \beta) f(z) \right)'. \tag{3.18}$$

If $f \in \Sigma(\alpha+1, \beta, k, \lambda; h)$, then it follows from (3.18) that

$$g(z) + \frac{z g'(z)}{\alpha} \prec h(z) \qquad (\Re \alpha \geq 0,\ \alpha \neq 0).$$

Hence an application of Lemma 2.1 yields

$$g(z) \prec \widetilde{h}(z) = \alpha z^{-\alpha} \int_0^z t^{\alpha-1} h(t)\, dt \prec h(z),$$

which shows that

$$f(z) \in \Sigma\left(\alpha, \beta, k, \lambda; \widetilde{h}\right) \subset \Sigma(\alpha, \beta, k, \lambda; h)$$

Theorem 3.5. *Let $\lambda > 0$, $\delta > 0$ and $f(z) \in \Sigma(\alpha, \beta, k, \lambda; \delta h + 1 - \delta)$. If $\delta \leq \delta_0$, where*

$$\delta_0 = \frac{1}{2} \left(1 - \frac{1}{\lambda} \int_0^1 \frac{u^{\frac{1}{\lambda}-1}}{1+u} du \right)^{-1} \tag{3.19}$$

then $f(z) \in \Sigma(\alpha, \beta, k, \lambda; h)$. The bound δ_0 is sharp when $h(z) = \frac{1}{1-z}$.

Proof. Let us define

$$g(z) = z \left(I^k L(\alpha, \beta) f(z) \right) \tag{3.20}$$

for $f(z) \in \Sigma(\alpha, \beta, k, \lambda; \delta h + 1 - \delta)$ with $\lambda > 0$, and $\delta > 0$. Then we have

$$g(z) + \lambda z g'(z) = (1+\lambda) z \left(I^k L(\alpha,\beta) f(z)\right) + \lambda z^2 \left(I^k L(\alpha,\beta) f(z)\right)' \prec \delta (h(z) - 1) + 1.$$

Hence an application of Lemma 2.1 yields that

$$g(z) \prec \frac{\delta}{\lambda} z^{-\frac{1}{\lambda}} \int_0^z t^{\frac{1}{\lambda}-1} h(t)\, dt + 1 - \delta = (h * \Psi)(z), \tag{3.21}$$

where

$$\Psi(z) = \frac{\delta}{\lambda} z^{-\frac{1}{\lambda}} \int_0^z \frac{t^{\frac{1}{\lambda}-1}}{1-t} dt + 1 - \delta. \tag{3.22}$$

If $0 < \delta \le \delta_0$, where $\delta_0 > 1$ is given by (3.19), then it follows from (3.21) that

$$\Re \Psi(z) = \frac{\delta}{\lambda} \int_0^1 u^{\frac{1}{\lambda}-1} \Re \left(\frac{1}{1-uz}\right) du + 1 - \delta > \frac{\delta}{\lambda} \int_0^1 \frac{u^{\frac{1}{\lambda}-1}}{1+u} du + 1 - \delta \ge \frac{1}{2} \quad (z \in \mathbb{U}).$$

Now, by using the Herglotz representation for $\Psi(z)$, from (3.20) and (3.21) we arrive at

$$z\left(I^k L(\alpha,\beta) f(z)\right) \prec (h * \Psi)(z) \prec h(z)$$

because $h(z)$ is convex univalent in $\mathbb{U}$. This shows that $f(z) \in \Sigma(\alpha, \beta, k, \lambda; h)$. For $h(z) = \frac{1}{1-z}$ and $f(z) \in \Sigma$ defined by

$$z\left(I^k L(\alpha,\beta) f(z)\right) = \frac{\delta}{\lambda} z^{-\frac{1}{\lambda}} \int_0^z \frac{t^{\frac{1}{\lambda}-1}}{1-t} dt + 1 - \delta,$$

it is easy to verify that

$$(1+\lambda) z \left(I^k L(\alpha,\beta) f(z)\right) + \lambda z^2 \left(I^k L(\alpha,\beta) f(z)\right)' = \delta h(z) + 1 - \delta.$$

Thus, $f(z) \in \Sigma(\alpha, \beta, k, \lambda; \delta h + 1 - \delta)$. Also, for $\delta > \delta_0$, we have

$$\Re z\left(I^k L(\alpha,\beta) f(z)\right) \to \frac{\delta}{\lambda} \int_0^1 \frac{u^{\frac{1}{\lambda}-1}}{1+u} du + 1 - \delta < \frac{1}{2} \qquad (z \to -1),$$

which implies that $f(z) \notin \Sigma(\alpha, \beta, k, \lambda; h)$. Hence the bound δ_0, cannot be increased when $h(z) = \frac{1}{1-z}$.

4. Convolution properties

Theorem 4.1. *Let $f(z) \in \Sigma(\alpha, \beta, k, \lambda; h)$, $g(z) \in \Sigma$ and*

$$\Re(z g(z)) > \frac{1}{2} \qquad (z \in \mathbb{U}).$$

Then,

$$(f * g)(z) \in \Sigma(\alpha, \beta, k, \lambda; h).$$

Proof. For $f(z) \in \Sigma(\alpha, \beta, k, \lambda; h)$ and $g \in \Sigma$, we have

$$(1+\lambda) z \left(I^k L(\alpha,\beta)(f*g)(z)\right) + \lambda z^2 \left(I^k L(\alpha,\beta)(f*g)(z)\right)'$$

$$= (1+\lambda) zg(z) * z\left(I^k L(\alpha,\beta) f(z)\right) + \lambda zg(z) * z^2 \left(I^k L(\alpha,\beta) f(z)\right)'$$

$$= zg(z) * \Psi(z), \tag{4.1}$$

where

$$\Psi(z) = (1+\lambda) z \left(I^k L(\alpha,\beta) f(z)\right) + \lambda z^2 \left(I^k L(\alpha,\beta) f(z)\right)' \prec h(z). \tag{4.2}$$

The remaining part of the proof of Theorem 4.1 is similar to that of Theorem 3.2 and hence we omit it.

Corollary 4.1. *Let $f(z) \in \Sigma(\alpha, \beta, k, \lambda; h)$ be given by (1.1) and let,*

$$\omega_m(z) = \frac{1}{z} + \sum_{n=1}^{m-1} a_n z^{n-1} \qquad (m \in N \setminus \{1\}).$$

Then function $\sigma_m(z) = \int_0^1 t\,\omega_m(tz)\,dt$ is also in the class $\Sigma(\alpha, \beta, k, \lambda; h)$.

Proof. Note that

$$\sigma_m(z) = \frac{1}{z} + \sum_{n=1}^{m-1} \frac{a_n}{n+1} z^{n-1} = (f * g_m)(z) \qquad (m \in N \setminus \{1\}), \tag{4.3}$$

where

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^{n-1} \in \Sigma(\alpha, \beta, k, \lambda; h)$$

and

$$g_m(z) = \frac{1}{z} + \sum_{n=1}^{m-1} \frac{z^{n-1}}{n+1} \in \Sigma.$$

Also, for $m \in N \setminus \{1\}$, it is known from [19] that

$$\Re\{zg_m(z)\} = \Re\left\{1 + \sum_{n=1}^{m-1} \frac{z^n}{n+1}\right\} > \frac{1}{2} \qquad (z \in \mathbb{U}). \tag{4.4}$$

In view of (4.3) and (4.4), an application of Theorem 4.1 leads to $\sigma_m(z) \in \Sigma(\alpha, \beta, k, \lambda; h)$.

Theorem 4.2. *Let* $f(z) \in \Sigma(\alpha, \beta, k, \lambda; h)$, $g(z) \in \Sigma$ *and*

$$z^2 g(z) \in R(\mathfrak{a}) \qquad (\mathfrak{a} < 1).$$

Then,

$$(f * g)(z) \in \Sigma(\alpha, \beta, k, \lambda; h).$$

Proof. For $f(z) \in \Sigma(\alpha, \beta, k, \lambda; h)$ and$g(z) \in \Sigma$, from (4.1) (used in the proof of Theorem 4.1), we can write:

$$(1+\lambda) z \left(I^k L(\alpha,\beta)(f*g)(z)\right) + \lambda z^2 \left(I^k L(\alpha,\beta)(f*g)(z)\right)'$$

$$= \frac{z^2 g(z) * z\Psi(z)}{z^2 g(z) * z} \qquad (z \in \mathbb{U}), \tag{4.5}$$

where $\Psi(z)$ is defined as in (4.2). Since $h(z)$ is convex univalent in $\mathbb{U}$, $\Psi(z) \prec h(z)$, $z^2 g(z) \in R(\mathfrak{a})$ and

$$z \in S^*(\mathfrak{a}) \qquad (\mathfrak{a} < 1),$$

it follows from (4.5) and Lemma 2.2 the desired result.

Taking $\mathfrak{a} = 0$ and $\mathfrak{a} = \frac{1}{2}$, Theorem 4.2 reduces to the following.

Corollary 4.2. *Let* $f(z) \in \Sigma(\alpha, \beta, k, \lambda; h)$ *and let* $g(z) \in \Sigma$ *satisfy either of the following conditions:*

(i) $z^2 g(z)$ *is convex univalent in* $\mathbb{U}$ *or*

(ii) $z^2 g(z) \in S^*\left(\frac{1}{2}\right)$. *Then* $(f * g)(z) \in \Sigma(\alpha, \beta, k, \lambda; h)$

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