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APPROXIMATION PROPERTIES OF THE GENERALIZED q-BERNSTEIN-SCHURER-KANTOROVICH OPERATORS

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Abstract. In this paper, a generalization of the q-Bernstein-Schurer-Kantorovich operators is considered. Some approximation results using the Korovkin type approximation theorem are established. The rate of convergence using the first and the second modulus of continuity is studied. The rate of convergence by means of the Lipschitz class is also investigated. Furthermore, the rate of convergence in terms of the modulus of continuity of the derivative of a function is estimated.

Keywords. q-integers; q-integral; Positive linear operators; Rate of convergence; Lipschitz function.

1. Introduction

By $C[a,b]$, we denote the space of continuous functions defined on $[a,b]$. In 1962, Schurer [14] presented the following generalization of the classical Bernstein operators

(1)
$$B_n^p(f;x) = \sum_{r=0}^{n+p} \binom{n+p}{r} x^r (1-x)^{n+p-r} f\left(\frac{r}{n}\right), \ x \in [0,1],$$

where $n \in N, p \in N_0 = \{0,1,2,...\}$ is fixed and $f \in C[0,p+1]$.

The introduction of q-calculus has emerged as a new venue of research in the field of approximation theory. In [4], Lupaş introduced the first q-analogue of the well-known Bernstein polynomials. Another q-analogue of the classical Bernstein polynomials was given by Phillips [12].

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For $0 < q < 1$, Muraru [5] introduced the q-companion of the Bernstein-Schurer operators as follows

$$B_n^p(f;q;x) = \sum_{r=0}^{n+p} \begin{bmatrix} n+p \\ r \end{bmatrix}_q x^r \prod_{s=0}^{n+p-r} (1-q^s x) f\left(\frac{[r]_q}{[n]_q}\right),\ x \in [0,1], \tag{2}$$

where $n \in N,\ p \in N_0 = \{0,1,2,...\}$ is fixed. She studied approximation properties and obtained the rate of convergence of these operators. Recently, approximation properties of q-analogue of various operators have been studied in [1], [2], [6]-[10] and [13].

2. Preliminaries

In this section we present some basics of the q-calculus. We refer the readers to [3] for all the following q-standard notations.

The q-integer $[n]_q$, the q-factorial $[n]_q!$ and the q-binomial coefficient of $n \in \mathbb{N}$ are defined by

$$[n]_q := \begin{cases} \frac{1-q^n}{1-q}, & \text{if } q \in \mathbb{R}^+ \setminus \{1\} \\ n, & \text{if } q = 1, \end{cases} \quad \text{for } n \in \mathbb{N} \text{ and } [0]_q = 0,$$

$$[n]_q! := \begin{cases} [n]_q[n-1]_q \cdots [1]_q, & n \geq 1, \\ 1, & n = 0, \end{cases}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!},$$

respectively. The q-analogue of $(1+x)^n$ is the polynomial

$$(1+x)_q^n := \begin{cases} (1+x)(1+qx)\cdots(1+q^{n-1}x) & n = 1,2,3,\cdots, \\ 1 & n = 0. \end{cases}$$

A q-analogue of the common Pochhammer symbol also called a q-shifted factorial is defined by $(x;q)_0 = 1\ (x;q)_n = \prod_{j=0}^{n-1}(1-q^j x),\ (x;q)_\infty = \prod_{j=0}^{\infty}(1-q^j x)$.

The Gauss binomial formula is given by

$$(x+a)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2} a^k x^{n-k}.$$

The q-Jackson integral over the interval $[0,b]$ is defined by

$$\int_0^b f(t)d_qt=(1-q)b\sum_{s=0}^{\infty}f(q^sb)q^s,\ 0<q<1. \tag{3}$$

3. Main results

In 2013, özarslan and Vedi [11] introduced the following sequence of operators $K_n^p(f;q;x):C[0,1]\to C[0,1]$,

$$K_n^p(f;q;x)=\sum_{r=0}^{n+p}\begin{bmatrix}n+p\\ r\end{bmatrix}x^r\prod_{s=0}^{n+p-r-1}(1-q^sx)\int_0^1 f\left(\frac{[r]}{[n+1]}+\frac{1+(q-1)[r]}{[n+1]}t\right)d_qt \tag{4}$$

for a real number $0<q<1$ and $f\in C[0,1+p]$. They investigated the approximation properties and estimated the rate of convergence of these operators.

Inspired by this work, we introduce the following operators

$$\begin{aligned}T_{n,p}^l(f;q;x) &= \frac{1}{[1+l]_q^{n+p}(1+l)}\sum_{k=0}^{n+p}\begin{bmatrix}n+p\\ k\end{bmatrix}_q x^k[1+l-x]_q^{n+p-k}\\ &\times \int_0^{1+l} f\Big(\frac{[k]_q}{[n+1]_q}+\frac{1+(q-1)[k]_q}{[n+1]_q}t\Big)d_qt,\end{aligned} \tag{5}$$

where $l\geq 0$, and call them as generalized q-Bernstein-Schurer-Kantorovich operators. We shall study approximation properties of these operators and estimate the rate of convergence. Following are the associated q-Bernstein-Schurer basis operators for the operators (5),

$$B_{n,p}^l(f;q;x)=\sum_{r=0}^{n+p}\begin{bmatrix}n+p\\ k\end{bmatrix}_q x^k[1+l-x]_q^{n+p-k}f\left(\frac{[k]_q}{[n]_q}\right),\ x\in[0,1].$$

We have the following lemmas.

Lemma 3.1. *Let $f(t)=1,t$ and t^2. Then the following are obtained*

(i) $B_{n,p}^l(1;q;x)=[1+l]_q^{n+p}$,

(ii) $B_{n,p}^l(t;q;x)=[1+l]_q^{n+p-1}\frac{[n+p]_q}{[n]_q}x$,

(iii) $B_{n,p}^l(t^2;q;x)=[1+l]_q^{n+p-2}\frac{[n+p]_q[n+p-1]_q}{[n]_q^2}qx^2+[1+l]_q^{n+p-1}\frac{[n+p]_q}{[n]_q^2}x.$

Proof. The proof is plain and straight so we omit it.

Lemma 3.2. *Let $T_{n,p}^l(f;q;x)$ be the operators given by (5). Then*

(i) $T_{n,p}^l(1;q;x)=1$,

(ii) $T^l_{n,p}(t;q;x) = \frac{(1+l)}{[2]_q[n+1]_q} + \frac{((2+l)q-l)}{[2]_q[n+1]_q}\frac{[n+p]_q}{[n+1]_q}x,$

(iii) $T^l_{n,p}(t^2;q;x) = \frac{1}{[n+1]_q^2[1+l]}\Big\{\frac{(1+l)^2}{[3]_q} + \Big(1+\frac{(1+l)((3+l)q^3+(3+l)q^2+(1-l)q-l-1)}{[2]_q[3]_q}\Big)\frac{[n+p]_q}{[1+l]_q}x$
$+\Big(1+\frac{(1+l)((3+l)q^3+(3+l)q^2+(1-l)q-l-1)}{[2]_q[3]_q}\Big)\frac{q[n+p]_q[n+p-1]_q}{[1+l]_q^2}x^2\Big\}.$

Proof. In order to prove the lemma, we shall use the following integrals

$$\int_0^{1+l} d_qt = 1+l.$$

Using this integral, (i) is obvious. Also,

$$\int_0^{1+l}\Big(\frac{[k]_q}{[n+1]_q}+\frac{1+(q-1)[k]_q}{[n+1]_q}t\Big)d_qt = \frac{(1+l)^2}{[2]_q[n+1]_q}+\frac{((2+l)q-l)(1+l)}{[2]_q[n+1]_q}[k]_q$$

and

$$\begin{aligned}
&\int_0^{1+l}\Big(\frac{[k]_q}{[n+1]_q}+\frac{1+(q-1)[k]_q}{[n+1]_q}t\Big)^2 d_qt\\
=\;& \frac{(1+l)}{[2]_q[n+1]_q^2}\Big(\frac{(1+l)^2}{[3]_q}+\Big(\frac{2(1+l)}{[2]_q}+\frac{2(q-1)(1+l)^2}{[3]_q}\Big)\\
\times\;& [k]_q+\Big(1+\frac{2(q-l)(1+l)}{[2]_q}+\frac{(q-l)^2(1+l)^2}{[3]_q}\Big)[k]_q^2\Big).
\end{aligned}$$

$$\begin{aligned}
T^l_{n,p}(t;q;x) \;=\;& \frac{1}{[1+l]_q^{n+p}(1+l)}\sum_{k=0}^{n+p}\begin{bmatrix} n+p\\ k\end{bmatrix}_q x^k[1+l-x]_q^{n+p-k}\int_0^{1+l}\Big(\frac{[k]_q}{[n+1]_q}\\
+\;& \frac{1+(q-1)[k]_q}{[n+1]_q}t\Big)d_qt\\
=\;& \frac{1}{[1+l]_q^{n+p}(1+l)}\sum_{k=0}^{n+p}\begin{bmatrix} n+p\\ k\end{bmatrix}_q x^k[1+l-x]_q^{n+p-k}\Big(\frac{(1+l)^2}{[2]_q[n+1]_q}\\
+\;& \frac{((2+l)q-l)(1+l)}{[2]_q[n+1]_q}[k]_q\Big)\\
=\;& \frac{1}{[1+l]_q^{n+p}(1+l)}\Big(\frac{(1+l)^2}{[2]_q[n+1]_q}\sum_{k=0}^{n+p}\begin{bmatrix} n+p\\ k\end{bmatrix}_q x^k[1+l-x]_q^{n+p-k}\\
+\;& \frac{((2+l)q-l)(1+l)}{[2]_q[n+1]_q}\sum_{k=0}^{n+p}\begin{bmatrix} n+p\\ k\end{bmatrix}_q x^k[1+l-x]_q^{n+p-k}[k]_q\Big)\\
=\;& \frac{(1+l)}{[2]_q[n+1]_q}+\frac{((2+l)q-l)}{[2]_q[n+1]_q}\frac{[n+p]_q}{[n+1]_q}x,
\end{aligned}$$

which proves (ii).

$$
\begin{aligned}
& T_{n,p}^{l}(t^2;q;x) \\
= \;& \frac{1}{[1+l]_q^{n+p}(1+l)} \sum_{k=0}^{n+p} \begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k [1+l-x]_q^{n+p-k} \\
\times \;& \int_0^{1+l} \Big(\frac{[k]_q}{[n+1]_q} + \frac{1+(q-1)[k]_q}{[n+1]_q} t \Big)^2 d_q t \\
= \;& \frac{1}{[1+l]_q^{n+p}(1+l)} \sum_{k=0}^{n+p} \begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k [1+l-x]_q^{n+p-k} \frac{(1+l)}{[n+1]_q^2} \Big\{ \frac{(1+l)^2}{[3]_q} + \Big(\frac{2(1+l)}{[2]_q} \\
+ \;& \frac{2(q-1)(1+l)^2}{[3]_q} \Big) [k]_q + \Big(1 + \frac{2(q-1)(1+l)}{[2]_q} + \frac{(q-1)^2(1+l)^2}{[3]_q} \Big) [k]_q^2 \Big\} \\
= \;& \frac{1}{[n+1]_q^2 [1+l]_q^{n+p}} \Big\{ \frac{(1+l)^2}{[3]_q} B_{n,p}^{l}(1;q;x) + [n]_q \Big(\frac{2(1+l)}{[2]_q} + \frac{2(q-1)(1+l)^2}{[3]_q} \Big) \\
\times \;& B_{n,p}^{l}(t;q;x) + [n]_q^2 \Big(1 + \frac{2(q-1)(1+l)}{[2]_q} + \frac{(q-1)^2(1+l)^2}{[3]_q} \Big) B_{n,p}^{l}(t^2;q;x) \Big\} \\
= \;& \frac{1}{[n+1]_q^2 [1+l]} \Big\{ \frac{(1+l)^2}{[3]_q} + \Big(\frac{2(1+l)}{[2]_q} + \frac{2(q-1)(1+l)^2}{[3]_q} \Big) \frac{[n+p]_q}{[1+l]_q} x \\
+ \;& \Big(1 + \frac{2(q-1)(1+l)}{[2]_q} + \frac{(q-1)^2(1+l)^2}{[3]_q} \Big) \frac{q[n+p]_q [n+p-1]_q}{[1+l]_q^2} x^2 \\
+ \;& \Big(1 + \frac{2(q-1)(1+l)}{[2]_q} + \frac{(q-1)^2(1+l)^2}{[3]_q} \Big) \frac{[n+p]_q}{[1+l]_q} x \Big\} \\
= \;& \frac{1}{[n+1]_q^2 [1+l]} \Big\{ \frac{(1+l)^2}{[3]_q} + \Big(1 + \frac{(1+l)((3+l)q^3+(3+l)q^2+(1-l)q-l-1)}{[2]_q[3]_q} \Big) \\
\times \;& \frac{[n+p]_q}{[1+l]_q} x + \Big(1 + \frac{(1+l)((3+l)q^3+(3+l)q^2+(1-l)q-l-1)}{[2]_q[3]_q} \Big) \\
\times \;& \frac{q[n+p]_q [n+p-1]_q}{[1+l]_q^2} x^2 \Big\}.
\end{aligned}
$$

This proves (iii).

Lemma 3.3. *Let $T_{n,p}^{l}(f;q;x)$ be the operators given by (5). Then*

(i) $T_{n,p}^{l}((t-x);q;x) = \frac{(1+l)}{[2]_q[n+1]_q} + \Big(\frac{(2+l)q-l}{[2]_q[1+l]_q} \frac{[n+p]_q}{[n+1]_q} - 1 \Big) x$,

(ii) $T_{n,p}^{l}((t-x)^2;q;x) = \frac{(1+l)^2}{[n+1]_q^2(1+l)[3]_q} + \frac{1}{[n+1]_q^2(1+l)} \Big(1 + \frac{(1+l)((3+l)q^3+(3+l)q^2+(1-l)q-l-1)}{[2]_q[3]_q} \frac{[n+p]_q}{[1+l]_q} \Big) \Big(x + \frac{q[n+p-1]_q}{[1+l]_q} x^2 \Big) - \frac{2}{[n+1]_q[2]_q} \Big((1+l)x + ((2+l)q-l) \frac{[n+p]_q}{[1+l]_q} x^2 \Big) + x^2.$

Proof. Making use of the Lemma 3.2, we obtain

$$\begin{aligned} T^l_{n,p}((t-x);q;x) &= T^l_{n,p}(t;q;x) - xT^l_{n,p}(1;q;x) \\ &= \frac{(1+l)}{[2]_q[n+1]_q} + \frac{((2+l)q-l)}{[2]_q[n+1]_q}\frac{[n+p]_q}{[n+1]_q}x - x \\ &= \frac{(1+l)}{[2]_q[n+1]_q} + \Big(\frac{((2+l)q-l)}{[2]_q[n+1]_q}\frac{[n+p]_q}{[n+1]_q} - 1\Big)x, \end{aligned}$$

which proves (i).

$$\begin{aligned} T^l_{n,p}((t-x)^2;q;x) &= T^l_{n,p}(t^2;q;x) - 2xT^l_{n,p}(t;q;x) + x^2T^l_{n,p}(1;q;x) \\ &= \frac{1}{[n+1]_q^2[1+l]}\Bigg\{\frac{(1+l)^2}{[3]_q} \\ &+ \Big(1+\frac{(1+l)((3+l)q^3+(3+l)q^2+(1-l)q-l-1)}{[2]_q[3]_q}\Big)\frac{[n+p]_q}{[1+l]_q}x \\ &+ \Big(1+\frac{(1+l)((3+l)q^3+(3+l)q^2+(1-l)q-l-1)}{[2]_q[3]_q}\Big)\frac{q[n+p]_q[n+p-1]_q}{[1+l]_q^2}x^2\Bigg\} \\ &- \frac{2(1+l)}{[n+1]_q[2]_q}x - \frac{2((2+l)q-l}{[1+l]_q[2]_q}\frac{[n+p]_q}{[n+1]_q}x^2 + x^2 \\ &= \frac{(1+l)^2}{[n+1]_q^2(1+l)[3]_q} + \frac{1}{[n+1]_q^2(1+l)} \\ &\times \Big(1+\frac{(1+l)((3+l)q^3+(3+l)q^2+(1-l)q-l-1)}{[2]_q[3]_q}\frac{[n+p]_q}{[1+l]_q}\Big) \\ &\times \frac{[n+p]_q}{[1+l]_q}\Big(x+\frac{q[n+p-1]_q}{[1+l]_q}x^2\Big) - \frac{2}{[n+1]_q[2]_q}\Big((1+l)x + ((2+l)q-l) \\ &\times \frac{[n+p]_q}{[1+l]_q}x^2\Big) + x^2. \end{aligned}$$

This proves (ii).

In view of the Korovkin's theorem, we have the following theorem.

Theorem 3.4. *Let* $q=(q_n)$ *be a sequence such that* $\lim_{n\to\infty} q_n = 1$ *and* $\lim_{n\to\infty}\frac{1}{[n]_{q_n}} = 0$*. Then*

$$\lim_{n\to\infty} \|T^l_{n,p}(f;q_n;.) - f(.)\|_{C[0,1]} = 0,$$

for all $f \in C[0,1+p]$.

4. Rate of convergence

In what follows, we shall calculate the rate of convergence of the operators (5) by means of modulus of continuity, elements of Lipschitz classes and the first and the second modulus of continuity of a function. Also, we compute the rate of convergence in terms of the first modulus of continuity of the derivative of a function.

Let $\delta > 0$. The first modulus of continuity of a function f on the interval $C[0,1+p]$ is defined as

$$\omega(f,\delta) = \max_{|y-x|\leq\delta} \mid f(y) - f(x) \mid, \; x,y \in [0,p+1]. \tag{6}$$

Let $f \in C[0,1+p]$ and $\delta > 0$. Then the following are well known

$$\lim_{\delta\to 0+} \omega(f,\delta) = 0$$

and

$$\mid f(y) - f(x) \mid \leq \left(\frac{\mid y-x \mid}{\delta} + 1\right)\omega(f,\delta).$$

We have the following theorem.

Theorem 4.1. *Let $T^l_{n,p}(f;q;x)$ be the operators defined by (5). Then for $f \in C[0,1+p]$, $0 < q < 1$, we have*

$$|T^l_{n,p}(f;q;x) - f(x)| \leq 2\omega\Big(f;(\delta_{n,q}(x))^{\frac{1}{2}}\Big),$$

where $\omega(f,\delta)$ is the modulus of continuity of f and

$$\delta_{n,q}(x) = T^l_{n,p}((t-x)^2;q;x)$$

as given in Lemma 3.3.

Proof. Making use of the linearity and the positivity of the operators, we obtain

$$\begin{aligned} |T^l_{n,p}(f;q;x) - f(x)| &\leq \frac{1}{[1+l]_q^{n+p}(1+l)} \sum_{k=0}^{n+p} \begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k [1+l-x]_q^{n+p-k} \\ &\times \int_0^{1+l} \mid f\Big(\frac{[k]_q}{[n+1]_q} + \frac{1+(q-1)[k]_q}{[n+1]_q}t\Big) - f(x) \mid d_q(t) \\ &\leq \sum_{k=0}^{n+p} \int_0^{1+l} \Big(1 + \frac{|\frac{[k]_q}{[n+1]_q} + \frac{1+(q-1)[k]_q}{[n+1]_q}t - x|}{\delta}\Big) \end{aligned}$$

$$
\begin{aligned}
&\times \quad \omega(f;\delta)\frac{1}{[1+l]_q^{n+p}(1+l)}\sum_{k=0}^{n+p}\begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k[1+l-x]_q^{n+p-k}d_q(t) \\
&= \quad \Big(\frac{1}{[1+l]_q^{n+p}(1+l)}\sum_{k=0}^{n+p}\begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k[1+l-x]_q^{n+p-k}\Big)\omega(f;\delta) \\
&+ \quad \Big(\frac{1}{[1+l]_q^{n+p}(1+l)}\sum_{k=0}^{n+p}\int_0^{1+l}|\frac{[k]_q}{[n+1]_q}+\frac{1+(q-1)[k]_q}{[n+1]_q}t-x| \\
&\times \quad \begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k[1+l-x]_q^{n+p-k}d_q(t)\Big)\frac{\omega(f;\delta)}{\delta}.
\end{aligned}
$$

Using the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
|T_{n,p}^l(f;q;x)-f(x)| \quad &\leq \quad \omega(f;\delta)+\frac{\omega(f;\delta)}{\delta}\sum_{k=0}^{n+p}\Big\{\int_0^{1+l}\Big(\frac{[k]_q}{[n+1]_q}+\frac{1+(q-1)[k]_q}{[n+1]_q}t-x\Big)^2 d_qt\Big\}^{\frac{1}{2}} \\
&\times \quad \Big\{\frac{1}{[1+l]_q^{n+p}(1+l)}\begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k[1+l-x]_q^{n+p-k}\Big\}.
\end{aligned}
$$

Again using the Cauchy-Schwarz inequality, the following is obtained

$$
\begin{aligned}
|T_{n,p}^l(f;q;x)-f(x)| \quad &\leq \quad \omega(f;\delta)+\frac{\omega(f;\delta)}{\delta}\Big\{\frac{1}{[1+l]_q^{n+p}(1+l)}\sum_{k=0}^{n+p}\int_0^{1+l}\Big(\frac{[k]_q}{[n+1]_q} \\
&+ \quad \frac{1+(q-1)[k]_q}{[n+1]_q}t-x\Big)^2 d_qt \times \begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k[1+l-x]_q^{n+p-k}\Big\}^{\frac{1}{2}} \\
&\times \quad \Big\{\frac{1}{[1+l]_q^{n+p}(1+l)}\sum_{k=0}^{n+p}\begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k[1+l-x]_q^{n+p-k}\Big\}^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&= \quad \omega(f;\delta)+\frac{\omega(f;\delta)}{\delta}\Big\{\frac{1}{[1+l]_q^{n+p}(1+l)}\sum_{k=0}^{n+p}\begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k[1+l-x]_q^{n+p-k} \\
&\times \quad \int_0^{1+l}\Big(\frac{[k]_q}{[n+1]_q}+\frac{1+(q-1)[k]_q}{[n+1]_q}t-x\Big)^2 d_qt\Big\}^{\frac{1}{2}} \\
&= \quad \omega(f;\delta)+\frac{\omega(f;\delta)}{\delta}\Big\{T_{n,p}^l((t-x)^2;q;x)-f(x)\Big\}^{\frac{1}{2}}.
\end{aligned}
$$

On choosing

$$\delta=\delta_{n,q}(x)=T_{n,p}^l((t-x)^2;q;x),$$

we obtain

$$|T_{n,p}^l(f;q;x)-f(x)|\leq 2\omega\Big(f;(\delta_{n,q}(x))^{\frac{1}{2}}\Big),$$

which completes the proof of the theorem.

In the following theorem, we compute the rate of convergence of the operators in (5) in terms of the elements of the Lipschitz class $Lip_M(\nu)$.

Let $f \in C[0,1+p]$, $M > 0$ and $0 < \nu \leq 1$. The class $Lip_M(\nu)$ is defined as

$$Lip_M(\nu) = \{f : \mid f(\zeta_1) - f(\zeta_2) \mid \leq M \mid \zeta_1 - \zeta_2 \mid^{\nu},\ (\zeta_1, \zeta_2 \in [0,1+p])\}. \tag{7}$$

Theorem 4.2. *Let $T^l_{n,p}(f;q;x)$ be the operators given by (5). Then for each $f \in Lip_M(\nu)$ satisfying (7), we have*

$$|T^l_{n,p}(f;q;x) - f(x)| \leq M \frac{(1+l)}{\sqrt{(1+l)^{\nu}}} \sqrt{(\delta_{n,q}(x))^{\nu}},$$

where $\delta_{n,q}(x)$ is given as in Theorem 4.1.

Proof. Using the linearity and the positivity of the operators, the following is obtained

$$\begin{aligned} |T^l_{n,p}(f;q;x) - f(x)| \ &\leq \ \frac{1}{[1+l]_q^{n+p}(1+l)} \sum_{k=0}^{n+p} \begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k [1+l-x]_q^{n+p-k} \\ &\times \int_0^{1+l} \mid f\Big(\frac{[k]_q}{[n+1]_q} + \frac{1+(q-1)[k]_q}{[n+1]_q} t\Big) - f(x) \mid d_q(t). \end{aligned}$$

In view of (7) and the Hölder's inequality with $p = \frac{2}{\nu}$ and $q = \frac{2}{2-\nu}$, we obtain

$$\begin{aligned} &\int_0^{1+l} \mid f\Big(\frac{[k]_q}{[n+1]_q} + \frac{1+(q-1)[k]_q}{[n+1]_q} t\Big) - f(x) \mid d_q(t) \\ &\leq M \int_0^{1+l} \mid \frac{[k]_q}{[n+1]_q} + \frac{1+(q-1)[k]_q}{[n+1]_q} t - x \mid^{\nu} d_q(t) \\ &= M \Big\{ \int_0^{1+l} \Big(\frac{[k]_q}{[n+1]_q} + \frac{1+(q-1)[k]_q}{[n+1]_q} t - x\Big)^2 d_q(t) \Big\}^{\frac{\nu}{2}} \Big\{ \int_0^{1+l} 1 d_q t \Big\}^{\frac{2-\nu}{2}} \\ &= M(1+l)^{\frac{2-\nu}{2}} \Big\{ \int_0^{1+l} \Big(\frac{[k]_q}{[n+1]_q} + \frac{1+(q-1)[k]_q}{[n+1]_q} t - x\Big)^2 d_q(t) \Big\}^{\frac{\nu}{2}}. \end{aligned}$$

Again applying the Hölder's inequality with $p = \frac{2}{\nu}$ and $q = \frac{2}{2-\nu}$, we get the following

$$\begin{aligned} |T^l_{n,p}(f;q;x) - f(x)| \leq M(1+l)^{\frac{2-\nu}{2}} \Big\{ \frac{1}{[1+l]_q^{n+p}(1+l)} \sum_{k=0}^{n+p} \int_0^{1+l} \Big(\frac{[k]_q}{[n+1]_q} \int_0^{1+l} \Big(\frac{[k]_q}{[n+1]_q} \\ + \frac{1+(q-1)[k]_q}{[n+1]_q} t - x\Big)^2 d_q(t) \Big\}^{\frac{\nu}{2}} \end{aligned}$$

$$= M\frac{(1+l)}{\sqrt{(1+l)^{\nu}}}\left\{T^{l}_{n,p}((t-x)^2;q;x)\right\}^{\frac{\nu}{2}}$$
$$= M\frac{(1+l)}{\sqrt{(1+l)^{\nu}}}\sqrt{(\delta_{n,q}(x))^{\nu}},$$

which completes the proof of the theorem.

By $C^2[0,1+p]$, we denote the space of all functions f in $C[0,1+p]$ such that $f', f'' \in C[0,1+p]$ equipped with the norm

$$\| f \| = \sup_{x\in C[0,1+p]} | f(x) | .$$

The Peetre's K-functional is defined by

$$K_2(f,\delta) = \inf_{g\in C^2[0,1+p]} \left\{\| f-g \| + \delta \| g'' \|\right\}.$$

The second modulus of smoothness of the function $f \in C[0,1+p]$, for $\delta > 0$, is defined by

$$\omega(f,\delta) = \sup_{0<h<\delta} | f(x+2h) - 2f(x+h) + f(x) |;\ x, x+h \in [0,1+p].$$

It is known that there exits a positive constant C such that

$$K_2(f,\delta) \le C\omega_2(f,\delta^{\frac{1}{2}}). \tag{8}$$

We prove the following theorem.

Theorem 4.3. *Let $0 \le x \le 1$, $0 < q < 1$ and $f \in C[0,1+p]$. Then the following holds*

$$|T^{l}_{n,p}(f;q;x) - f(x)| \le C\omega_2(f,(\alpha_{n,q}(x))^{\frac{1}{2}}) + \omega(f,\beta_{n,q}),$$

where $C > 0$ is a constant, $p \in N_0$ is fixed,

$$\begin{aligned}
\alpha_{n,q}(x) &= \frac{(1+l)^2}{[2]_q^2[n+1]_q^2} + \frac{(1+l)^2}{(1+l)[n+1]_q^2[3]_q} \\
&+ \frac{1}{[n+1]_q^2(1+l)}\Big(1 + \frac{(1+l)((3+l)q^3+(3+l)q^2+(1-l)q-l-1)}{[2]_q[3]_q}\frac{[n+p]_q}{[1+l]_q} \\
&+ \frac{2(1+l)}{[2]_q[n+1]_q}\frac{((2+l)q-l)}{[2]_q[n+1]_q}\frac{[n+p]_q}{[n+1]_q} - \frac{4(1+l)}{[n+1]_q^2[3]_q}\Big)x \\
&+ \Big(\frac{((2+l)q-l)^2}{[2]_q^2[n+1]_q^2}\frac{[n+p]_q^2}{[n+1]_q^2}q^2 + \frac{1}{[n+1]_q^2(1+l)}
\end{aligned}$$

$$\times \quad \Big(1+\frac{(1+l)((3+l)q^3+(3+l)q^2+(1-l)q-l-1)}{[2]_q[3]_q}\frac{[n+p]_q}{[1+l]_q}\frac{[n+p-1]_q}{[1+l]_q}q$$

$$- \quad \frac{4((2+l)q-l)}{[2]_q[n+1]_q}\frac{[n+p]_q}{[1+l]_q}+2\Big)x^2 \tag{9}$$

and

$$\beta_{n,q}=\frac{(1+l)}{[2]_q[n+1]_q}+\Big(\frac{((2+l)q-l)}{[2]_q[n+1]_q}\frac{[n+p]_q}{[n+1]_q}q-1\Big)x. \tag{10}$$

Proof. Let us define the following auxiliary operators

$$T^{*l}_{n,p}(f;q;x)=T^{l}_{n,p}(f;q;x)-f\Big(\frac{(1+l)}{[2]_q[n+1]_q}+\frac{((2+l)q-l)}{[2]_q[n+1]_q}\frac{[n+p]_q}{[n+1]_q}qx\Big)+f(x). \tag{11}$$

By Lemma 3.2 (i) and Lemma 3.3 (i), it follows that

$$T^{*l}_{n,p}(f;q;x)=1,\quad T^{*l}_{n,p}((t-x);q;x)=0. \tag{12}$$

Then by the Taylor's formula, for $g\in C^2[0,1+p]$, we can write

$$g(y)=g(x)+(y-x)g'(x)+\int_x^y (y-t)g''(t)dt,\ 0\le y\le 1.$$

In view of (12), for every $0<x<1$, the following is obtained

$$\begin{aligned}|T^{*l}_{n,p}(g;q;x)-g(x)| &= |T^{*l}_{n,p}(g(y)-g(x);q;x)|\\ &= |g'(x)T^{*l}_{n,p}((t-x);q;x)+T^{*l}_{n,p}\Big(\int_x^y (y-t)g''(t)dt;q;x\Big)|\\ &= |T^{*l}_{n,p}\Big(\int_x^y (y-t)g''(t)dt;q;x\Big)|.\end{aligned}$$

By (11), we get

$$\begin{aligned}&|T^{*l}_{n,p}(g;q;x)-g(x)|\\ &\quad= |T^{l}_{n,p}\Big(\int_x^y (y-t)g''(t)dt;q;x\Big)-\int_x^{\frac{(1+l)}{[2]_q[n+1]_q}+\frac{((2+l)q-l)}{[2]_q[n+1]_q}\frac{[n+p]_q}{[n+1]_q}x}\\ &\quad\times \Big(\frac{(1+l)}{[2]_q[n+1]_q}+\frac{((2+l)q-l)}{[2]_q[n+1]_q}\frac{[n+p]_q}{[n+1]_q}x-t\Big)g''(t)dt|\\ &\quad\le |T^{l}_{n,p}\Big(\int_x^y (y-t)g''(t)dt;q;x\Big)|+|\int_x^{\frac{(1+l)}{[2]_q[n+1]_q}+\frac{((2+l)q-l)}{[2]_q[n+1]_q}\frac{[n+p]_q}{[n+1]_q}x}\\ &\quad\times \Big(\frac{(1+l)}{[2]_q[n+1]_q}+\frac{((2+l)q-l)}{[2]_q[n+1]_q}\frac{[n+p]_q}{[n+1]_q}x-t\Big)g''(t)dt|.\end{aligned}$$

Since

$$|T^{l}_{n,p}\Big(\int_x^y (y-t)g''(t)dt;q;x\Big)| \leq \|g\| T^{l}_{n,p}((y-x)^2;q;x),$$

$$|\int_x^{\frac{(1+l)}{[2]_q[n+1]_q}+\frac{((2+l)q-l)}{[2]_q[n+1]_q}\frac{[n+p]_q}{[n+1]_q}x}\Big(\frac{(1+l)}{[2]_q[n+1]_q}+\frac{((2+l)q-l)}{[2]_q[n+1]_q}\frac{[n+p]_q}{[n+1]_q}x-t\Big)g''(t)dt|$$
$$\leq \Big(\frac{(1+l)}{[2]_q[n+1]_q}+\Big(\frac{((2+l)q-l)}{[2]_q[n+1]_q}\frac{[n+p]_q}{[n+1]_q}-1\Big)x\Big)^2,$$

the following is obtained

$$|T^{*l}_{n,p}(g;q;x)-g(x)| \leq \|g''\|\Big\{T^{l}_{n,p}((y-x)^2;q;x)+\Big(\frac{(1+l)}{[2]_q[n+1]_q}$$
$$+\Big(\frac{((2+l)q-l)}{[2]_q[n+1]_q}\frac{[n+p]_q}{[n+1]_q}-1\Big)x\Big)^2\Big\}.$$

Using Lemma 3.3, one has

$$\begin{aligned}
|T^{*l}_{n,p}(g;q;x)-g(x)| &\leq \|g''\|\Big\{\frac{(1+l)}{[2]_q[n+1]_q}+\Big(\frac{(2+l)q-l}{[2]_q[1+l]_q}\frac{[n+p]_q}{[n+1]_q}-1\Big)+\frac{(1+l)^2}{[n+1]_q^2(1+l)[3]_q} \\
&+ \frac{1}{[n+1]_q^2(1+l)}\Big(1+\frac{(1+l)((3+l)q^3+(3+l)q^2+(1-l)q-l-1)}{[2]_q[3]_q} \\
&\times \frac{[n+p]_q}{[1+l]_q}\Big)\Big(x+\frac{q[n+p-1]_q}{[1+l]_q}x^2\Big)-\frac{2}{[n+1]_q[2]_q}\Big((1+l)x+((2+l)q-l) \\
&\times \frac{[n+p]_q}{[1+l]_q}x^2\Big)+x^2\Big\}. \qquad (13)
\end{aligned}$$

Since

$$|T^{*l}_{n,p}(f;q;.)| \leq 3,$$

so for all $f \in C[0,1+p]$ and $g \in C^2[0,1+p]$, from (13) we write

$$\begin{aligned}
|T^{l}_{n,p}(f;q;x)-f(x)| &\leq |T^{*l}_{n,p}(f-g;q;x)-(f-g)(x)|+|T^{*l}_{n,p}(g;q;x)-g(x)| \\
&+ |f\Big(\frac{(1+l)}{[2]_q[n+1]_q}+\frac{((2+l)q-l)}{[2]_q[n+1]_q}\frac{[n+p]_q}{[n+1]_q}x\Big)-f(x)| \\
&\leq 4\|f-g\|+\alpha_{n,q}(x)\|g''\|+|f\Big(\frac{(1+l)}{[2]_q[n+1]_q}+\frac{((2+l)q-l)}{[2]_q[n+1]_q}\frac{[n+p]_q}{[n+1]_q}x\Big) \\
&- f(x)| \\
&\leq 4(\|f-g\|+\alpha_{n,q}(x)\|g''\|)+\omega(f,\beta_{n,q}(x)),
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}|T_{n,p}^{l}(f;q;x)-f(x)| &\leq 4K(f,\alpha_{n,q}(x))+\omega(f,\beta_{n,q}(x))\\ &\leq C\omega_2(f,(\alpha_{n,q}(x))^{\frac{1}{2}})+\omega(f,\beta_{n,q}(x)),\end{aligned}$$

where

$$\begin{aligned}\alpha_{n,q}(x) &= \frac{(1+l)^2}{[2]_q^2[n+1]_q^2}+\frac{(1+l)^2}{(1+l)[n+1]_q^2[3]_q}+\frac{1}{[n+1]_q^2(1+l)}\\ &\times \Big(1+\frac{(1+l)((3+l)q^3+(3+l)q^2+(1-l)q-l-1)}{[2]_q[3]_q}\frac{[n+p]_q}{[1+l]_q}+\frac{2(1+l)}{[2]_q[n+1]_q}\\ &\times \frac{((2+l)q-l)}{[2]_q[n+1]_q}\frac{[n+p]_q}{[n+1]_q}-\frac{4(1+l)}{[n+1]_q^2[3]_q}\Big)x+\Big(\frac{((2+l)q-l)^2}{[2]_q^2[n+1]_q^2}\frac{[n+p]_q^2}{[n+1]_q^2}q^2\\ &+ \Big(1+\frac{(1+l)((3+l)q^3+(3+l)q^2+(1-l)q-l-1)}{[2]_q[3]_q}\frac{[n+p]_q}{[1+l]_q}\frac{[n+p-1]_q}{[1+l]_q}q\\ &- \frac{4((2+l)q-l)}{[2]_q[n+1]_q}\frac{[n+p]_q}{[1+l]_q}+2\Big)x^2\times\frac{1}{[n+1]_q^2(1+l)}\end{aligned}$$

and

$$\beta_{n,q}=\frac{(1+l)}{[2]_q[n+1]_q}+\Big(\frac{((2+l)q-l)}{[2]_q[n+1]_q}\frac{[n+p]_q}{[n+1]_q}q-1\Big)x,$$

which completes the proof of the theorem.

Next we estimate the rate of convergence of the operators (5) using the modulus of continuity of the derivative of the function.

Theorem 4.4. *Let $T_{n,p}^{l}(f;q;x)$ be the operators defined by (5) and let $0<q<1$, $p\in N_0$ be fixed. Suppose f' is continuous and $\omega(f';\delta)$ is the modulus of continuity of f' on $[0,1+p]$. Then we have*

$$|T_{n,p}^{l}(f;q;x)-f(x)|\leq M\gamma_{n,q}(x)+\omega(f';\delta)(1+\sqrt{\delta_{n,q}^{p}(x)},$$

where $M>0$ is a constant such that $|f'(x)|\leq M$, for all $0\leq x\leq 1+p$,

$$\begin{aligned}\delta_{n,q}^{p}(x) &= \frac{(1+l)^2}{[n+1]_q^2(1+l)[3]_q}+\frac{1}{[n+1]_q^2(1+l)}\Big(1+\frac{(1+l)((3+l)q^3+(3+l)q^2+(1-l)q-l-1)}{[2]_q[3]_q}\\ &\times \frac{[n+p]_q}{[1+l]_q}\Big)\Big(x+\frac{q[n+p-1]_q}{[1+l]_q}x^2\Big)-\frac{2}{[n+1]_q[2]_q}\Big((1+l)x+((2+l)q-l)\frac{[n+p]_q}{[1+l]_q}x^2\Big)+x^2\end{aligned}$$

and

$$\gamma_{n,q}(x)=\frac{(1+l)}{[2]_q[n+1]_q}+\Big(\frac{((2+l)q-l)}{[2]_q[n+1]_q}\frac{[n+p]_q}{[n+1]_q}-1\Big)x. \tag{14}$$

Proof. By the mean value theorem, we can write

$$f\Big(\frac{[k]_q}{[n+1]_q}+\frac{1+(q-1)[k]_q}{[n+1]_q}t\Big)-f(x)=\Big(\frac{[k]_q}{[n+1]_q}+\frac{1+(q-1)[k]_q}{[n+1]_q}t\Big)f'(\zeta)$$

$$=\Big(\frac{[k]_q}{[n+1]_q}+\frac{1+(q-1)[k]_q}{[n+1]_q}t-x\Big)f'(x)+\Big(\frac{[k]_q}{[n+1]_q}+\frac{1+(q-1)[k]_q}{[n+1]_q}t-x\Big)(f'(\zeta)-f'(x)),$$

where

$$x<\zeta<\frac{[k]_q}{[n+1]_q}+\frac{1+(q-1)[k]_q}{[n+1]_q}t.$$

Thus the following is obtained

$$\begin{aligned}
|T^l_{n,p}(f;q;x)-f(x)| &= \Big|\frac{f'(x)}{[1+l]_q^{n+p}(1+l)}\sum_{k=0}^{n+p}\int_0^{1+l}\Big(\frac{[k]_q}{[n+1]_q}+\frac{1+(q-1)[k]_q}{[n+1]_q}t-x\Big)\\
&\times \begin{bmatrix} n+p\\ k\end{bmatrix}_q x^k[1+l-x]_q^{n+p-k}d_qt+\frac{1}{[1+l]_q^{n+p}(1+l)}\sum_{k=0}^{n+p}\int_0^{1+l}\Big(\frac{[k]_q}{[n+1]_q}\\
&+ \frac{1+(q-1)[k]_q}{[n+1]_q}t-x\Big)+\frac{(f'(\zeta)-f'(x))}{[1+l]_q^{n+p}(1+l)}\begin{bmatrix} n+p\\ k\end{bmatrix}_q x^k[1+l-x]_q^{n+p-k}d_qt\Big|\\
&\leq |f'(x)|T^l_{n,p}((t-x);q;x)+\frac{1}{[1+l]_q^{n+p}(1+l)}\sum_{k=0}^{n+p}\int_0^{1+l}\Big(\frac{[k]_q}{[n+1]_q}\\
&+ \frac{1+(q-1)[k]_q}{[n+1]_q}t-x\Big)\frac{(f'(\zeta)-f'(x))}{[1+l]_q^{n+p}(1+l)}\begin{bmatrix} n+p\\ k\end{bmatrix}_q x^k[1+l-x]_q^{n+p-k}d_qt\\
&\leq M\gamma_{n,q}(x)+\frac{1}{[1+l]_q^{n+p}(1+l)}\sum_{k=0}^{n+p}\int_0^{1+l}\Big(\frac{[k]_q}{[n+1]_q}+\frac{1+(q-1)[k]_q}{[n+1]_q}t-x\Big)\\
&\times \frac{(f'(\zeta)-f'(x))}{[1+l]_q^{n+p}(1+l)}\begin{bmatrix} n+p\\ k\end{bmatrix}_q x^k[1+l-x]_q^{n+p-k}d_qt,
\end{aligned}$$

where $\gamma_{n,q}(x)$ is given by (14). Hence we obtain the following

$$\begin{aligned}
|T^l_{n,p}(f;q;x)-f(x)| &\leq M\gamma_{n,q}(x)+\frac{1}{[1+l]_q^{n+p}(1+l)}\sum_{k=0}^{n+p}\int_0^{1+l}\omega(f';\delta)\\
&\times \Big(\frac{\big(\frac{[k]_q}{[n+1]_q}+\frac{1+(q-1)[k]_q}{[n+1]_q}t-x\big)}{\delta}+1\Big)\Big(\frac{[k]_q}{[n+1]_q}+\frac{1+(q-1)[k]_q}{[n+1]_q}t-x\Big)\Big),
\end{aligned}$$

since

$$\zeta - x \leq \frac{[k]_q}{[n+1]_q} + \frac{1+(q-1)[k]_q}{[n+1]_q}t - x.$$

In the first term applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&|T_{n,p}^l(f;q;x) - f(x)| \\
\leq\ & M\gamma_{n,q}(x) + \frac{\omega(f',\delta)}{[1+l]_q^{n+p}(1+l)} \sum_{k=0}^{n+p} \int_0^{1+l} \Big(\frac{[k]_q}{[n+1]_q} + \frac{1+(q-1)[k]_q}{[n+1]_q}t - x) \Big) \\
\times\ & \begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k [1+l-x]_q^{n+p-k} d_q t + \frac{\omega(f',\delta)}{\delta} \frac{1}{[1+l]_q^{n+p}(1+l)} \\
\times\ & \sum_{k=0}^{n+p} \int_0^{1+l} \Big(\frac{[k]_q}{[n+1]_q} + \frac{1+(q-1)[k]_q}{[n+1]_q}t - x \Big)^2 \begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k [1+l-x]_q^{n+p-k} d_q t
\end{aligned}$$

$$\begin{aligned}
\leq\ & M\gamma_{n,q}(x) + \frac{\omega(f',\delta)}{[1+l]_q^{n+p}(1+l)} \Big(\sum_{k=0}^{n+p} \int_0^{1+l} \Big(\frac{[k]_q}{[n+1]_q} + \frac{1+(q-1)[k]_q}{[n+1]_q}t - x) \Big)^2 \\
\times\ & \frac{1}{[1+l]_q^{n+p}(1+l)} \begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k [1+l-x]_q^{n+p-k} d_q t \Big)^{\frac{1}{2}} \frac{\omega(f',\delta)}{\delta} \frac{1}{[1+l]_q^{n+p}(1+l)} \\
\times\ & \sum_{k=0}^{n+p} \int_0^{1+l} \Big(\frac{[k]_q}{[n+1]_q} + \frac{1+(q-1)[k]_q}{[n+1]_q}t - x \Big)^2 \begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k [1+l-x]_q^{n+p-k} d_q t \\
=\ & M\gamma_{n,q}(x) + \omega(f',\delta)\sqrt{T_{n,p}^l((t-x)^2;q;x)} + \frac{\omega(f',\delta)}{\delta} T_{n,p}^l((t-x)^2;q;x) \\
=\ & M\gamma_{n,q}(x) + \omega(f',\delta)(1+\sqrt{\delta_{n,q}^p(x)}),
\end{aligned}$$

where

$$\begin{aligned}
\delta\ =\ & \delta_{n,q}^p(x) \\
=\ & \frac{(1+l)^2}{[n+1]_q^2(1+l)[3]_q} + \frac{1}{[n+1]_q^2(1+l)} \Big(1 + \frac{(1+l)((3+l)q^3+(3+l)q^2+(1-l)q-l-1)}{[2]_q[3]_q} \\
\times\ & \frac{[n+p]_q}{[1+l]_q} \Big) \Big(x + \frac{q[n+p-1]_q}{[1+l]_q} x^2 \Big) - \frac{2}{[n+1]_q[2]_q} \Big((1+l)x + ((2+l)q-l) \frac{[n+p]_q}{[1+l]_q} x^2 \Big) + x^2.
\end{aligned}$$

Eventually, we obtain

$$|T_{n,p}^l(f;q;x) - f(x)| \leq M\gamma_{n,q}(x) + \omega(f';\delta)(1+\sqrt{\delta_{n,q}^p(x)},$$

which completes the proof.

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