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HYBRID PROJECTION METHODS FOR EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS OF INFINITE FAMILY OF MULTIVALUED ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. The purpose of this paper is to prove a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of infinite family of multivalued asymptotically nonexpansive mappings in Hilbert spaces based on a new hybrid method.

Keywords. Multivalued asymptotically nonexpansive mappings; Equilibrium problem; Shrinking projection method; Fixed point.

1. Introduction

An element $p \in K$ is called a *fixed point* of a single-valued mapping T if $p = Tp$ and of a multivalued mapping T if $p \in Tp$. The set of fixed points of T is denoted by $F(T)$.

Let X be a real Banach space. A subset K of X is called *proximinal* if for each $x \in X$, there exists an element $k \in K$ such that $d(x,k) = d(x,K)$, where $d(x,K) = \inf\{\|x-y\| : y \in K\}$ is the distance from a point x to the set K.

Let X be a uniformly convex real Banach space, and let K be a nonempty closed convex subset of X. In the sequel, we denote by $CB(K)$ the family of all nonempty closed and bounded subsets of K, by $P(K)$ the family of all nonempty proximinal and bounded subsets of K.

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Let $T : K \to P(K)$ be a multivalued mapping. Define $P_T(x) := \{y \in T(x) : \|x-y\| = d(x,T(x))\}$ for all $x \in K$.

The *Hausdorff metric* $H(\cdot,\cdot)$ on $CB(X)$ is defined by

$$H(A,B) = max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\}$$

for all $A,B \in CB(X)$.

A multivalued mapping $T : K \to CB(K)$ is said to be *nonexpansive* if $H(Tx,Ty) \leq \|x-y\|$ for all $x,y \in K$.

Definition 1.1. Let X be a uniformly convex real Banach space and K be a nonempty closed convex subset of X, and let $CB(K)$ be a family of nonempty closed bounded subsets of K. A multivalued mapping $T : K \to CB(K)$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$H(T^n x, T^n y) \leq k_n\|x-y\|, \quad \forall x,y \in K.$$

The study of fixed points for multivalued nonexpansive mappings using the Hausdorff metric was initiated by Markin [2] (see also [1]). Later, an interesting and rich fixed point theory for such kind of maps was developed and it also has been applied to control theory, convex optimization, differential inclusion, and economics (see, [3] and reference scited therein).

It is known that Mann iteration has only weak convergence even in the Hilbert spaces. To overcome this problem, Takahashi *et al.* [4] introduced the following iteration scheme, which is usually called the shrinking projection method. Let H be a real Hilbert space and D be a nonempty closed convex subset of H. let $\{\alpha_n\}$ be a sequence in $(0,1)$ and $x_0 \in H$. For $C_1 = D$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ of D as follows:

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T_n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ \ n \geq 1, \end{cases}$$

where P_{C_n} is the metric projection of H onto C_n and $\{T_n\}$ is a family of nonexpansive mappings. They proved that the sequence $\{x_n\}$ converges strongly to $z = P_{F(T)}x_0$, where $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$.

Recently the shrinking projection method has been studied widely by many authors, for example, see [5]-[18] and the references therein.

In the recent years, the problem of finding a common element of the set of solutions of equilibrium problems and the set of common fixed points of single-valued nonexpansive mappings in the framework of Hilbert spaces and Banach spaces has been intensively studied by many authors, for instance, see [5, 19-27] and the references cited theorems.

In this paper, we use the shrinking projection method to define a new hybrid method for equilibrium problems and fixed point problem for infinite family of multivalued asymptotically nonexpansive mappings in Hilbert spaces. We obtain a strong convergence theorem for the sequences generated by the proposed algorithm without the assumption of compactness of the domain and other conditions imposing on the mappings.

2. Preliminaries

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let D be a nonempty closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in D, denoted by P_Dx, such that $\|x-P_Dx\| \leq \|x-y\|$, $\forall y \in D$. P_D is called the *metric projection* of H onto D. It is known that P_D is a nonexpansive mapping of H onto D. It is also know that P_D satisfies$\langle x-y, P_Dx-P_Dy\rangle \geq \|P_Dx-P_Dy\|^2$ for every $x,y \in H$. Moreover, P_Dx is characterized by the properties: $P_Dx \in D$ and $\langle x-P_Dx, P_Dx-y\rangle \geq 0$ for all $y \in D$.

Lemma 2.1. [28] *Let D be a nonempty closed convex subset of a real Hilbert space H and $P_D : H \to D$ be the metric projection from H onto D . Then the following inequality holds:*

$$\|y-P_Dx\|^2 + \|x-P_Dx\|^2 \leq \|x-y\|^2, \ \forall x \in H, \forall y \in D.$$

Lemma 2.2. [29] *Let H be a real Hilbert space. Then the following equations hold:*

(i) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle, \ \forall x,y \in H;$

(ii) $\|tx+(1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \ \forall t \in [0,1]$ *and* $x,y \in H.$

Lemma 2.3. [9]. *Let X be a uniformly convex Banach space, $r > 0$ be a positive number and $B_r(0) := \{x \in X : \|x\| \leq r\}$ be a closed ball of X. Then for any given infinite subset $\{x_n\}_{n=1}^{\infty} \subset$*

$B_r(0)$ and for any given sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers with $\sum_{n=1}^{\infty}\alpha_n = 1$, there exists a continuous, strictly increasing and convex function $g : [0,2r) \to [0,\infty)$ with $g(0) = 0$ such that for any $i, j \in N$ with $i < j$,

$$\Big\|\sum_{n=1}^{\infty}\alpha_n x_n\Big\|^2 \le \sum_{n=1}^{\infty}\alpha_n\|x_n\|^2 - \alpha_i\alpha_j g(\|x_i - x_j\|)$$

Lemma 2.4. [30] *Let D be a nonempty closed and convex subset of a real Hilbert space H. Given $x,y,z \in H$ and $a \in R$, the set $\{v \in D : \|y-v\|^2 \le \|x-v\|^2 + \langle z,v\rangle + a\}$ is convex and closed.*

Let $f : D \times D \to R$ be a bifunction. Combettes and Hirstoaga [12] introduced the following equilibrium problem:

$$Find\ \ x \in D\ \ such\ \ that\ \ f(x,y) \ge 0,\ \ \forall y \in D. \tag{2.1}$$

The set of solutions of (2.1) is denoted by $EP(f)$.

For solving the equilibrium problem, we assume that the bifunction f satisfies the following conditions:

(A1) $f(x,x) = 0$ for all $x \in D$;

(A2) F is monotone, that is, $f(x,y) + f(y,x) \le 0$ for all $x,y \in D$;

(A3) for each $x,y,z \in D, \limsup_{t\to 0} f(tz+(1-t)x,y) \le f(x,y)$;

(A4) $f(x,\cdot)$ is convex and lower semicontinuous for each $x \in D$;

Lemma 2.5. [26] *Let D be a nonempty closed and convex subset of a real Hilbert space H. Let $f : D \times D \to R$ be a bifunction satisfying conditions (A1)-(A4). For $r > 0$ and $x \in D$, define a mapping $T_r : H \to D$ as follows:*

$$T_r(x) = \{z \in D : f(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \ge 0, \forall y \in D\}$$

for all $x \in H$. Then the following conclusions hold:

(1) T_r is single-valued;

(2) T_r is firmly nonexpansive, that is, $\|T_r(x) - T_r(y)\|^2 \le \langle T_r(x) - T_r(y), x-y\rangle$, $\forall x,y \in H$;

(3) $F(T_r) = EP(f)$;

(4) $EP(f)$ is closed and convex.

Lemma 2.6. *Let D be a closed convex subset of a real Hilbert space H. Let $T : D \to P(D)$ be a multivalued asymptotically nonexpansive mapping with $F(T) \neq \emptyset$ such that P_T is nonexpansive. Then $F(T)$ is a closed and convex subset of D.*

Proof. First, we will show that $F(T)$ is closed. Let $\{x_n\}$ be a sequence in $F(P_T)$ such that $x_n \to x$ as $n \to \infty$. We have

$$
\begin{aligned}
d(x, P_T x) &\leq d(x, x_n) + d(x_n, P_T x) \\
&\leq d(x, x_n) + H(P_T x_n, P_T x) \\
&\leq d(x, x_n) + d(x_n, x) \\
&= 2d(x, x_n).
\end{aligned}
$$

It follows that $d(x, P_T x) = 0$, so $x \in F(P_T) = F(T)$. Next, we show that $F(T)$ is convex. Let $p = tp_1 + (1-t)p_2$, where $p_1, p_2 \in F(P_T)$ and $t \in (0,1)$. Let $z \in P_T p$, by Lemma 2.2, we have

$$
\begin{aligned}
\|p - z\|^2 &= t\|z - p_1\|^2 + (1-t)\|z - p_2\|^2 - t(1-t)\|p_1 - p_2\|^2 \\
&= td(z, P_T p_1)^2 + (1-t)d(z, P_T p_2)^2 - t(1-t)\|p_1 - p_2\|^2 \\
&\leq tH(P_T p, P_T p_1)^2 + (1-t)H(P_T p, P_T p_2)^2 - t(1-t)\|p_1 - p_2\|^2 \\
&\leq t\|p - p_1\|^2 + (1-t)\|p - p_2\|^2 - t(1-t)\|p_1 - p_2\|^2 \\
&= t(1-t)^2\|p_1 - p_2\|^2 + t^2(1-t)\|p_1 - p_2\|^2 - t(1-t)\|p_1 - p_2\|^2 = 0.
\end{aligned}
$$

Hence $p = z$. Therefore, $p \in F(P_T) = F(T)$.

3. Main results

Theorem 3.1. *Let D be a nonempty closed and convex subset of a real Hilbert space H. Let $f : D \times D \to R$ be a bifunction satisfying conditions (A1)-(A4), and Let $T_i : D \to P(D)$ be multivalued asymptotically nonexpansive mappings for all $i \in N$ with sequence $\{k_{n,i}\} \subset [1, \infty), k_{n,i} \to 1$ as $n \to \infty$. Assume $\Omega := \bigcap_{i=1}^{\infty} F(T_i) \bigcap EP(f) \neq \emptyset$ such that for each $i \geq 1$, P_{T_i} is nonexpansive. Define the sequence $\{x_n\}$ as follows: $x_0 \in D = C_0$,*

$$\begin{cases} u_n \in D \ \ such \ \ that \ \ f(u_n, y) + \dfrac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \ \ \forall y \in D, \\ y_n = \alpha_{n,0} u_n + \sum\limits_{i=1}^{\infty} \alpha_{n,i} x_{n,i}, \\ C_{n+1} = \{ z \in C_n : \|y_n - z\| \leq \|x_n - z\| \}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ \ n \geq 0, \end{cases} \tag{3.1}$$

where the sequences $r_n \in (0, \infty)$ *with* $\liminf\limits_{n\to\infty} r_n > 0$ *and* $x_{n,i} \in P_{T_i} u_n$ *for each* $i \in N$*, and* $\{\alpha_{n,i}\}$ *is a real sequence in* $[0,1)$ *satisfies the following conditions:*

(a) $\sum\limits_{i=0}^{\infty} \alpha_{n,i} = 1, \ \ \forall n \geq 1$

(b) $\liminf\limits_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0, \ \ \forall i \in N$

Then $\{x_n\}$ *converges strongly to* $P_\Omega x_0$*.*

Proof. We split the proof into six steps.

Step 1. Show that $P_{C_{n+1}} x_0$ is well defined for every $x_0 \in D$.

By Lemma 2.5-2.6, we obtain that $EP(f)$ and $\bigcap\limits_{i=1}^{\infty} F(T_i)$ are closed and convex subset of D. Hence Ω is a closed and convex subset of D. It follows from Lemma 2.4 that C_{n+1} is a closed and convex for each $n \geq 0$. Let $v \in \Omega$. Then $P_{T_i}(v) = \{v\}$ for all $i \in N$. Since $u_n = T_{r_n} x_n$, we have $\|u_n - v\| = \|T_{r_n} x_n - T_{r_n} v\| \leq \|x_n - v\|$, for every $n \geq 0$. Then

$$\begin{aligned} \|y_n - v\| &= \|\alpha_{n,0} u_n + \sum_{i=1}^{\infty} \alpha_{n,i} x_{n,i} - v\| \\ &\leq \alpha_{n,0} \|u_n - v\| + \sum_{i=1}^{\infty} \alpha_{n,i} \|x_{n,i} - v\| \\ &= \alpha_{n,0} \|u_n - v\| + \sum_{i=1}^{\infty} \alpha_{n,i} d(x_{n,i}, P_{T_i} v) \\ &\leq \alpha_{n,0} \|u_n - v\| + \sum_{i=1}^{\infty} \alpha_{n,i} H(P_{T_i} u_n, P_{T_i} v) \\ &\leq \alpha_{n,0} \|u_n - v\| + \sum_{i=1}^{\infty} \alpha_{n,i} \|u_n - v\| \\ &= \|u_n - v\| \\ &\leq \|x_n - v\|. \end{aligned} \tag{3.2}$$

Hence $v \in C_{n+1}$, so that $\Omega \subset C_{n+1}$. Therefore, $P_{C_{n+1}}x_0$ is well defined.

Step 2. Show that $\lim_{n\to\infty} \|x_n - x_0\|$ exists.

Since Ω is a nonempty closed convex subset of H, there exists a unique $v \in \Omega$ such that $v = P_\Omega x_0$. Since $x_n = P_{C_n}x_0$ and $x_{n+1} \in C_{n+1} \subset C_n,\ \forall n \geq 0$, we have

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|,\ \forall n \geq 0.$$

On the other hand, as $v \in \Omega \subset C_n$, we obtain

$$\|x_n - x_0\| \leq \|v - x_0\|,\ \forall n \geq 0.$$

It follows that the sequence $\{x_n\}$ is bounded and nondecreasing. Therefore, $\lim_{n\to\infty} \|x_n - x_0\|$ exists.

Step 3. Show that $\lim_{n\to\infty} x_n = w \in D$.

For $m > n$, by the definition of C_n, we get $x_m = P_{C_m}x_0 \in C_m \subset C_n$. By applying Lemma 2.1, we have

$$\|x_m - x_n\|^2 \leq \|x_m - x_0\|^2 - \|x_n - x_0\|^2.$$

Since $\lim_{n\to\infty} \|x_n - x_0\|$ exists, it follows that $\{x_n\}$ is a Cauchy sequence in D. Hence there exists $w \in D$ such that $\lim_{n\to\infty} x_n = w$.

Step 4. Show that $\|x_{n,i} - x_n\| \to 0$ as $n \to \infty$ for every $i \in N$.

From $x_{n+1} \in C_{n+1}$, we have

$$\begin{aligned} \|x_n - y_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \\ &\leq 2\|x_n - x_{n+1}\| \to 0\ \ as\ \ n \to \infty. \end{aligned} \tag{3.3}$$

Since $v \in \Omega$ and $\{x_n\}$ is bounded, by virtue of Lemma 2.3 and (3.2), we get

$$\begin{aligned}
\|y_n - v\|^2 &= \|\alpha_{n,0}(u_n - v) + \sum_{i=1}^{\infty} \alpha_{n,i}(x_{n,i} - v)\|^2 \\
&\le \alpha_{n,0}\|u_n - v\|^2 + \sum_{i=1}^{\infty} \alpha_{n,i}\|x_{n,i} - v\|^2 - \alpha_{n,i}\alpha_{n,0}\|x_{n,i} - u_n\|^2 \\
&= \alpha_{n,0}\|u_n - v\|^2 + \sum_{i=1}^{\infty} \alpha_{n,i} d(x_{n,i}, P_{T_i}v)^2 - \alpha_{n,i}\alpha_{n,0}\|x_{n,i} - u_n\|^2 \\
&\le \alpha_{n,0}\|u_n - v\|^2 + \sum_{i=1}^{\infty} \alpha_{n,i} H(P_{T_i}u_n, P_{T_i}v)^2 - \alpha_{n,i}\alpha_{n,0}\|x_{n,i} - u_n\|^2 \\
&\le \alpha_{n,0}\|u_n - v\|^2 + \sum_{i=1}^{\infty} \alpha_{n,i}\|u_n - v\|^2 - \alpha_{n,i}\alpha_{n,0}\|x_{n,i} - u_n\|^2 \\
&= \|u_n - v\|^2 - \alpha_{n,i}\alpha_{n,0}\|x_{n,i} - u_n\|^2 \\
&\le \|x_n - v\|^2 - \alpha_{n,i}\alpha_{n,0}\|x_{n,i} - u_n\|^2.
\end{aligned} \tag{3.4}$$

This implies that

$$\begin{aligned}
\alpha_{n,i}\alpha_{n,0}\|x_{n,i} - u_n\|^2 &\le \|x_n - v\|^2 - \|y_n - v\|^2 \\
&\le M\|x_n - y_n\|,
\end{aligned}$$

where $M = \sup_{n\ge 0}\{\|x_n - v\| + \|y_n - v\|\}$. By the given control condition on $\{\alpha_{n,i}\}$ and (3.3), we obtain

$$\lim_{n\to\infty} \|x_{n,i} - u_n\| = 0, \ \ \forall i \in N. \tag{3.5}$$

It follows from Lemma 2.5 that

$$\begin{aligned}
\|u_n - v\|^2 &= \|T_{r_n}x_n - T_{r_n}v\|^2 \\
&\le \langle T_{r_n}x_n - T_{r_n}v, x_n - v\rangle \\
&= \langle u_n - v, x_n - v\rangle \\
&= \frac{1}{2}\{\|u_n - v\|^2 + \|x_n - v\|^2 - \|u_n - x_n\|^2\}.
\end{aligned}$$

Hence $\|u_n - v\|^2 \le \|x_n - v\|^2 - \|u_n - x_n\|^2$. By (3.4), we get

$$\|y_n - v\|^2 \le \|u_n - v\|^2.$$

Therefore, $\|y_n - v\|^2 \leq \|x_n - v\|^2 - \|u_n - x_n\|^2$. This implies that

$$\|u_n - x_n\|^2 \leq \|x_n - v\|^2 - \|y_n - v\|^2$$
$$\leq M\|x_n - y_n\|,$$

where $M = \sup_{n\geq 0}\{\|x_n - v\| + \|y_n - v\|\}$. From (3.3), we get $\lim_{n\to\infty} \|u_n - x_n\| = 0$. Hence it follows from (3.5) that

$$\|x_{n,i} - x_n\| \leq \|x_{n,i} - u_n\| + \|u_n - x_n\| \to 0 \ \ as \ \ n \to \infty.$$

Step 5. Show that $w \in \Omega$.

By the assumption that $\liminf_{n\to\infty} r_n > 0$, we have

$$\|\frac{x_n - u_n}{r_n}\| = \frac{1}{r_n}\|x_n - u_n\| \to 0 \ \ as \ \ n \to \infty. \tag{3.6}$$

Since $\lim_{n\to\infty} x_n = w$, we obtain $\lim_{n\to\infty} u_n = w$. We will show that $w \in EP(f)$. Since $u_n = T_{r_n}x_n$, we have

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \ \forall y \in D.$$

It follows by (A2) that

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq f(y, u_n), \ \ \forall y \in D.$$

Hence

$$\langle y - u_n, \frac{u_n - x_n}{r_n}\rangle \geq f(y, u_n), \ \ \forall y \in D.$$

It follows from (3.6) and (A4) that

$$f(y, w) \leq 0, \ \ \forall y \in D.$$

For t with $0 < t \leq 1$ and $y \in D$, let $y_t = ty + (1-t)w$. Since $y, w \in D$ and D is convex, then $y_t \in D$ and hence

$$f(y_t, w) \leq 0.$$

By (A1) and (A4) this implies that

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, w)$$
$$\leq tf(y_t, y).$$

Dividing by t, we have

$$f(y_t, y) \geq 0, \ \ \forall y \in D.$$

Letting $t \to 0$, from the condition (A$_4$), we obtain that

$$f(w, y) \geq 0, \ \ \forall y \in D.$$

This implies that $w \in EP(f)$. Next, we will show that $w \in \bigcap_{i=1}^{\infty} F(T_i)$, In fact for each $i \in N$ we have

$$\begin{aligned} d(w, P_{T_i}w) &\leq d(w, x_n) + d(x_n, x_{n,i}) + d(x_{n,i}, P_{T_i}w) \\ &\leq d(w, x_n) + d(x_n, x_{n,i}) + H(P_{T_i}u_n, P_{T_i}w) \\ &\leq d(w, x_n) + d(x_n, x_{n,i}) + d(u_n, w). \end{aligned}$$

By Step 3, Step 4 and (3.6), we have $d(w, P_{T_i}w) = 0$. Hence $w \in P_{T_i}w$ for all $i \in N$, This shows that $w \in T_i w$.

Step 6 Show that $w = P_\Omega x_0$.

Since $x_n = P_{C_n}x_0$, we get

$$\langle z - x_n, x_0 - x_n \rangle \leq 0, \ \ \forall z \in C_n.$$

From $w \in \Omega \subset C_n$, we have

$$\langle z - w, x_0 - w \rangle \leq 0, \ \ \forall z \in \Omega.$$

Hence we have $w = P_\Omega x_0$. This completes the proof.

Theorem 3.2. *Let D be a nonempty closed and convex subset of a real Hilbert space H. Let $T_i : D \to P(D)$ be multivalued asymptotically nonexpansive mappings for all $i \in N$ with sequence $\{k_{n,i}\} \subset [1, \infty), k_{n,i} \to 1$ as $n \to \infty$ such that for each $i \geq 1$, P_{T_i} is nonexpansive. Assume $\Omega := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Define the sequence $\{x_n\}$ as follows: $x_0 \in D = C_0$,*

$$\begin{cases} y_n = \alpha_{n,0}x_n + \sum_{i=1}^{\infty} \alpha_{n,i}x_{n,i}, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ \ n \geq 0, \end{cases} \tag{3.7}$$

where $x_{n,i} \in P_{T_i}u_n$ for $i \in N$ and $\{\alpha_{n,i}\}$ is a real sequence in $[0, 1)$ satisfies the following conditions:

(a) $\sum_{i=0}^{\infty} \alpha_{n,i} = 1, \ \forall n \geq 1$

(b) $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0, \ \forall i \in N$

Then $\{x_n\}$ *converges strongly to* $P_{\Omega}x_0$.

Proof. Putting $f = 0, \ r_n = 1, \ u_n = x_n$ in Theorem 3.1, the desired result can be obtained from Theorem 3.1 directly .

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