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FIXED POINT THEOREMS FOR CYCLIC CONTRACTIONS ON CONE DISLOCATED QUASI-b-METRIC SPACES WITH BANACH ALGEBRAS

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Abstract. In this paper, we introduce the concept of cone dislocated quasi-b-metric spaces with Banach algebras and establish some fixed point theorems of cyclic contraction mappings under some natural conditions. As consequences, we extend some known fixed point results in the literature.

Keywords. Cone dislocated quasi-b-metric space; Banach algebra; Fixed point; Cyclic contraction.

1. Introduction-Preliminaries

In 1989, Bakhtin [1] introduced b-metric space as a generalization of metric space. Since then, more other generalized b-metric spaces such as quasi-b-metric spaces [2], b-metric-like spaces [3] and quasi-b-metric-like spaces [4] were introduced. A large number of papers on fixed point results in the setting of b-metric spaces have appeared, we refer the reader to [5-13] and the references mentioned therein.

The notion of dislocated (metric-like) metric spaces was introduced by Hitzler and seda [14] in 2000 as a generalization of a metric space. Zeyada *et al.* [15] generalized the results of [14] and introduced the concept of dislocated quasi-metric spaces.

Recently, Klin-eam and Suanoom [16] introduced the concept of dislocated quasi-b-metric spaces which generalized the concept of quasi-b-metric spaces and b-metric spaces, and verified some fixed point theorems in such spaces.

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Definition 1.1. [16] Let X be a non-empty set. Suppose that the mapping $d : X \times X \to [0,\infty)$ such that constant $s \geq 1$ satisfies the following conditions:

(i) $d(x,y) = d(y,x) = 0$ implies $x = y$ for all $x,y \in X$;

(ii) $d(x,y) \leq s[d(x,z) + d(z,y)]$ for all $x,y,z \in X$.

The pair (X,d) is called a dislocated quasi-b-metric space. The number s is called the coefficient of (X,d).

Remark 1.2. It is obvious that b-metric spaces and quasi-b-metric spaces are dislocated quasi-b-metric spaces, but the converse is not true.

In [17], Huang and Zhang introduced cone metric spaces in which is a generalization of metric spaces. After that, based on the work of [17], more and more fixed point results of some mappings with certain contractive property on cone metric spaces appeared, for example, see [18-21]. In [22], in order to generalized the Banach contraction principle to a more general form, Liu and Xu introduced the concept of cone metric spaces over Banach algebras by replacing Banach spaces with Banach algebras.

Let $\mathscr{A}$ always be a real Banach algebra, that is, $\mathscr{A}$ is a real Banach space in which an operator of multiplication is defined, subject to the following properties (for all $x,y,z \in \mathscr{A}$, $\alpha \in \mathbb{R}$) :

(1) $(xy)z = x(yz)$;

(2) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;

(3) $\alpha(xy) = (\alpha x)y = x(\alpha y)$;

(4) $\|xy\| \leq \|x\|\|y\|$.

In the following, we assume that a Banach algebra has a unit (*i.e.*, a multiplicative identity) e such that $ex = xe = x$ for all $x \in \mathscr{A}$. An element $x \in \mathscr{A}$ is said to be invertible if there is an inverse element $y \in \mathscr{A}$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}. For more details, we refer the reader to [23].

Proposition 1.3. [23] *Let $\mathscr{A}$ be a Banach algebra with a unit e, and $x \in \mathscr{A}$. If the spectral radius $\rho(x)$ of x is less than 1, i.e.,*

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|x^n\|^{\frac{1}{n}} < 1,$$

then $e - x$ is invertible. Actually,

$$(e-x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

Lemma 1.4. [24] *Let $x,y \in \mathscr{A}$. If x and y commute, then the spectral radius ρ satisfies the following properties:*

(i) $\rho(xy) \leq \rho(x)\rho(y)$;
(ii) $\rho(x+y) \leq \rho(x)+\rho(y)$;
(iii) $|\rho(x)-\rho(y)| \leq \rho(x-y)$.

A non-empty closed subset P of a Banach algebra $\mathscr{A}$ is called a cone if

(1) $\{\theta, e\} \subset P$;
(2) $\alpha P+\beta P \subset P$ for all non-negative real numbers α, β;
(3) $P^2 = PP \subset P$;
(4) $P \cap (-P) = \{\theta\}$,

where θ denotes the null of the Banach algebra $\mathscr{A}$.

For a given cone $P \subset \mathscr{A}$, a partial ordering '$\preceq$' with respect to P can be defined by $x \preceq y$ if and only if $y-x \in P$. $x \prec y$ will stand for $x \preceq y$ and $x \neq y$, while $x \ll y$ stands for $y-x \in int(P)$, where $int(P)$ denotes the interior of P. P is called a solid cone if $int(P) \neq \emptyset$.

Definition 1.5. [17] Let X be a non-empty set and $\mathscr{A}$ be a real Banach algebra. Assume that the mapping $d : X \times X \to \mathscr{A}$ satisfies :

1. $\theta \preceq d(x,y)$ for all $x,y \in X$, and $d(x,y) = d(y,x) = \theta$ if and only if $x = y$;
2. $d(x,y) = d(y,x)$ for all $x,y \in X$;
3. $d(x,y) \preceq d(x,z)+d(z,y)$ for all $x,y,z \in X$.

Then d is called a cone metric on X, and (X,d) is called a cone metric space over the Banach algebra $\mathscr{A}$.

See [17] for some examples of cone metric spaces over Banach algebras.

In this paper, motivated by the above works, we replace the Banach space by a Banach algebra and introduce cone dislocated quasi-b-metric space over the Banach algebras. In this way, we shall prove some fixed point theorems of cyclic contraction mappings under some natural conditions. Our main theorem extends existing results in the recent literature. Finally, as an application of our results, we give a concrete example.

Definition 1.6. Let X be a non-empty set and $\mathscr{A}$ be a real Banach algebra. Assume that the mapping $d : X \times X \to \mathscr{A}$ such that $s \in P$ (cone $P \subset \mathscr{A}$) with $\rho(s) \geq 1$ satisfies the following conditions :

(1) $\theta \preceq d(x,y)$, and $d(x,y) = d(y,x) = \theta$ implies $x = y$, for all $x, y \in X$;
(2) $d(x,y) \preceq s[d(x,z) + d(z,y)]$ for all $x, y, z \in X$.

Then d is called a cone dislocated quasi-b-metric on X, and (X,d) is called a cone dislocated quasi-b-metric space over the Banach algebra $\mathscr{A}$.

Now we give an example as follows:

Example 1.7. Let $\mathscr{A} = \mathbb{R}^2$. For each $x = (x_1, x_2) \in \mathscr{A}$, $\|x\| = \|(x_1, x_2)\| = |x_1| + |x_2|$. The multiplication is defined by

$$xy = (x_1, x_2)(y_1, y_2) = (x_1 y_1, x_1 y_2 + x_2 y_1).$$

Then $\mathscr{A}$ is a Banach algebra with unit $e = (1,0)$. Let $P = \{(x_1, x_2) \in \mathbb{R}^2 | x_1, x_2 \geq 0\}$. Then $P \subset \mathscr{A}$ is normal with a normal constant $M = 1$.

Let $X = \mathbb{R}^2$ and let

$$\begin{aligned} d(x,y) &= d((x_1,x_2),(y_1,y_2)) \\ &= (|x_1 - y_1|^2 + m|x_1|^2 + n|y_1|^2, |x_2 - y_2|^2 + m|x_2|^2 + n|y_2|^2) \in \mathscr{A}, \end{aligned}$$

where $m, n \geq 0$ with $m \neq n$. Then (X,d) is a cone dislocated quasi-b-metric space with the coefficient $s = (2,0) \in P$. Indeed, let $(x_1,x_2), (y_1,y_2), (z_1,z_2) \in X$. If

$$d((x_1,x_2),(y_1,y_2)) = d((y_1,y_2),(x_1,x_2)) = \theta,$$

then

$$|x_1 - y_1|^2 + m|x_1|^2 + n|y_1|^2 = 0, \quad |x_2 - y_2|^2 + m|x_2|^2 + n|y_2|^2 = 0.$$

It implies that $|x_1-y_1|^2=0$ and $|x_2-y_2|^2=0$. So $x_1=y_1$ and $x_2=y_2$, that is $x=y$. Moreover, we have

$$\begin{aligned} d(x,y) &= (|x_1-y_1|^2+m|x_1|^2+n|y_1|^2, |x_2-y_2|^2+m|x_2|^2+n|y_2|^2) \\ &\preceq ((|x_1-z_1|+|z_1-y_1|)^2+m(|x_1|+|z_1|)^2+n(|y_1|+|z_1|)^2, \\ &\quad (|x_2-z_2|+|z_2-y_2|)^2+m(|x_2|+|z_2|)^2+n(|y_2|+|z_2|)^2) \\ &\preceq (2(|x_1-z_1|^2+|z_1-y_1|^2)+2m(|x_1|^2+|z_1|^2)+2n(|y_1|^2+|z_1|^2), \\ &\quad (2(|x_2-z_2|^2+|z_2-y_2|^2)+2m(|x_2|^2+|z_2|^2)+2n(|y_2|^2+|z_2|^2) \\ &= (2,0)[(|x_1-z_1|^2+m|x_1|^2+n|z_1|^2, |x_2-z_2|^2+m|x_2|^2+n|z_2|^2) \\ &\quad +(|z_1-y_1|^2+m|z_1|^2+n|y_1|^2, |z_2-y_2|^2+m|z_2|^2+n|y_2|^2)] \\ &= (2,0)[d(x,z)+d(z,y)]. \end{aligned}$$

Then (X,d) is a cone dislocated quasi-b-metric space over the Banach algebra $\mathscr{A}$ with the efficient $s=(2,0)$.

According to Zoto *et al*. [25], we have

Definition 1.8. Let (X,d) be a cone dislocated quasi-b-metric space over the Banach algebra $\mathscr{A}$, $x\in X$ and $\{x_n\}$ a sequence in X. Then

1. $\{x_n\}$ converges to x whenever for each $c\in\mathscr{A}$ with $\theta\ll c$, there is a natural number N such that $d(x_n,x)\ll c$ for all $n>N$. We denote this by $\lim_{n\to\infty}x_n=x$ or $x_n\to x$.
2. $\{x_n\}$ is a Cauchy sequence if for each $c\in\mathscr{A}$ with $\theta\ll c$, there is a natural number N such that $d(x_n,x_m)\ll c$ for all $n,m>N$.
3. (X,d) is complete if every Cauchy sequence is convergent.

2. Main results

In this section, we prove some fixed point theorems of cyclic contraction mappings in the setting of a cone dislocated quasi-b-metric spaces with Banach algebras.

Let A and B be nonempty subsets of a metric space (X,d), $T:A\cup B\to A\cup B$. T is called a cyclic map iff $T(A)\subset B$ and $T(B)\subset A$.

Lemma 2.1. [26] *If E is a real Banach space with a solid cone P and if $\|x_n\|\to 0$ ($n\to\infty$), then for any $\theta\ll c$, there exists $N\in\mathbb{N}$ such that, for any $n>N$, we have $x_n\ll c$.*

Theorem 2.2. *Let A and B be nonempty subsets of a complete cone dislocated quasi-b-metric space (X,d) over the Banach algebra $\mathscr{A}$ and let P be the underlying solid cone. Assume that the cyclic mapping $T : A\cup B \to A\cup B$ satisfies the following contraction condition*

$$d(Tx,Ty) \preceq kd(x,y), \quad \forall x\in A,\ y\in B, \quad \text{or} \quad \forall x\in B,\ y\in A, \tag{2.1}$$

where $k\in P$ with $\rho(k) < \frac{1}{\rho(s)}$, $s\in P$ is as in Definition 1.6, s and k commute. Then T has a unique fixed point in $A\cap B$.

Proof. Let $x\in A$ (fix). By using the condition (2.1), we have

$$d(T^2x,Tx) = d(T(Tx),Tx) \preceq kd(Tx,x)$$

and

$$d(Tx,T^2x) = d(Tx,T(Tx)) \preceq kd(x,Tx).$$

Thus,

$$d(T^2x,Tx) \preceq kD, \quad d(Tx,T^2x) \preceq kD, \tag{2.2}$$

where $D = d(Tx,x)\vee d(x,Tx)\in\mathscr{A}$. Inductively, by using (2.1) and (2.2), we obtain

$$d(T^{n+1}x,T^nx) \preceq k^nD, \quad d(T^nx,T^{n+1}x) \preceq k^nD. \tag{2.3}$$

Let $n,m\in\mathbb{N}$ with $m>n$, by using the Definition 1.6 and (2.1), we have

$$\begin{aligned}
d(T^nx,T^mx) &\preceq s[d(T^nx,T^{n+1}x)+d(T^{n+1}x,T^mx)]\\
&\preceq sd(T^nx,T^{n+1}x)+s^2d(T^{n+1}x,T^{n+2}x)+\cdots+s^{m-n}d(T^{m-1}x,T^mx)\\
&\preceq sd(T^nx,T^{n+1}x)+s^2kd(T^nx,T^{n+1}x)+\cdots+s^{m-n}k^{m-n-1}d(T^nx,T^{n+1}x)\\
&\preceq (sk^n+s^2k^{n+1}+\cdots+s^{m-n}k^{m-1})D \preceq (s^nk^n+s^{n+1}k^{n+1}+\cdots+s^{m-1}k^{m-1})D\\
&= [(sk)^n+(sk)^{n+1}+\cdots+(sk)^{m-1})D\\
&\preceq \left(\sum_{i=0}^{\infty}(sk)^i\right)(sk)^nD = (e-sk)^{-1}(sk)^nD.
\end{aligned} \tag{2.4}$$

By Lemma 1.4 and condition $\rho(k) < \frac{1}{\rho(s)} \le 1$, we have $\rho(sk) \le \rho(s)\rho(k) < 1$. Thus

$$\|(sk)^n\| \to 0 \quad n\to\infty, \quad \text{and} \quad \|(e-sk)^{-1}\| < \infty. \tag{2.5}$$

By Lemma 2.1 and the fact that $\|(e-sk)^{-1}(sk)^n D\| \to 0$ ($n \to \infty$) (by (2.5)), it follows that for any $c \in \mathscr{A}$ with $\theta \ll c$, there exists $N_1 \in \mathbb{N}$ such that, for any $m > n > N_1$, we have

$$d(T^n x, T^m x) \preceq (e-sk)^{-1}(sk)^n D \ll c. \tag{2.6}$$

Similarly, there exists $N_2 \in \mathbb{N}$ such that, for any $m > n > N_2$, we obtain

$$d(T^m x, T^n x) \preceq (e-sk)^{-1}(sk)^n D \ll c. \tag{2.7}$$

Combine (2.6) and (2.7), we know that the sequence $\{x_n\} = \{T^n x\}$ is a Cauchy sequence. By the completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. We note that $\{x_{2n}\}$ is a sequence in A and $\{x_{2n+1}\}$ is a sequence in B, which both sequence tend to the same limit x^*. Since A and B are closed, we have $x^* \in A \cap B$, and then $A \cap B \neq \emptyset$. We now show that $Tx^* = x^*$. From Definition 1.6 and (2.1), we get

$$\begin{aligned} d(Tx^*, x^*) &\preceq s[d(Tx^*, Tx_n) + d(Tx_n, x^*)] \\ &\preceq s[kd(x^*, x_n) + d(x_{n+1}, x^*)]. \end{aligned} \tag{2.8}$$

Since

$$\|s[kd(x^*, x_n) + d(x_{n+1}, x^*)]\| \leq \|s\|(\|k\|\|d(x^*, x_n)\| + \|d(x_{n+1}, x^*)\|) \to 0, \tag{2.9}$$

as $n \to \infty$, we have by (2.8) and (2.9) that for any $c \in \mathscr{A}$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that, for any $n > N$ such that

$$d(Tx^*, x^*) \ll c,$$

which implies that $d(Tx^*, x^*) = \theta$. Similarly, we can get that $d(x^*, Tx^*) = \theta$. Hence,

$$d(Tx^*, x^*) = d(x^*, Tx^*) = \theta \Longrightarrow Tx^* = x^*.$$

So, x^* is a fixed point of T. If there exists another fixed point $y^* \in A \cap B$ of T, then

$$d(x^*, y^*) = d(Tx^*, Ty^*) \preceq kd(x^*, y^*),$$

which implies that

$$(e-k)d(x^*, y^*) \preceq \theta.$$

Multiplying both sides above by

$$(e-k)^{-1} = \sum_{i=0}^{\infty} k^i \succeq \theta, \tag{2.10}$$

we obtain $d(x^*,y^*) \preceq \theta$. Hence $d(x^*,y^*)=\theta$. Similarly, we can obtain $d(y^*,x^*)=\theta$. So, we have $x^*=y^*$. The proof is completed.

If (X,d) is a cone metric metric spaces with Banach algebras (see Definition 1.5), then $s=e$ (s as in Definition 1.6) and $\rho(s)=1$. By Theorem 2.2, we have

Corollary 2.3. *Let A and B be nonempty subsets of a complete cone metric space (X,d) and let P be the underlying solid cone. Assume that the cyclic mapping $T: A\cup B\to A\cup B$ satisfies the condition*

$$d(Tx,Ty)\preceq kd(x,y), \quad \forall x\in A,\ y\in B, \quad \text{or} \quad \forall x\in B,\ y\in A,$$

where $k\in\mathscr{A}$ with $\rho(k)<1$. Then T has a unique fixed point in $A\cap B$.

Theorem 2.4. *Let A and B be nonempty subsets of a complete cone dislocated quasi-b-metric space (X,d) over the Banach algebra $\mathscr{A}$ and let P be the underlying solid cone. Assume that the cyclic mapping $T: A\cup B\to A\cup B$ satisfies the condition*

$$d(Tx,Ty)\preceq k(d(x,Tx)+d(y,Ty)), \quad \forall x\in A,\ y\in B, \quad \text{or} \quad \forall x\in B,\ y\in A, \tag{2.11}$$

where $k\in P$ with $\rho(k)<\frac{1}{2\rho(s)}$, $s\in P$ is as in Definition 1.6, s and k commute. Then T has a unique fixed point in $A\cap B$.

Proof. Let $x\in A$ (fix), from (2.11), we have

$$\begin{aligned} d(Tx,T^2x) &= d(Tx,T(Tx)) \\ &\preceq k(d(x,Tx)+d(Tx,T^2x)), \end{aligned}$$

which implies that

$$d(Tx,T^2x)\preceq (e-k)^{-1}kd(x,Tx). \tag{2.12}$$

Moreover, by (2.11) and (2.12), and note that $(e-k)^{-1}$ and k commute, we obtain

$$\begin{aligned} d(T^2x,Tx) &= d(T(T(x)),Tx) \\ &\preceq k(d(Tx,T^2x)+d(x,Tx)) \\ &\preceq k[k(e-k)^{-1}d(x,Tx)+(e-k)(e-k)^{-1}d(x,Tx)] \\ &= k(e-k)^{-1}d(x,Tx). \end{aligned} \tag{2.13}$$

By using (2.11), (2.12) and (2.13), we get

$$d(T^2x,T^3x)\preceq (k(e-k)^{-1})^2 d(x,Tx)$$

and

$$d(T^3x,T^2x)\preceq (k(e-k)^{-1})^2 d(x,Tx).$$

Inductively, for all $n\in\mathbb{N}$, we have

$$d(T^{n+1}x,T^nx)\preceq (k(e-k)^{-1})^n d(x,Tx) \tag{2.14}$$

and

$$d(T^nx,T^{n+1}x)\preceq (k(e-k)^{-1})^n d(x,Tx). \tag{2.15}$$

Let $n,m\in\mathbb{N}$ with $m>n$, by using the definition 1.6, (2.11) and (2.14), we have

$$\begin{aligned}
d(T^mx,T^nx) &\preceq s[d(T^mx,T^{n+1}x)+d(T^{n+1}x,T^nx)]\\
&\preceq sd(T^{n+1}x,T^nx)+s^2d(T^{n+2}x,T^{n+1}x)+\cdots+s^{m-n}d(T^mx,T^{m-1}x)\\
&\preceq sd(T^{n+1}x,T^nx)+s^2kd(T^{n+1}x,T^nx)+\cdots+s^{m-n}k^{m-n-1}d(T^{n+1}x,T^nx)\\
&\preceq s[1+sk+\cdots+(sk)^{m-n-1}](k(e-k)^{-1})^n d(x,Tx)\\
&\preceq s\sum_{i=0}^{\infty}(sk)^i(k(e-k)^{-1})^n d(x,Tx)\\
&= s(e-sk)^{-1}(k(e-k)^{-1})^n d(x,Tx).
\end{aligned} \tag{2.16}$$

Let n be large enough such that

$$\|k^{n+i}\|\le \beta^{n+i}, \quad \forall i\ge 0,$$

where $\beta\in\mathbb{R}$ such that

$$\rho(k)=\lim_{n\to\infty}\|k^n\|^{\frac{1}{n}}<\beta<\frac{1}{2\rho(s)}\le\frac{1}{2}.$$

Set

$$(e-k)^{-n}=\left(\sum_{i=0}^{\infty}k^i\right)^n=\sum_{i=0}^{\infty}\alpha_i^{(n)}k^i,$$

where $\alpha_i^{(n)} \in \mathbb{R}$, $n, i \geq 0$. It is easy to see that $\alpha_i^{(n)} \geq 0$ for all $n, i \geq 0$. Noting that k and $(e-k)^{-1}$ commute, we obtain

$$\begin{aligned} \|(k(e-k)^{-1})^n\| = \|k^n(e-k)^{-n}\| &= \left\| \sum_{i=0}^{\infty} \alpha_i^{(n)} k^{n+i} \right\| \\ &\leq \sum_{i=0}^{\infty} \alpha_i^{(n)} \|k^{n+i}\| \leq \sum_{i=0}^{\infty} \alpha_i^{(n)} \beta^{n+i} \\ &= \beta^n \left(\sum_{i=0}^{\infty} \beta^i \right)^n \\ &= \left(\frac{\beta}{1-\beta} \right)^n \to 0, \quad n \to \infty. \end{aligned} \tag{2.17}$$

From (2.17) and (2.5), we have

$$\|s(e-sk)^{-1}(k(e-k)^{-1})^n d(x,Tx)\| \to 0, \quad n \to \infty. \tag{2.18}$$

By Lemma 2.1 and (2.18), it follows that for any $c \in \mathscr{A}$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that, for any $m > n > N$, we have

$$d(T^m x, T^n x) \ll c.$$

So, $d(T^m x, T^n x) \to \theta$. Similarly, we have $d(T^n x, T^m x) \to \theta$. Hence $\{T^n x\}$ is a Cauchy sequence. Since (X,d) is complete, we have $\{T^n x\}$ converges to some $u \in X$. Noting that $\{T^{2n} x\}$ is a sequence in A, $\{T^{2n-1} x\}$ is a sequence in B, and A and B are closed, we have $u \in A \cap B$, and then $A \cap B \neq \emptyset$. In the following, we will show that $Tu = u$. By (2.11) and (2.15), we get

$$\begin{aligned} d(u,Tu) &\preceq s[d(u,T^{n+1}x) + d(T^{n+1}x,Tu)] \\ &\preceq sd(u,T^{n+1}x) + sk(d(T^n x,T^{n+1}x) + d(u,Tu)) \\ &\preceq sd(u,T^{n+1}x) + skd(u,Tu) + sk(k(e-k)^{-1})^n d(x,Tx). \end{aligned}$$

That is,

$$(e-sk)d(u,Tu) \preceq sd(u,T^{n+1}x) + sk(k(e-k)^{-1})^n d(x,Tx). \tag{2.19}$$

Since

$$\begin{aligned} &\|(e-sk)^{-1}[sd(u,T^{n+1}x) + sk(k(e-k)^{-1})^n d(x,Tx)]\| \\ &\quad \leq \|(e-sk)^{-1}\| [\|s\| \|d(u,T^{n+1}x)\| + \|s\| \|k\| \|(k(e-k)^{-1})^n\| \|d(x,Tx)\|] \to 0, \end{aligned} \tag{2.20}$$

as $n \to \infty$, we obtain by (2.19) and (2.20) that for any $c \in \mathscr{A}$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that, for any $n > N$ such that

$$d(u,Tu) \ll c,$$

which implies that $d(u,Tu) = \theta$. On the other hand, we have

$$\begin{aligned} d(Tu,u) &\preceq s[d(Tu,T^{n+1}x) + d(T^{n+1}x,u)] \\ &\preceq sk(d(u,Tu) + d(T^n x, T^{n+1}x)) + sd(T^{n+1}x,u) \qquad (2.21) \\ &\preceq sk(k(e-k)^{-1})^n d(x,Tx) + sd(T^{n+1}x,u). \end{aligned}$$

Since

$$\begin{aligned} &\|sk(k(e-k)^{-1})^n d(x,Tx) + sd(T^{n+1}x,u)\| \\ &\quad \leq \|s\|\|k\|\|(k(e-k)^{-1})^n\|\|d(x,Tx)\| + \|s\|\|d(T^{n+1}x,u)\| \to 0, \quad n \to \infty, \end{aligned}$$

we have by Lemma 2.1 and (2.21) that $d(Tu,u) = \theta$. Hence, we get

$$d(u,Tu) = d(Tu,u) = \theta \Longrightarrow Tu = u.$$

So, u is a fixed point of T. Now if v is another fixed point of T, then

$$d(u,v) = d(Tu,Tv) \preceq k(d(u,Tu) + d(v,Tv)) = \theta,$$

and

$$d(v,u) = d(Tv,Tu) \preceq k(d(v,Tv) + d(u,Tu)) = \theta.$$

So, we get that $d(u,v) = d(v,u) = \theta \Longrightarrow u = v$. Therefore, the fixed point is unique. This completes the proof.

If (X,d) is a cone metric metric spaces with Banach algebras (see Definition 1.6), then $s = e$ (s is as in Definition 1.6) and $\rho(s) = 1$. By Theorem 2.4, we have

Corollary 2.5. *Let A and B be nonempty subsets of a complete cone metric space (X,d) and let P be the underlying solid cone. Assume that the cyclic mapping $T : A \cup B \to A \cup B$ satisfies the condition*

$$d(Tx,Ty) \preceq k(d(x,Tx) + d(y,Ty)), \quad \forall x \in A,\ y \in B, \quad \text{or} \quad \forall x \in B,\ y \in A,$$

where $k \in P$ with $\rho(k) < \frac{1}{2}$. Then T has a unique fixed point in $A \cap B$.

Remark 2.6. Obviously, Theorem 2.9 in [16] is a special case of Theorem 2.2, and Theorem 2.12 in [16] is a special case of Theorem 2.4. Corollaries 2.3 and 2.5 extends the some corresponding results of [27, 28].

The following example illustrates Theorem 2.2.

Example 2.7. Let a cone dislocated quasi-b-metric space (X,d) be as in Example 1.7 with $m=\frac{1}{5}$ and $n=\frac{1}{6}$. Let $A=(-\infty,0]\times(-\infty,0]$ and $B=[0,\infty)\times[0,\infty)$, and let $T:A\cup B\to A\cup B$ defined by

$$T(x_1,x_2)=\left(-\frac{\sqrt{2}x_1}{3},-\frac{x_2}{3}-\alpha x_1\right),$$

where α can be any large positive real number. Obviously, if $x\in A$, then $Tx\in B$, and if $x\in B$, then $Tx\in A$. Thus the map T is cyclic on X because $T(A)\subset B$ and $T(B)\subset A$.

From Example 1.7, we have

$$\begin{aligned}
&d(T(x_1,x_2),T(y_1,y_2))\\
&=\left(\left|\frac{-\sqrt{2}x_1}{3}-\frac{-\sqrt{2}y_1}{3}\right|^2+\frac{|\frac{\sqrt{2}x_1}{\sqrt{3}}|^2}{5}+\frac{|\frac{\sqrt{2}y_1|}{3}|^2}{6},\right.\\
&\left.\left|\left(\frac{-x_2}{3}-\alpha x_1\right)-\left(\frac{-y_2}{3}-\alpha y_1\right)\right|^2+\frac{|\frac{x_2}{3}+\alpha x_1|^2}{5}+\frac{\frac{y_2}{3}+\alpha y_1|^2}{6}\right)\\
&\preceq\left(\frac{2}{9}|x_1-y_1|^2+\frac{2}{45}|x_1|^2+\frac{2}{54}|y_1|^2,\right.\\
&\left.2\left(\frac{1}{9}|x_2-y_2|^2+\alpha^2|x_1-y_1|^2\right)+\frac{2}{5}\left(\frac{|x_2|^2}{9}+\alpha^2|x_1|^2\right)+\frac{2}{6}\left(\frac{|y_2|^2}{9}+\alpha^2|y_1|^2\right)\right)\\
&=\left(\frac{2}{9},2\alpha^2\right)\left(|x_1-y_1|^2+\frac{|x_1|^2}{5}+\frac{|y_1|^2}{6},|x_2-y_2|^2+\frac{|x_2|^2}{5}+\frac{|y_2|^2}{6}\right)\\
&=kd((x_1,x_2),(y_1,y_2)),
\end{aligned}$$

where $k=\left(\frac{2}{9},2\alpha^2\right)\in P$. Since

$$\left\|\left(\frac{2}{9},2\alpha^2\right)^n\right\|^{\frac{1}{n}}=\left\|\left(\left(\frac{2}{9}\right)^n,2\alpha^2 n\left(\frac{2}{9}\right)^{n-1}\right)\right\|^{\frac{1}{n}}\to\frac{2}{9}<\frac{1}{2}=\frac{1}{\rho(s)},$$

as $n\to\infty$, that is $\rho(k)<\frac{1}{\rho(s)}$, by Theorem 2.2, we have that 0 is the unique fixed point of T.

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References

[1] I. Bakhtin, The contraction principle in quasimetric spaces, Funct. Anal. 30 (1989), 26-37.

[2] M. Shah, N. Hussain, Nonlinear contractions in partially ordered quasi b-metric spaces, Commun. Korean Math. Soc. 27 (2012), 117-128.

[3] M. Alghamdi, N. Hussain, P. Salimi, Fixed point and coupled fixed point theorems on b-metric-like spaces, J. Inequal. Appl. 2013 (2013), 402.

[4] C. Zhu, C. Chen, X. Zhang, Some results in quasi-b-metric-like spaces, J. Inequal. Appl. 2014 (2014), 437.

[5] H. Aydi, M. Bota, E. Karap.nar, S. Moradi, A common fixed point for weak φ-contractions on b-metric spaces, Fixed Point Theory 13 (2012), 337-346.

[6] M. Bota, E. Karapinar, O. Mlesnite, Ulam-Hyers stability results for fixed point problems via α-ψ-contractive mappings in b-metric spaces, Abstr. Appl. Anal. 2013 (2013), 825293.

[7] M. Kutbi, E. Karapinar, J. Ahmad, A. Azam, Some fixed point results for multi-valued mappings in b-metric spaces, J. Inequal. Appl. 2014 (2014), 126.

[8] J. Roshan, V. Parvaneh, I. Altun, Some coincidence point results in ordered b-metric spaces and applications in a system of integral equations, Appl. Math. Comput. 226 (2014), 725-737.

[9] H. Huang, S. Radenović, J. Vujaković, On some recent coincidence and immediate consequences in partially ordered b-metric spaces, Fixed Point Theory Appl. 2015 (2015), 63.

[10] S. Radenović, T. Dosnović, T. Aleksić-Lampert, Z. Golubović, A note on some recent fixed point results for cyclic contractions in b-metric spaces and an application to integral equations, Appl. Math. Comput. 273 (2016), 155-164.

[11] H. Huaping, S. Radenović, Common fixed point theorems of generalized Lipschitz mappings in cone b-metric spaces over Banach algebras and applications, J. Nonlinear Sci. Appl. 8 (2015), 787-799.

[12] H. Huaping, S. Radenović, Common fixed point theorems of generalized Lipschitz mappings in cone metric spaces over Banach algebras, Appl. Math. Inf. Sci. 9 (2015), 2983-2990.

[13] A. Azam, N. Mehmood, J. Ahmad, S. Radenović, Multivalued Fixed Point Theorems in Cone b-Metric Spaces, J. Inequal. Appl. 2013 (2013), 582.

[14] P. Hitzler, A. Seda, Dislocated topologies, J. Electr. Eng. 51 (2000), 3-7.

[15] F. Zeyada, G. Hassan, M. Ahmad, A generalization of fixed point theorem due to Hitzler and Seda in dislocated quasi-metric space, Arab. J. Sci. Eng. 31 (2005), 111-114.

[16] C. Klin-eam, C. Suanoom1, Dislocated quasi-b-metric spaces and fixed point theorems for cyclic contractions, Fixed Point Theory Appl. 2015 (2015), 74.

[17] L. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl. 332 (2007), 1468-1476.

[18] M. Abbas, B. Rhoades, T. Nazir, Common fixed points for four maps in cone metric spaces, Appl. Math. Comput. 216 (2010), 80-86.

[19] M. Khamsi, Remarks on cone metric spaces and fixed point theorems of contractive mappings, Fixed Point Theory Appl. 2010(2010), 315398.

[20] P. Proinov, A unified theory of cone metric spaces and its applications to the fixed point theory, Fixed Point Theory Appl. 2013 (2013), 103.

[21] W. Du, A note on cone metric fixed point theory and its equivalence, Nonlinear Anal. 72 (2010), 2259-2261.

[22] H. Liu, S. Xu, Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings, Fixed Point Theory Appl. 2013 (2013), 320.

[23] W. Rudin, Functional Analysis, 2nd edn. McGraw-Hill, New York (1991)

[24] S. Xu, S. Radenovic, Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebras without assumption of normality, Fixed Point Theory Appl. 2014 (2014), 102.

[25] K. Zoto, P. Kumari, E. Hoxha, Some fixed point theorems and cyclic contractions in dislocated and dislocated quasi-metric spaces, Am. J. Numer. Anal. 2 (2014), 79-84.

[26] S. Radenović, B. Rhoades, Fixed point theorem for two non-self mappings in cone metric spaces, Comput. Math. Appl. 57 (2009), 1701-1707.

[27] W. Kirk, P. Srinivasan, P. Veeramani, Fixed points for mapping satisfying cyclic contractive conditions, Fixed Point Theory 4 (2003), 79-89.

[28] E. Karapinar, I. Erhan, Best proximity on different type contractions, Appl. Math. Inf. Sci. 5 (2010), 558-569.