J. Nonlinear Funct. Anal. 2016 (2016), Article ID 31

MATHRES
Journal of Nonlinear Functional Analysis
Available online at http://jnfa.mathres.org

DHAGE ITERATION METHOD FOR NONLINEAR FIRST ORDER ORDINARY HYBRID DIFFERENTIAL EQUATIONS WITH MIXED PERTURBATION OF SECOND TYPE AND MAXIMA

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Abstract. In this paper, the author proves the existence as well as approximation of the solutions for an initial value problem of first order ordinary nonlinear hybrid differential equations with maxima. An algorithm for the solutions is developed and it is shown that certain sequence of successive approximations converges monotonically to the solution of the related hybrid differential equations under some suitable mixed hybrid conditions. We base our results on the Dhage iteration principle embodied in a recent hybrid fixed point theorem of Dhage (2014) in a partially ordered normed linear space. An example is also provided to illustrate the hypotheses and abstract theory developed in this paper.

Keywords. Hybrid differential equation; Initial value problem; Differential equations with maxima; Dhage iteration method; Existence and approximate solution.

1. Introduction

The significance of the differential equations with maxima lies in the real world problems of automatic regulation of the technical systems and such differential equations form a special class of functional differential equations in which the present state of the unknown function related to the systems depends upon the maximum value of the past state in some past interval of time. See Magomedov [23, 24] and the reerences therein. Again, Mishkis [25] pointed out

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Received May 1, 2016

the need to study the differential equations with maxima and since then several classes of ordinary and partial differential equations with maxima have been discussed in the literature for different qualitative aspects of the solutions. A few details on the topic appears in the monograph of Bainov and Hristova [1] and the research papers by Otrocol and Rus [21], Dhage and Otrocol [20] and the references therein. Similarly, hybrid differential equations are introduced in Dhage [4] to cover different dynamic systems of the real world problems. Therefore, the study of hybrid differential equations with maxima would definitely enrich the area of dynamic systems and applications. The class of hybrid differential equations involving three nonlinearities with mixed perturbation of second type may be found in the works of Dhage [2, 3] wherein such hybrid differential equations are discussed for different aspects of the solutions. See Dhage and Lakshmikantham [19], Dhage and Dhage [16], Dhage *et.al.* [18] and the references therein. The purpose of the present paper is to blend these two ideas together and discuss the hybrid differential equations with maxima for existence and numerical aspects of the solutions. It is well-known that the hybrid differential equations can be tackled using the Dhage iteration method embodied in the hybrid fixed point theory initiated by Dhage [2, 5, 7] which also yields the algorithms for the solutions. Therefore, it is of interest to establish algorithms for the hybrid differential equations with maxima for existence and approximation of the solutions along similar lines. The novelty of the present paper lies in the fact that our problem as well as our method is new to the literature in the theory of nonlinear differential equations with maxima.

2. Statement of the problem

Given a closed and bounded interval $J=[t_0,t_0+a]$ of the real line $\mathbb{R}$ for some $t_0,a\in\mathbb{R}$ with $t_0\geq 0, a>0$, consider the initial value problem (in short IVP) of first order ordinary nonlinear hybrid differential equation (in short HDE) with maxima viz.,

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)-h\big(t,x(t),X(t)\big)}{f\big(t,x(t),X(t)\big)}\right]+\lambda\left[\frac{x(t)-h\big(t,x(t),X(t)\big)}{f\big(t,x(t),X(t)\big)}\right] &= g\big(t,x(t),X(t)\big),\ t\in J,\\ x(t_0)=\alpha_0\in\mathbb{R}_+, \end{aligned}\right\} \tag{2.1}$$

for $\lambda \in \mathbb{R}$, $\lambda > 0$, where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ and $h, g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $X(t) = \max\limits_{t_0 \le \xi \le t} x(\xi)$ for $t \in J$.

By a *solution* of the HDE (2.1) we mean a function $x \in C(J, \mathbb{R})$ that satisfies

(i) $t \mapsto \dfrac{x(t) - h(t, x(t), X(t))}{f(t, x(t), X(t))}$ is differentiable function, and
(ii) x satisfies the equations in (2.1) on J,

where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J.

The HDE (2.1) is a mixed perturbation of second type of a nonlinear differential equation with maxima and includes the following important classes of hybrid differential equations with maxima as special cases.

1. When $f(t,x,y) = 1$ and $h(t,x,y) = 0$ for all $t \in J$ and $x, y \in \mathbb{R}$, the HDE (2.1)reduces to the following known nonlinear differential equation with maxima

$$\left.\begin{aligned} x'(t) + \lambda x(t) = g(t, x(t), X(t)),\ t \in J, \\ x(t_0) = \alpha_0 \in \mathbb{R}_+. \end{aligned}\right\} \tag{2.2}$$

The above nonlinear differential equation with maxima (2.2) has already been discussed in the literature for existence and uniqueness of the solutions via classical methods of Schauder and Banach fixed point principles. See Bainov and Hristova [1] and the references therein. Here our method is different and constructive.

2. Again, when $f(t,x,y) = 1$ for all $t \in J$ and $x, y \in \mathbb{R}$, the HDE (2.1)reduces to the following HDE without maxima,

$$\left.\begin{aligned} & \frac{d}{dt}\big[x(t) - h\big(t, x(t), X(t)\big)\big] \\ & \qquad + \lambda\,\big[x(t) - h\big(t, x(t), X(t)\big)\big] = g\big(t, x(t), X(t)\big),\ t \in J, \\ & x(t_0) = \alpha_0 \in \mathbb{R}, \end{aligned}\right\} \tag{2.3}$$

which is new but could also be discussed as in Dhage [10, 11] via Dhage iteration method and established the existence and approximation result under some mixed partial Lipschitz and partial compactness conditions.

3. If $h(t,x,y)=0$ for all $t\in J$ and $x,y\in\mathbb{R}$, then the HDE (2.1)reduces to the following HDE with maxima,

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{x(t)}{f\big(t,x(t),X(t)\big)}\right]+\lambda\left[\frac{x(t)}{f\big(t,x(t),X(t)\big)}\right] &= g\big(t,x(t),X(t)\big),\ t\in J,\\ x(t_0)=\alpha_0\in\mathbb{R}_+, \end{aligned}\right\} \tag{2.4}$$

which has been discussed in Dhage and Dhage [17] under same mixed hypotheses and arguments with appropriate modifications via Dhage iteration method for the existence and approximation result of positive solutions.

4. If we take $h(t,x,y)=0$, $f(t,x,y)=1$ and $g(t,x,y)=py+F(t)$ for all $t\in J$ and $x,y\in\mathbb{R}$ in the HDE (2.1), then it reduces to the standard linear differential equation of automatic regulation,

$$\left.\begin{aligned} x'(t)+\lambda x(t)=pX(t)+F(t),\ t\in J,\\ x(t_0)=\alpha_0\in\mathbb{R}_+, \end{aligned}\right\} \tag{2.5}$$

for all $t\in J$, where $\lambda>0$, $p>0$ are constants and $F:J\to\mathbb{R}$ is a continuous perturbation function. The differential equation with maxima (2.5) is the motivation for development of the subject of differential equations with maxima.

Again, the HDEs (2.3) and (2.4) include several other nonlinear differential equations studied in the literature as the special cases. Therefore, the existence and approximation result for the HDE (2.1)is more general which includes the existence and approximations results for all the above differential equations with maxima as special cases. In the following section we give some preliminaries which will be used in the subsequent sections of the paper.

3. Auxiliary results

Unless otherwise mentioned, throughout this paper that follows, let E denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm $\|\cdot\|$ in which the addition and the scalar multiplication by positive real numbers are preserved by $\preceq$. A few details of a partially ordered normed linear space appear in Dhage [5, 7], Heikkilä and Lakshmikantham [22] and the references therein.

Two elements x and y in E are said to be *comparable* if either the relation $x \preceq y$ or $y \preceq x$ holds. A non-empty subset C of E is called a *chain* or *totally ordered* if all the elements of C are comparable. It is known that E is *regular* if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Heikkilä and Lakshmikantham [22] and the references therein.

We need the following definitions in the sequel.

Definition 3.1. A mapping $\mathscr{T} : E \to E$ is called *isotone* or *nondecreasing* if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathscr{T}x \preceq \mathscr{T}y$ for all $x, y \in E$.

Definition 3.2. [7] A mapping $\mathscr{T} : E \to E$ is called *partially continuous* at a point $a \in E$ if for $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|\mathscr{T}x - \mathscr{T}a\| < \varepsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathscr{T}$ is called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathscr{T}$ is partially continuous on E, then it is continuous on every chain C contained in E.

Definition 3.3. [7] A non-empty subset S of the partially ordered Banach space E is called *partially bounded* if every chain C in S is bounded. An operator $\mathscr{T} : E \to E$ is called *partially bounded* if every chain C in $T(E)$ is bounded. $\mathscr{T}$ is called *uniformly partially bounded* if all chains C in $\mathscr{T}(E)$ are bounded by a unique constant. $\mathscr{T}$ is called *bounded* if $T(E)$ is a bounded subset of E.

Definition 3.4. [7] A non-empty subset S of the partially ordered Banach space E is called *partially compact* if every chain C in S is compact. An operator $\mathscr{T} : E \to E$ is called *partially compact* if every chain or totally ordered set C in $\mathscr{T}(E)$ is a relatively compact subset of E. $\mathscr{T}$ is called *uniformly partially compact* if $\mathscr{T}(E)$ is a uniformly partially bounded and partially compact on E. $\mathscr{T}$ is called *partially totally bounded* if for any bounded subset S of E, $\mathscr{T}(S)$ is a relatively compact subset of E. If $\mathscr{T}$ is partially continuous and partially totally bounded, then it is called *partially completely continuous* on E.

Remark 3.1. Suppose that $\mathscr{T}$ is a nondecreasing operator on E into itself. Then $\mathscr{T}$ is a partially bounded or partially compact if $\mathscr{T}(C)$ is a bounded or relatively compact subset of E for each chain C in E.

Definition 3.5. [3] The order relation $\preceq$ and the metric d on a non-empty set E are said to be *compatible* if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E,\preceq,\|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible. A subset S of E is called *Janhavi* if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are compatible in it. In particular, if $S = E$, then E is called a *Janhavi metric* or *Janhavi Banach space*.

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has compatibility property. Similarly, the finite dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property and so is a *Janhavi Banach space*.

Definition 3.6. [7] An upper semi-continuous and nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathscr{D}$-function provided $\psi(0) = 0$. Let $(E,\preceq,\|\cdot\|)$ be a partially ordered normed linear space. A mapping $\mathscr{T} : E \to E$ is called *partially nonlinear $\mathscr{D}$-Lipschitz* if there exists a $\mathscr{D}$-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|\mathscr{T}x - \mathscr{T}y\| \leq \psi(\|x-y\|)$$

for all comparable elements $x, y \in E$. If $\psi(r) = k\,r$, $k > 0$, then $\mathscr{T}$is called a partially Lipschitz with a Lipschitz constant k.

Let $(E,\preceq,\|\cdot\|)$ be a partially ordered normed linear algebra. Denote

$$E^+ = \big\{x \in E \mid x \succeq \theta, \text{ where } \theta \text{ is the zero element of } E\big\}$$

and

$$\mathscr{K} = \{E^+ \subset E \mid uv \in E^+ \text{ for all } u, v \in E^+\}.$$

The elements of the set $\mathscr{K}$ are called the positive vectors in E. The following lemma follows immediately from the definition of the set $\mathscr{K}$ which is often times used in the hybrid fixed point theory of Banach algebras and applications to nonlinear differential and integral equations.

Lemma 3.1. [3] *If $u_1,u_2,v_1,v_2 \in \mathscr{K}$ are such that $u_1 \preceq v_1$ and $u_2 \preceq v_2$, then $u_1u_2 \preceq v_1v_2$.*

Definition 3.7. An operator $\mathscr{T} : E \to E$ is said to be positive if the range $R(\mathscr{T})$ of $\mathscr{T}$ is such that $R(\mathscr{T}) \subseteq \mathscr{K}$.

The central idea of *Dhage iteration principle* or method (in short DIP or DIM) developed in Dhage [5, 6, 7, 8] may be described as "*the monotonic convergence of the sequence of successive approximations to the solutions of a nonlinear equation beginning with a lower or an upper solution of the equation as its initial or first approximation*." The above convergence principle forms a useful tool in the subject of existence theory of nonlinear analysis. It is known that the Dhage iteration method is different from other iterations methods and embodied in the following applicable hybrid fixed point theorem of Dhage [7] which is the key tool for our work contained in the present paper. A few other hybrid fixed point theorems containing the Dhage iteration method along with their applications appear in Dhage [7, 8, 9, 10] and Dhage and Dhage [12, 13, 14, 15, 16].

Theorem 3.1. [Dhage [5, 7]] *Let $\big(E,\preceq,\|\cdot\|\big)$ be a regular partially ordered complete normed linear algebra such that every compact chain C of E is Janhavi. Let $\mathscr{A},\mathscr{B} : E \to \mathscr{K}$ and $\mathscr{C} : E \to E$ be three nondecreasing operators such that*

(a) *$\mathscr{A}$ and $\mathscr{C}$ are partially bounded and partially nonlinear $\mathscr{D}$-Lipschitz with $\mathscr{D}$-functions $\psi_{\mathscr{A}}$ and $\psi_{\mathscr{C}}$ respectively,*

(b) *$\mathscr{B}$ is partially continuous and uniformly partially compact,*

(c) *$0 < M\psi_{\mathscr{A}}(r)+\psi_{\mathscr{C}}(r) < r$, $r>0$, where $M = \sup\{\|\mathscr{B}(C)\| : C$ is a chain in $E\}$, and*

(d) *there exists an element $\alpha_0 \in X$ such that $\alpha_0 \preceq \mathscr{A}\alpha_0\mathscr{B}\alpha_0+\mathscr{C}\alpha_0$ or $\alpha_0 \succeq \mathscr{A}\alpha_0\mathscr{B}\alpha_0+\mathscr{C}\alpha_0$.*

Then the operator equation

$$\mathscr{A}x\mathscr{B}x+\mathscr{C}x = x$$

has a solution x^* *and the sequence* $\{x_n\}$ *of successive iterations defined by* $x_{n+1} = \mathscr{A}x_n\mathscr{B}x_n + \mathscr{C}x_n$, $n = 0, 1, \ldots$; *converges monotonically to* x^*.

Remark 3.2. The condition that every compact chain C of E is Janhavi holds if every partially compact subset of E possesses the compatibility property with respect to the order relation $\preceq$ and the norm $\|\cdot\|$ in it.

Remark 3.3. We remark that hypothesis (a) of Theorem 3.1 implies that the operators $\mathscr{A}$ and $\mathscr{C}$ are partially continuous and consequently all the operators $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ in the theorem are partially continuous on E. The regularity of E in above Theorem 3.1 may be replaced with a stronger continuity condition of the operators $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ on E which is a result proved in Dhage [5, 6].

In the following section we prove our main existence and approximation result of this paper.

4. Main results

The HDE (2.1) is considered in the function space $C(J,\mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J,\mathbb{R})$ by

$$\|x\| = \sup_{t\in J} |x(t)| \tag{4.1}$$

and

$$x \leq y \iff x(t) \leq y(t) \tag{4.2}$$

for all $t \in J$ respectively. Clearly, $C(J,\mathbb{R})$ is a Banach algebra with respect to above supremum norm and is also partially ordered w.r.t. the above partially order relation $\leq$. It is known that the partially ordered Banach algebra $C(J,\mathbb{R})$ has some nice properties w.r.t. the above order relation in it. The following lemma follows by an application of Arzelá-Ascoli theorem.

Lemma 4.1. *Let* $\big(C(J,\mathbb{R}), \leq, \|\cdot\|\big)$ *be a partially ordered Banach space with the norm* $\|\cdot\|$ *and the order relation* $\leq$ *defined by (4.1) and (4.2) respectively. Then every partially compact subset* S *of* $C(J,\mathbb{R})$ *is Janhavi.*

Proof. The proof of the lemma is given in Dhage and Dhage [12, 13, 14, 15] and so we omit the details of it.

We need the following definition in what follows.

Definition 4.1. A function $u \in C(J,\mathbb{R})$ is said to be a lower solution of the HDE (2.1) if the function $t \mapsto \dfrac{x(t)-h(t,u(t),U(t))}{f(t,u(t),U(t))}$ is differentiable and satisfies

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{u(t)-h\big(t,u(t),U(t)\big))}{f\big(t,u(t),U(t)\big)}\right]+\lambda\left[\frac{u(t)-h\big(t,u(t),U(t)\big)}{f\big(t,u(t),U(t)\big)}\right] &\leq g\big(t,u(t),U(t)\big), \\ u(t_0) &\leq \alpha_0, \end{aligned}\right\}$$

for all $t \in J$, where $U(t) = \max\limits_{t_0 \leq \xi \leq t} u(\xi)$ for $t \in J$. Similarly, a function $v \in C(J,\mathbb{R})$ is said to be an upper solution of the HDE (2.1) if it satisfies the above property and inequalities with reverse sign.

We consider the following set of assumptions in what follows:

(A_0) The map $x \mapsto \dfrac{x-h(t,x,x)}{f(t,x,x)}$ is injection for each $t \in J$.

(A_1) f defines a function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$.

(A_2) There exists a constant $M_f > 0$ such that $0 < f(t,x,y) \leq M_f$ for all $t \in J$ and $x,y \in \mathbb{R}$.

(A_3) There exists a $\mathscr{D}$-function φ such that

$$0 \leq f(t,x_1,x_2) - f(t,y_1,y_2) \leq \varphi\big(\max\{x_1 - y_1\,,\, x_2 - y_2\}\big),$$

for all $t \in J$ and $x_1,x_2,y_1,y_2 \in \mathbb{R}$, $x_1 \geq y_1$ and $x_2 \geq y_2$.

(B_1) g defines a function $g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$.

(B_2) There exists a constant $M_g > 0$ such that $g(t,x,y) \leq M_g$ for all $t \in J$ and $x,y \in \mathbb{R}$.

(B_3) $g(t,x,y)$ is nondecreasing in x and y for all $t \in J$.

(C_1) The function h satisfies the inequality $\alpha_0 \geq h(t_0,\alpha_0,\alpha_0)$.

(C_2) There exists a constant $M_h > 0$ such that $|h(t,x,y)| \leq M_h$ for all $t \in J$ and $x,y \in \mathbb{R}$.

(C_3) There exists a $\mathscr{D}$-function φ such that

$$0 \leq h(t,x_1,x_2) - h(t,y_1,y_2) \leq \omega\big(\max\{x_1 - y_1\,,\, x_2 - y_2\}\big),$$

for all $t \in J$ and $x_1,x_2,y_1,y_2 \in \mathbb{R}$, $x_1 \geq y_1$ and $x_2 \geq y_2$.

(D_1) The HDE (2.1) has a lower solution $u \in C(J,\mathbb{R})$.

Remark 4.1. Notice that Hypothesis (A_0) holds in particular if the function $x \mapsto \dfrac{x-h(t,x,x)}{f(t,x,x)}$ is increasing for each $t \in J$.

Lemma 4.2. *Suppose that hypothesis (A_0) holds. Then a function $x \in C(J,\mathbb{R})$ is a solution of the HDE (2.1)if and only if it is a solution of the nonlinear hybrid integral equation (in short HIE),*

$$\begin{aligned} x(t) &= h\big(t,x(t),X(t)\big) \\ &\quad + [f(t,x(t),X(t))]\left(ce^{-\lambda t} + \int_{t_0}^{t} e^{-\lambda(t-s)} g(s,x(s),X(s))\, ds\right) \end{aligned} \tag{4.3}$$

for all $t \in J$, where $c = \dfrac{\big(\alpha_0 - h(t_0,\alpha_0,\alpha_0)\big)e^{\lambda t_0}}{f(t_0,\alpha_0,\alpha_0)}$. Moreover $c > 0$ if the hypothesis (C_1) holds.

Theorem 4.1. *Assume that hypotheses (A_0)-(A_3), (B_1)-(B_3), (C_1)-(C_3) and (D_1) hold. If*

$$\left(\frac{\alpha_0 - h(t_0,\alpha_0,\alpha_0)}{f(t_0,\alpha_0,\alpha_0)} + M_g\, a\right)\varphi(r) + \omega(r) < r,\ r > 0, \tag{4.4}$$

then the HDE (2.1) has a solution x^* defined on J and the sequence $\{x_n\}_{n=1}^{\infty}$ of successive approximations defined by

$$\begin{aligned} x_{n+1}(t) &= h\big(t,x_n(t),X_n(t)\big) \\ &\quad + [f(t,x_n(t),X_n(t))]\left(ce^{-\lambda t} + \int_{t_0}^{t} e^{-\lambda(t-s)} g(s,x_n(s),X_n(s))\, ds\right), \end{aligned} \tag{4.5}$$

where $x_1 = u$, converges monotonically to x^*.

Proof. Set $E = C(J,\mathbb{R})$. Then, by Lemma 4.1, every compact chain C in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ in E.

Define three operators $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ on E by

$$\mathscr{A}x(t) = f\big(t,x(t),X(t)\big),\ t \in J, \tag{4.6}$$

$$\mathscr{B}x(t) = ce^{-\lambda t} + \int_{t_0}^{t} e^{-\lambda(t-s)} g\big(s,x(s),X(s)\big)\, ds,\ t \in J, \tag{4.7}$$

and

$$\mathscr{C}x(t) = f\big(t,x(t),X(t)\big),\ t \in J. \tag{4.8}$$

From the continuity of the integral, it follows that $\mathscr{A}$ and $\mathscr{B}$ define the maps $\mathscr{A},\mathscr{B}: E \to \mathscr{K}$. Now by Lemma 4.2, the HDE (2.1) is equivalent to the operator equation

$$\mathscr{A}x(t)\mathscr{B}x(t)+\mathscr{C}x = x(t),\ t\in J. \tag{4.9}$$

We shall show that the operators $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ satisfy all the conditions of Theorem 3.1. This is achieved in the series of following steps.

Step I: *$\mathscr{A}$ and $\mathscr{B}$ are nondecreasing on E.*

Let $x,y\in E$ be such that $x\geq y$. Then $x(t)\geq y(t)$ for all $t\in J$. Since y is continuous on $[t_0,t]$, there exists a $\xi^*\in[t_0,t]$ such that $y(\xi^*)=\max\limits_{t_0\leq\xi\leq t} y(\xi)$. By definition of $\leq$, one has $x(\xi^*)\geq y(\xi^*)$. Consequently, we obtain

$$X(t)=\max_{t_0\leq\xi\leq t} x(\xi)=x(\xi^*)\geq y(\xi^*)=\max_{t_0\leq\xi\leq t} y(\xi)=Y(t)$$

for each $t\in J$. Then by hypothesis (A_3), we obtain

$$\mathscr{A}x(t)=f\big(t,x(t),X(t)\big)\geq f\big(t,x(t),Y(t)\big)=\mathscr{A}y(t),$$

for all $t\in J$. This shows that $\mathscr{A}$ is nondecreasing operator on E into E. Similarly using hypothesis (B_3), it is shown that the operator $\mathscr{B}$ is also nondecreasing on E into itself. Thus, $\mathscr{A}$ and $\mathscr{B}$ are nondecreasing positive operators on E into itself.

Step II: *$\mathscr{A}$ is partially bounded and partially $\mathscr{D}$-Lipschitz on E.*

Let $x\in E$ be arbitrary. Then by (A_2),

$$|\mathscr{A}x(t)|\leq \big|f\big(t,x(t),X(t)\big)\big|\leq M_f,$$

for all $t\in J$. Taking supremum over t, we obtain $\|\mathscr{A}x\|\leq M_f$ and so, $\mathscr{A}$ is bounded. This further implies that $\mathscr{A}$ is partially bounded on E.

Next, let $x,y\in E$ be such that $x\geq y$. Then, we have

$$|x(t)-y(t)|\leq|X(t)-Y(t)|$$

and that

$$\begin{aligned}|X(t)-Y(t)| &= X(t)-Y(t)\\ &= \max_{t_0\le\xi\le t} x(\xi) - \max_{t_0\le\xi\le t} y(\xi)\\ &\le \max_{t_0\le\xi\le t} [x(\xi)-y(\xi)]\\ &= \max_{t_0\le\xi\le t} |x(\xi)-y(\xi)|\\ &\le \|x-y\|\end{aligned}$$

for each $t\in J$. As a result, we obtain

$$\begin{aligned}|\mathscr{A}x(t)-\mathscr{A}y(t)| &= \big|f(t,x(t),X(t))-f(t,y(t),Y(t))\big|\\ &\le \varphi\left(\max\{|x(t)-y(t)|\,,|X(t)-Y(t)|\}\right)\\ &\le \varphi(\|x-y\|),\end{aligned}$$

for all $t\in J$. Taking supremum over t, we obtain

$$\|\mathscr{A}x-\mathscr{A}y\|\le\varphi(\|x-y\|),$$

for all $x,y\in E$ with $x\ge y$. Hence, $\mathscr{A}$ is a partial nonlinear $\mathscr{D}$-Lipschitz on E with a $\mathscr{D}$-function φ and which further implies that $\mathscr{A}$ is a partially continuous operator on E. Similarly, it can be shown that $\mathscr{C}$ is a partial nonlinear $\mathscr{D}$-Lipschitz on E with a $\mathscr{D}$-function ω and which is again a partially continuous operator on E.

Step III: *$\mathscr{B}$ is partially continuous on E.*

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a chain C of E such that $x_n\to x$ for all $n\in\mathbb{N}$. Then, by dominated convergence theorem, we have

$$\begin{aligned}\lim_{n\to\infty}\mathscr{B}x_n(t) &= \lim_{n\to\infty} ce^{-\lambda t}+\lim_{n\to\infty}\int_{t_0}^{t} e^{-\lambda(t-s)}g(s,x_n(s),X_n(s))\,ds\\ &= ce^{-\lambda t}+\int_{t_0}^{t} e^{-\lambda(t-s)}\left[\lim_{n\to\infty} g(s,x_n(s),X_n(s))\right]ds\\ &= ce^{-\lambda t}+\int_{t_0}^{t} e^{-\lambda(t-s)}g(s,x(s),X(s))\,ds\\ &= \mathscr{B}x(t),\end{aligned}$$

for all $t \in J$. This shows that $\mathscr{B}x_n$ converges to $\mathscr{B}x$ pointwise on J.

Next, we will show that $\{\mathscr{B}x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then

$$\begin{aligned}
|\mathscr{B}x_n(t_2) - \mathscr{B}x_n(t_1)| &\le \left| ce^{-\lambda t_1} - ce^{-\lambda t_2} \right| \\
&\quad + \left| \int_{t_0}^{t_1} e^{-\lambda(t_1-s)} g(s, x_n(s), X_n(s))\, ds - \int_{t_0}^{t_1} e^{-\lambda(t_2-s)} g(s, x_n(s), X_n(s))\, ds \right| \\
&\quad + \left| \int_{t_0}^{t_1} e^{-\lambda(t_2-s)} g(s, x_n(s), X_n(s))\, ds - \int_{t_0}^{t_2} e^{-\lambda(t_2-s)} g(s, x_n(s), X_n(s))\, ds \right| \\
&\le \left| ce^{-\lambda t_1} - ce^{-\lambda t_2} \right| + \left| \int_{t_0}^{t_1} \left| e^{-\lambda(t_1-s)} - e^{-\lambda(t_2-s)} \right| |g(s, x_n(s), X_n(s))|\, ds \right| \\
&\quad + \left| \int_{t_2}^{t_1} |g(s, x_n(s), X_n(s))|\, ds \right| \\
&\le \left| ce^{-\lambda t_1} - ce^{-\lambda t_2} \right| + M_g \int_{t_0}^{t_0+a} \left| e^{-\lambda(t_1-s)} - e^{-\lambda(t_2-s)} \right| ds \\
&\quad + M_g\, |t_1 - t_2| \\
&\to 0 \quad as \quad t_2 - t_1 \to 0
\end{aligned}$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathscr{B}x_n \to \mathscr{B}x$ is uniform and hence $\mathscr{B}$ is partially continuous on E.

Step IV: *$\mathscr{B}$ is uniformly partially compact operator on E.*

Let C be an arbitrary chain in E. We show that $\mathscr{B}(C)$ is a uniformly bounded and equicontinuous set in E. First we show that $\mathscr{B}(C)$ is uniformly bounded. Let $y \in \mathscr{B}(C)$ be any element. Then there is an element $x \in C$ such that $y = \mathscr{B}x$. Now, by hypothesis (B$_2$),

$$\begin{aligned}
|y(t)| &\le \left| ce^{-\lambda t} + \int_{t_0}^{t} e^{-\lambda(t-s)} g(s, x(s), X(s))\, ds \right| \\
&\le \left| ce^{-\lambda t} \right| + \left| \int_{t_0}^{t} e^{-\lambda(t-s)} g(s, x(s), X(s))\, ds \right| \\
&\le \left| \frac{\alpha_0 - h(t_0, \alpha_0, \alpha_0)}{f(t_0, \alpha_0, \alpha_0)} \right| + \int_{t_0}^{t_0+a} |g(s, x(s), X(s))|\, ds \\
&\le \left| \frac{\alpha_0 - h(t_0, \alpha_0, \alpha_0)}{f(t_0, \alpha_0, \alpha_0)} \right| + M_g\, a = M,
\end{aligned}$$

for all $t \in J$. Taking supremum over t, we obtain $\|y\| = \|\mathscr{B}x\| \leq M$ for all $y \in \mathscr{B}(C)$. Hence, $\mathscr{B}(C)$ is a uniformly bounded subset of E. Moreover, $\|\mathscr{B}(C)\| \leq M$ for all chains C in E. Hence, $\mathscr{B}$ is a uniformly partially bounded operator on E.

Next, we will show that $\mathscr{B}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then, for any $y \in \mathscr{B}(C)$, one has

$$
\begin{aligned}
|y(t_2) - y(t_1)| &= |\mathscr{B}x(t_2) - \mathscr{B}x(t_1)| \\
&\leq \left| ce^{-\lambda t_1} - ce^{-\lambda t_2} \right| \\
&\quad + \left| \int_{t_0}^{t_1} e^{-\lambda(t_1 - s)} g(s, x(s), X(s))\, ds - \int_{t_0}^{t_1} e^{-\lambda(t_2 - s)} g(s, x(s), X(s))\, ds \right| \\
&\quad + \left| \int_{t_0}^{t_1} e^{-\lambda(t_2 - s)} g(s, x(s), X(s))\, ds - \int_{t_0}^{t_2} e^{-\lambda(t_2 - s)} g(s, x(s), X(s))\, ds \right| \\
&\leq \left| ce^{-\lambda t_1} - ce^{-\lambda t_2} \right| + \left| \int_{t_0}^{t_1} \left| e^{-\lambda(t_1 - s)} - e^{-\lambda(t_2 - s)} \right| |g(s, x(s), X(s))|\, ds \right| \\
&\quad + \left| \int_{t_2}^{t_1} |g(s, x(s), X(s))|\, ds \right| \\
&\leq \left| ce^{-\lambda t_1} - ce^{-\lambda t_2} \right| + M_g \int_{t_0}^{t_0 + a} \left| e^{-\lambda(t_1 - s)} - e^{-\lambda(t_2 - s)} \right| ds \\
&\quad + M_g |t_1 - t_2| \\
&\to 0 \quad as \quad t_2 - t_1 \to 0
\end{aligned}
$$

uniformly for all $y \in \mathscr{B}(C)$. Hence $\mathscr{B}(C)$ is an equicontinuous subset of E. Now, $\mathscr{B}(C)$ is a uniformly bounded and equicontinuous set of functions in E, so it is compact. Consequently, $\mathscr{B}$ is a uniformly partially compact operator on E into itself.

Step V: *u satisfies the operator inequality $u \leq \mathscr{A}u\mathscr{B}u + \mathscr{C}u$.*

By hypothesis (B_4), the HDE (2.1) has a lower solution u defined on J. Then, we have

$$
\left.
\begin{aligned}
\frac{d}{dt}\left[\frac{u(t) - h(t, u(t), U(t))}{f(t, u(t), U(t))} \right] + \lambda \left[\frac{u(t) - h(t, u(t), U(t))}{f(t, u(t), U(t))} \right] &\leq g(t, u(t), U(t)), \\
u(t_0) &\leq \alpha_0,
\end{aligned}
\right\} \tag{4.10}
$$

for all $t \in J$. Multiplying the above inequality (4.7) by the integrating factor $e^{\lambda t}$, we obtain

$$
\left(e^{\lambda t} \frac{u(t) - h(t, u(t), U(t))}{f(t, u(t), U(t))} \right)' \leq e^{\lambda t} g(t, u(t), U(t)), \tag{4.11}
$$

for all $t \in J$. A direct integration of (4.8) from t_0 to t yields

$$\begin{aligned} u(t) &\le h\big(t, u(t), U(t))\big) \\ &\quad + \big[f(t, u(t), U(t))\big] \left(c e^{-\lambda t} + \int_{t_0}^{t} e^{-\lambda(t-s)} g(s, u(s), U(s))\, ds \right), \end{aligned} \tag{4.12}$$

for all $t \in J$. From definitions of the operators $\mathscr{A}$ and $\mathscr{B}$ it follows that $u(t) \le \mathscr{A}u(t)\,\mathscr{B}u(t) + \mathscr{C}u(t)$, for all $t \in J$. Hence $u \le \mathscr{A}u\,\mathscr{B}u + \mathscr{C}u$.

Step VI: *$\mathscr{D}$-function φ and ω satisfy the growth condition*

$$0 < M\psi_{\mathscr{A}}(r) + \psi_{\mathscr{A}}(r) < r,\ r > 0.$$

Finally, the $\mathscr{D}$-function φ of the operator $\mathscr{A}$ satisfies the inequality given in hypothesis (d) of Theorem 3.1. Now from the estimate given in Step IV, it follows that

$$M\psi_{\mathscr{A}}(r) + \psi_{\mathscr{C}}(r) \le \left(\frac{\alpha_0 - h(t_0, \alpha_0, \alpha_0)}{f(t_0, \alpha_0, \alpha_0)} + M_g\, a \right) \varphi(r) + \omega(r) < r$$

for all $r > 0$.

Thus $\mathscr{A}$ and $\mathscr{B}$ satisfy all the conditions of Theorem 3.1 and we apply it to conclude that the operator equation $\mathscr{A}x\,\mathscr{B}x + \mathscr{C}x = x$ has a solution. Consequently the integral equation and the HDE (2.1) has a positive solution x^* defined on J. Furthermore, the sequence $\{x_n\}_{n=1}^{\infty}$ of successive approximations defined by (4.5) converges monotonically to x^*. This completes the proof.

Remark 4.2. The conclusion of Theorem 4.1 also remains true if we replace the hypothesis (D_1) with the following:

(D_2) The HDE (2.1) has an upper solution $v \in C(J, \mathbb{R})$.

The proof under the new hypothesis is similar to the proof of Theorem 4.1 with appropriate modifications.

Remark 4.3. We note that if the HDE (2.1) has a lower solution u as well as an upper solution v such that $u \le v$, then under the given conditions of Theorem 4.1 it has corresponding solutions x_* and x^* and these solutions satisfy $x_* \le x^*$. Hence they are the minimal and maximal solutions of the HDE (2.1)in the vector segment $[u,v]$ of the Banach space $E = C(J,\mathbb{R})$, where the vector segment $[u,v]$ is a set in $C(J,\mathbb{R})$ defined by

$$[u,v] = \{x \in C(J,\mathbb{R}) \mid u \leq x \leq v\}.$$

This is because the order relation $\leq$ defined by (4.2) is equivalent to the order relation defined by the order cone $\mathscr{K} = \{x \in C(J,\mathbb{R}) \mid x \geq \theta\}$ which is a closed set in $C(J,\mathbb{R})$.

Note that we have proved only the existence of solution for the HDE (2.1) by sequence of successive approximations. However, other qualitative aspects of the solutions such as minimal and maximal solutions etc., could also be proved via Dhage iteration method. Finally we give a numerical example to illustrate the hypotheses and the main abstract result formulated in Theorem 4.1.

5. An example

Given a closed and bounded interval $J = [0,1]$ in $\mathbb{R}$, consider the HDE

$$\left.\begin{aligned} &\frac{d}{dt}\left[\frac{x(t) - \frac{1}{12}\big[\tan^{-1}x(t) + \tan^{-1}X(t)\big]}{3 + \frac{1}{2}\big[\tan^{-1}x(t) + \tan^{-1}X(t)\big]}\right] \\ &\qquad + \left[\frac{x(t) - \frac{1}{12}\big[\tan^{-1}x(t) + \tan^{-1}X(t)\big]}{3 + \frac{1}{2}\big[\tan^{-1}x(t) + \tan^{-1}X(t)\big]}\right] = \frac{1}{24}\Big[2 + \tanh x(t) + \tanh X(t)\Big], \\ &x(0) = 1. \end{aligned}\right\} \tag{5.1}$$

for all $t \in J$,.

Here, $\lambda = 1$ and the functions h, f and g are given by

$$h(t,x,y) = \frac{1}{12}\big[\tan^{-1}x + \tan^{-1}y\big], \quad f(t,x,y) = 3 + \frac{1}{2}\big[\tan^{-1}x + \tan^{-1}y\big]$$

and

$$g(t,x,y) = \frac{1}{24}\big[2 + \tanh x + \tanh y\big]$$

for all $t \in J$ and $x, y \in \mathbb{R}$. We show that the functions f, h and g satisfy all the hypotheses of

Theorem 4.1. First we show that f satisfies the hypotheses (A_0)-(A_2). Now,

$$\frac{\partial}{\partial x}\Big[\frac{x-h(t,x,x)}{f(t,x,x)}\Big] = \frac{d}{dx}\Big[\frac{x-\frac{1}{6}\tan^{-1}x}{3+\tan^{-1}x}\Big] \geq \frac{2+\tan^{-1}x}{25} > 0,$$

for all $x \in \mathbb{R}$ and $t \in J$, so that the function $x \mapsto \dfrac{x-h(t,x,x)}{f(t,x,x)}$ is increasing in $\mathbb{R}$ for each $t \in J$. Therefore, hypothesis (A_0) holds. Next, let $x_1, x_2, y_1, y_2 \in \mathbb{R}$ be such that $x_1 \geq y_1$ and $x_2 \geq y_2$. Then,

$$\begin{aligned} 0 &\leq h(t,x_1,x_2) - h(t,y_1,y_2) \\ &\leq \frac{1}{12}\Big[\tan^{-1}x_2 - \tan^{-1}y_2\Big] + \frac{1}{12}\Big[\tan^{-1}x_2 - \tan^{-1}y_2\Big] \\ &= \frac{1}{6}\cdot\frac{1}{1+\xi^2}\max\{x_1-y_1, x_2-y_2\} \end{aligned}$$

for some ξ with $x_1 > \xi > y_1$, $x_2 > \xi > y_2$, showing that h satisfies the hypothesis (C_2) with $\mathscr{D}$-function ω given by

$$\omega(r) = \frac{1}{6}\cdot\frac{r}{1+\xi^2},$$

where $\xi \neq 0$. Again,

$$|h(t,x,y)| = \frac{1}{12}\cdot\big|\tan^{-1}x + \tan^{-1}y\big| \leq \frac{\pi}{12},$$

for all $t \in J$ and $x \in \mathbb{R}$. This shows that h satisfies hypothesis (C_1) with $M_h = \dfrac{\pi}{12}$.

Similarly, we have

$$\begin{aligned} 0 &\leq f(t,x_1,x_2) - f(t,y_1,y_2) \\ &\leq \frac{1}{12}\Big[\tan^{-1}x_2 - \tan^{-1}y_2\Big] + \frac{1}{12}\Big[\tan^{-1}x_2 - \tan^{-1}y_2\Big] \\ &= \frac{1}{6}\cdot\frac{1}{1+\xi^2}\max\{x_1-y_1, x_2-y_2\} \end{aligned}$$

for some ξ with $x_1 > \xi > y_1$, $x_2 > \xi > y_2$,, showing that f satisfies the hypothesis (A_2) with $\mathscr{D}$-function φ given by $\varphi(r) = \dfrac{1}{1+\xi^2}r$. Next, we have

$$0 \leq g(t,x,y) = \frac{1}{24}\big[2+\tanh x + \tanh y\big] \leq \frac{1}{6},$$

for all $t \in J$ and $x, y \in \mathbb{R}$, so that the hypothesis (B_1) holds with $M_g = \dfrac{1}{6}$. Again, since the function $(x,y) \mapsto \dfrac{1}{18}\big[2+\tanh x + \tanh y\big]$ is nondecreasing in $\mathbb{R}\times\mathbb{R}$ and so the hypothesis (B_2)

is satisfied. Furthermore, we have

$$\left(c+\frac{(e^{\lambda T}-1)M_g}{\lambda}\right)\varphi(r)+\omega(r)=\left(\frac{73}{318}+\frac{1}{3}\right)\cdot\frac{r}{1+\xi^2}r+\frac{1}{6}\cdot\frac{r}{1+\xi^2}<r$$

for each $r>0$ and $\xi\neq 0$. Finally, the function $u(t)=-e^{-t}-2$ is a lower solution of the HDE (5.1) defined on $J=[0,1]$.

Thus the functions f and g satisfy all the conditions of Theorem 4.1. Hence we apply and conclude that the HDE (5.1) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by

$$\begin{aligned} x_0 &= -e^{-t}-2,\\ x_{n+1}(t) &= \frac{1}{12}\big[\tan^{-1}x(t)+\tan^{-1}X_n(t)\big]\\ &\quad+\left[3+\frac{1}{2}\Big(\tan^{-1}x(t)+\tan^{-1}X_n(t)\Big)\right]\times\\ &\quad\times\left(\frac{73}{318}e^{-t}+\frac{1}{24}e^{-t}\int_0^t e^s\big[2+\tanh x(s)+\tanh X_n(s)\big]\,ds\right), \end{aligned}$$

for each $t\in J$, converges monotonically to x^*. A similar conclusion also remains true if we replace the lower solution u with the upper solution $v(t)=e^{-t}+3$, $t\in J$.

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