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POSITIVE SOLUTIONS FOR A SINGULAR SEMIPOSITONE DYNAMIC SYSTEM WITH M-POINT BOUNDARY CONDITIONS ON TIME SCALES

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Abstract. In this paper, under suitable conditions by applying fixed point theorems in cones, we give the existence of two positive solutions for the system of second order singular semipositone m-point boundary value problem on time scales. We emphasize that the nonlinear term may take a negative value and be singular. As an application, we also give an example to illustrate our results.

Keywords. Nonlinear operator; Positive solution; Fixed point theorem; Semipositone problem; Time scale.

1. Introduction

In recent years, multi-point boundary value problems for second order and higher order ordinary differential equations and systems, arise from many fields in physics, biology and chemistry, play very important role in both theory and applications [9, 10, 15, 16, 19].

In [8], Henderson and Luca investigated the existence and multiplicity of positive solutions of multi point boundary value problems for systems of nonlinear higher order ordinary differential equations:

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$$u^n(t)+f(t,v(t))=0, \quad t\in(0,T), \quad n\geq 2,$$

$$v^m(t)+g(t,u(t))=0, \quad t\in(0,T), \quad m\geq 2,$$

$$u(0)=u'(0)=...=u^{(n-2)}(0)=0 \quad u(T)=\sum_1^{p-2} a_i u(\xi_i), \quad p\geq 3,$$

$$v(0)=v'(0)=...=v^{(n-2)}(0)=0 \quad v(T)=\sum_1^{q-2} a_i v(\xi_i), \quad q\geq 3.$$

Problems where the nonlinear term have some singularities are referred to as singular problems in the literature and this type differential systems appear in the study of gas dynamics, fluid mechanics, the theory of boundary layer and so on. Because of its interest to physics, singular problems have attracted extensive study in recent years [4, 12, 13, 17].

For example, Asif and Khan [4] studied the existence of positive solution to a nonlinear singular system with four-point boundary conditions of the type

$$-x(t)''=f(t,x(t),y(t)), \quad t\in(0,1),$$

$$-y(t)''=g(t,x(t),y(t)), \quad t\in(0,1),$$

$$x(0)=0, \ \ x(1)=\alpha y(\xi), \ \ y(0)=0, \ \ y(1)=\beta x(\eta).$$

In [12], Liu and Yan considered the following singular boundary value problem of Sturm Liouville differential system:

$$(p(t)x(t)')'+\lambda f(t,x(t),y(t))=0, \quad t\in(0,1),$$

$$(p(t)y(t)')'+\lambda g(t,x(t),y(t))=0, \quad t\in(0,1),$$

$$\alpha x(0)-\beta p(0)x'(0)=\gamma x(1)-\delta p(1)x'(1)=0,$$

$$\alpha y(0)-\beta p(0)y'(0)=\gamma y(1)-\delta p(1)y'(1)=0.$$

Although an increasing interest has been observed in investigating the existence of positive solutions of dynamic equations on measure chains [1, 2, 17, 18, 20], very little work has been done on the existence of positive solutions of dynamic systems on measure chains [7, 11].

In [14], Prasad, Rao and Bharathi interested in the existence of positive solutions to the system of dynamic equations:

$$(-1)^n u^{\triangle^{2n}}(t) + \lambda p(t) f(v(\sigma(t))) = 0, \quad t \in [a,b],$$

$$(-1)^n v^{\triangle^{2n}}(t) + \mu q(t) f(u(\sigma(t))) = 0, \quad t \in [a,b],$$

$$\alpha_{i+1} u^{\triangle^{2i}}(a) - \beta_{i+1} u^{\triangle^{2i+1}}(a) = 0,\ \gamma_{i+1} u^{\triangle^{2i}}(\sigma(b)) + \delta_{i+1} u^{\triangle^{2i+1}}(\sigma(b)) = 0,$$

$$\alpha_{i+1} v^{\triangle^{2i}}(a) - \beta_{i+1} v^{\triangle^{2i+1}}(a) = 0,\ \gamma_{i+1} v^{\triangle^{2i}}(\sigma(b)) + \delta_{i+1} v^{\triangle^{2i+1}}(\sigma(b)) = 0.$$

Problems of this type where the nonlinear term may change sign are referred to as semipositone problems in the literature. Semipositone differential systems appear in the study of chemical reactors [3].

Motivated by the above works, in this paper, we shall consider the nonlinear singular semipositone system of m-point boundary value problem (SSS),

$$\begin{cases} -[p(t)u_i^{\triangle}(t)]^{\nabla} + q(t)u_i(t) = f_i(t, u_1(t), u_2(t)) + h_i(t), \quad t \in (a,b), \quad i = 1,2, \\ \alpha u_i(a) - \beta u_i^{[\triangle]}(a) = \sum_{k=1}^{m-2} \alpha_k u_i(\xi_k), \quad \gamma u_i(b) + \delta u_i^{[\triangle]}(b) = \sum_{k=1}^{m-2} \beta_k u_i(\xi_k), \quad i = 1,2, \end{cases} \tag{1.1}$$

where $\alpha, \beta, \gamma, \delta, \xi_k, \alpha_k, \beta_k$ (for $k \in \{1,2,...,m-2\}$) are complex constants such that $|\alpha| + |\beta| \neq 0$, $|\gamma| + |\delta| \neq 0$ and $\xi_k \in \mathrm{T} \backslash \{a,b\}$, $q : \mathrm{T} \to \mathscr{C}$ is a continuous function, $p : \mathrm{T} \to \mathscr{C}$ is $\nabla-$differentiable on T_k, $p(t) \neq 0$ for all $t \in \mathrm{T}$, $p^{\nabla} : \mathrm{T}_k \to \mathscr{C}$ is continuous, f_1 and $f_2 : [a,b] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous and h_1 and $h_2 : (a,b) \to (-\infty,\infty)$ are Lebesgue integrable and may have finitely many singularities in $[a,b]$.

By an interval (a,b), we mean the intersection of the real interval (a,b) with the given time scale T. Some preliminary definitions and theorems on time scales can be found in the books [5, 6].

Compared to previous work in this field, this study presented three new features. Firstly, the nonlinear term is allowed to change sign and tend to negative infinity. Secondly, is allowed to have finitely many singularities in $[a,b]$. Lastly, the boundary condition taken up generalizes the conditions of many problems in the literature. By using the cone theory technique, we establish some sufficient conditions for the existence of multiple positive solutions to the SSS (1.1). The rest of the paper is organized as follows. In Section 2, we give some inequalities for Green's function and some results which are needed later. Criteria for the existence of two positive solutions of the SSS (1.1) are established in Section 3, using fixed point index theorem. In addition, an example is given to illustrate the applications of main result.

2. Preliminaries

We shall work in the space $E = C([a,b];R) \times C([a,b];R)$. The space E is a Banach space if it is endowed with the norm as follows:

$$\|(u,v)\| = \|u\| + \|v\|, \qquad \|u\| = \max_{t\in[a,b]} |u(t)|, \quad \|v\| = \max_{t\in[a,b]} |v(t)|,$$

for any $(u,v) \in E$.

For any $u = (u_1,u_2), v = (v_1,v_2) \in E$, we denote $u \le v \Leftrightarrow u_i(t) \le v_i(t), \quad t \in [a,b], \quad i = 1,2.$

In the following, let us define a cone P of E by

$$P = \{(u_1,u_2) \in E : u_i(t) \ge g(t)\|u_i\|,\ t \in [a,b],\ i = 1,2\},$$

where g is defined by

$$g(t) := \min_{t\in[a,b]} \{\frac{\phi_1(t)}{\phi_1(b)}, \frac{\phi_2(t)}{\phi_2(a)}\}, \tag{2.1}$$

and ϕ_1, ϕ_2 are the solutions of the linear problems

$$[p(t)\phi_1^{\triangle}(t)]^{\nabla} - q(t)\phi_1(t) = 0,\ t \in (a,b),$$
$$\phi_1(a) = \beta,\ \phi_1^{[\triangle]}(a) = \alpha,$$

and

$$[p(t)\phi_2^{\triangle}(t)]^{\nabla} - q(t)\phi_2(t) = 0,\ t \in (a,b),$$
$$\phi_2(b) = \delta,\ \phi_2^{[\triangle]}(b) = -\gamma,$$

respectively.

Let $G(t,s)$ be the Green's function for the boundary value problem

$$-[p(t)u^{\triangle}(t)]^{\nabla} + q(t)u(t) = 0,\ t \in (a,b),$$
$$\alpha u(a) - \beta u^{[\triangle]}(a) = 0,$$
$$\gamma u(b) + \delta u^{[\triangle]}(b) = 0,$$

is given by

$$G(t,s) = \frac{1}{d}\begin{cases} \phi_1(s)\phi_2(t), & a \le s \le t \le b, \\ \phi_1(t)\phi_2(s) & a \le t \le s \le b, \end{cases} \tag{2.2}$$

where $d = -W_t(\phi_1, \phi_2) = p(t)[\phi_1^{\triangle}(t)\phi_2(t) - \phi_1(t)\phi_2^{\triangle}(t)]$.

Let we define

$$\Omega := \begin{vmatrix} -\sum_{k=1}^{m-2} \alpha_k \phi_1(\xi_k) & d - \sum_{k=1}^{m-2} \alpha_k \phi_2(\xi_k) \\ d - \sum_{k=1}^{m-2} \beta_k \phi_1(\xi_k) & -\sum_{k=1}^{m-2} \beta_k \phi_2(\xi_k) \end{vmatrix}$$

and assume that the following conditions are satisfied:

(H_1) $p(t) > 0$, $q(t) \geq 0$,

(H_2) $\alpha, \gamma \geq 0$, $\beta, \delta > 0$, α_k, $\beta_k \geq 0$ for $k \in \{1, 2, ..., m-2\}$,

(H_3) If $q(t) \equiv 0$, then $\alpha + \gamma > 0$,

(H_4) $\Omega < 0$, $d - \sum_{k=1}^{m-2} \alpha_k \phi_2(\xi_k) > 0$, $d - \sum_{k=1}^{m-2} \beta_k \phi_1(\xi_k) > 0$.

To prove the main results, we will employ following lemmas.

Lemma 2.1. [5] *Under the conditions (H_1) and (H_2), the solutions $\phi_1(t)$ and $\phi_2(t)$ posses the following properties:*

$$\phi_1(t), \phi_2(t) \geq 0, \quad \phi_1^{[\triangle]}(t) \geq 0, \quad \phi_2^{[\triangle]}(t) \leq 0, \qquad t \in [a, b].$$

Lemma 2.2. [5] *If the conditions $(H_1) - (H_3)$ are hold, then $G(t,s) > 0$ for $t, s \in [a, b]$.*

Lemma 2.3. *Assume that $(H_1) - (H_3)$ hold. Then*

$$g(t)G(s,s) \leq G(t,s) \leq G(s,s), \quad t, s \in [a, b],$$

where g is given in (2.1).

Proof. It follows from Lemma 2.1 that $\phi_1(t)$ is increasing and $\phi_2(t)$ is decreasing on $t \in [a, b]$. Then we have $G(t,s) \leq G(s,s)$. Besides this inequality, for all $t, s \in [a, b]$, we have

$$\frac{G(t,s)}{G(s,s)} = \begin{cases} \dfrac{\phi_2(t)}{\phi_2(s)}, & s \leq t \\ \dfrac{\phi_1(t)}{\phi_1(s)}, & t \leq s \end{cases} \geq \begin{cases} \dfrac{\phi_2(t)}{\phi_2(a)}, & s \leq t \\ \dfrac{\phi_1(t)}{\phi_1(b)}, & t \leq s \end{cases} \geq g(t).$$

This completes the proof.

We consider the following boundary value problem

$$-[p(t)u^{\triangle}(t)]^{\nabla} + q(t)u(t) = y(t)\ , \qquad t \in (a, b), \tag{2.3}$$

$$\alpha u(a) - \beta u^{[\triangle]}(a) = \sum_{k=1}^{m-2} \alpha_k u(\xi_k), \quad \gamma u(b) + \delta u^{[\triangle]}(b) = \sum_{k=1}^{m-2} \beta_k u(\xi_k). \tag{2.4}$$

Lemma 2.4. [18] *Let the conditions* $(H_1)-(H_3)$ *be hold. Assume that* $\Omega :\neq 0$. *Then for* $y \in C([a,b])$, *the boundary value problem* $(2.3)-(2.4)$ *has a unique solution*

$$u(t) = \int_a^b G(t,s)y(s)\nabla s + A(y)\phi_1(t) + B(y)\phi_2(t),$$

where $G(t,s)$ *is given in* (2.2) ,

$$A(y) := \frac{1}{\Omega}\begin{vmatrix} \sum_{k=1}^{m-2}\alpha_k \int_a^b G(\xi_k,s)y(s)\nabla s & d-\sum_{k=1}^{m-2}\alpha_k\phi_2(\xi_k) \\ \sum_{k=1}^{m-2}\beta_k \int_a^b G(\xi_k,s)y(s)\nabla s & -\sum_{k=1}^{m-2}\beta_k\phi_2(\xi_k) \end{vmatrix},$$

$$B(y) := \frac{1}{\Omega}\begin{vmatrix} -\sum_{k=1}^{m-2}\alpha_k\phi_1(\xi_k) & \sum_{k=1}^{m-2}\alpha_k \int_a^b G(\xi_k,s)y(s)\nabla s \\ d-\sum_{k=1}^{m-2}\beta_k\phi_1(\xi_k) & \sum_{k=1}^{m-2}\beta_k \int_a^b G(\xi_k,s)y(s)\nabla s \end{vmatrix}.$$

Lemma 2.5. [18] *Let* $(H_1)-(H_4)$ *hold. If* $y \in C([a,b],[0,\infty))$, *then the solution* u *of the boundary value problem* $(2.3)-(2.4)$ *satisfies* $u(t) \geq 0$, *for* $t \in [a,b]$.

Lemma 2.6. *If* $\int_a^b G(s,s)y(s)\nabla s < \infty$, *then the following inequalities are satisfied:*

$$A(y) \leq A\int_a^b G(s,s)y(s)\nabla s, \qquad B(y) \leq B\int_a^b G(s,s)y(s)\nabla s,$$

where

$$A = \frac{1}{\Omega}\begin{vmatrix} \sum_{k=1}^{m-2}\alpha_k & d-\sum_{k=1}^{m-2}\alpha_k\phi_2(\xi_k) \\ \sum_{k=1}^{m-2}\beta_k & -\sum_{k=1}^{m-2}\beta_k\phi_2(\xi_k) \end{vmatrix},$$

$$B = \frac{1}{\Omega}\begin{vmatrix} -\sum_{k=1}^{m-2}\alpha_k\phi_1(\xi_k) & \sum_{k=1}^{m-2}\alpha_k \\ d-\sum_{k=1}^{m-2}\beta_k\phi_1(\xi_k) & \sum_{k=1}^{m-2}\beta_k \end{vmatrix}.$$

Proof. It can be easily proven. We here omit the proof.

3. Main results

In this section, we apply the following fixed point index theorem to prove the existence of two positive solutions for the SSS (1.1).

Theorem 3.1. *Let* $E = (E, \|.\|)$ *be a Banach space,* Ω *be a bounded open subset of* E *with* $0 \in \Omega$, $P \subset E$ *be a cone in* E *and* $F : P \cap \overline{\Omega} \to P$ *be a completely continuous operator.*

(i) *Suppose that* $Fu \neq \lambda u, \quad \forall u \in \partial\Omega \cap P,\ \lambda \geq 1$. *Then* $i(F, \Omega \cap P, P) = 1$.

(ii) Suppose that $Fu \nleq u, \quad \forall u \in \partial\Omega \cap P$. *Then* $i(F, \Omega \cap P, P) = 0$.

For using this theorem now we have to do some preparations. In the rest of the paper, we assume that the following conditions are satisfied:

(H_5) $f_1, f_2 : [a,b] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous,

(H_6) $h_1, h_2 : (a,b) \to (-\infty,\infty)$ are Lebesgue integrable such that

$\int_a^b h_{i\_}(s)\nabla s > 0, \quad i = 1,2,$

where $h_{i\_}(t) = \max\{-h_i(t), 0\}$.

Let $w_i(t) = \int_a^b G(t,s)h_{i\_}(s)\nabla s + A(h_{i\_})\phi_1(t) + B(h_{i\_})\phi_2(t)$, $t \in [a,b]$, $i = 1,2$. Using the expression of the Green's function, the definition of the function g, the properties of ϕ_1 and ϕ_2, the assumption (H_6) and Lemma 2.6, we obtain

$$\begin{aligned}
w_i(t) &= \int_a^b G(t,s)h_{i_}(s)\nabla s + A(h_{i_})\phi_1(t) + B(h_{i_})\phi_2(t) \\
&= \frac{1}{d}\int_a^t \phi_1(s)\phi_2(t)h_{i_}(s)\nabla s + \frac{1}{d}\int_t^b \phi_1(t)\phi_2(s)h_{i_}(s)\nabla s + A(h_{i_})\phi_1(t) + B(h_{i_})\phi_2(t) \\
&\le \frac{1}{d}\int_a^t \phi_1(t)\phi_2(t)h_{i_}(s)\nabla s + \frac{1}{d}\int_t^b \phi_1(t)\phi_2(t)h_{i_}(s)\nabla s + A\phi_1(t)\int_a^b G(s,s)h_{i_}(s)\nabla s \\
&\quad + B\phi_2(t)\int_a^b G(s,s)h_{i_}(s)\nabla s \\
&= \frac{1}{d}\int_a^b \phi_1(t)\phi_2(t)h_{i_}(s)\nabla s + A\phi_1(t)\int_a^b G(s,s)h_{i_}(s)\nabla s + B\phi_2(t)\int_a^b G(s,s)h_{i_}(s)\nabla s \\
&\le \frac{1}{d}\phi_1(b)\phi_2(a)g(t)\int_a^b h_{i_}(s)\nabla s + \frac{1}{\phi_2(t)}A\phi_1(b)\phi_2(a)g(t)\int_a^b G(s,s)h_{i_}(s)\nabla s \\
&\quad + \frac{1}{\phi_1(t)}B\phi_1(b)\phi_2(a)g(t)\int_a^b G(s,s)h_{i_}(s)\nabla s \\
&\le \frac{1}{d}\phi_1(b)\phi_2(a)g(t)\int_a^b h_{i_}(s)\nabla s + \frac{1}{\phi_2(b)}A\phi_1(b)\phi_2(a)g(t)\int_a^b G(s,s)h_{i_}(s)\nabla s \\
&\quad + \frac{1}{\phi_1(a)}B\phi_1(b)\phi_2(a)g(t)\int_a^b G(s,s)h_{i_}(s)\nabla s \\
&= [\frac{1}{d}\int_a^b h_{i_}(s)\nabla s + (\frac{A}{\phi_2(b)} + \frac{B}{\phi_1(a)})\int_a^b G(s,s)h_{i_}(s)\nabla s]\phi_1(b)\phi_2(a)g(t) < +\infty, \ i = 1,2.
\end{aligned}$$

Therefore, we can write

$$w_i(t) \le C_i g(t), \quad t \in [a,b], \quad i = 1,2, \tag{3.1}$$

where g is given in (2.1) and $C_i = [\frac{1}{d}\int_a^b h_{i_-}(s)\nabla s + (\frac{A}{\phi_2(b)} + \frac{B}{\phi_1(a)})\int_a^b G(s,s)h_{i_-}(s)\nabla s]\phi_1(b)\phi_2(a)$.

So $w_i(t), i=1,2$ are well defined in E. By direct computation, we have

$$\alpha w_i(a) - \beta w_i^{[\triangle]}(a) = \sum_{k=1}^{m-2}\alpha_k w_i(\xi_k), \gamma w_i(b) + \delta w_i^{[\triangle]}(b) = \sum_{k=1}^{m-2}\beta_k w_i(\xi_k), i=1,2,$$

which imply that $w_i(t), i=1,2$ are positive solutions of the following boundary value problems:

$$-[p(t)u_1^{\triangle}(t)]^{\nabla} + q(t)u_1(t) = h_{1_-}(t), \quad t \in (a,b),$$
$$\alpha u_1(a) - \beta u_1^{[\triangle]}(a) = \sum_{k=1}^{m-2}\alpha_k u_1(\xi_k), \qquad \gamma u_1(b) + \delta u_1^{[\triangle]}(b) = \sum_{k=1}^{m-2}\beta_k u_1(\xi_k),$$

and

$$-[p(t)u_2^{\triangle}(t)]^{\nabla} + q(t)u_2(t) = h_{2_-}(t), \quad t \in (a,b),$$
$$\alpha u_2(a) - \beta u_2^{[\triangle]}(a) = \sum_{k=1}^{m-2}\alpha_k u_2(\xi_k), \qquad \gamma u_2(b) + \delta u_2^{[\triangle]}(b) = \sum_{k=1}^{m-2}\beta_k u_2(\xi_k),$$

respectively.

For any $u(t) \in C([a,b])$, let us define a function $[.]^*$ by

$$[u(t)]^* = \begin{cases} u(t), & u(t) \geq 0, \\ 0 & u(t) < 0. \end{cases}$$

Now, we consider the following dynamic system

$$\begin{cases} -[p(t)u_i^{\triangle}(t)]^{\nabla} + q(t)u_i(t) = f_i(t, [(u_1 - w_1)(t)]^*, [(u_2 - w_2)(t)]^*) + h_{i_+}(t),\ t \in (a,b), i=1,2, \\ \alpha u_i(a) - \beta u_i^{[\triangle]}(a) = \sum_{k=1}^{m-2}\alpha_k u_i(\xi_k), \quad \gamma u_i(b) + \delta u_i^{[\triangle]}(b) = \sum_{k=1}^{m-2}\beta_k u_i(\xi_k), \quad i=1,2, \end{cases} \tag{3.2}$$

where $h_{i_+}(t) = \max\{h_i(t), 0\}$ and we define the operator $F: E \to E$ by

$F(u_1,u_2) = (F_1(u_1,u_2), F_2(u_1,u_2))$, where $F_i(u_1,u_2)(t) = \int_a^b G(t,s)[f_i(s, [u_1(s) - w_1(s)]^*, [u_2(s) - w_2(s)]^*) + h_{i_+}(s)]\nabla s + A(f_i + h_{i_+})\phi_1(t)$

$$+B(f_i + h_{i_+})\phi_2(t), \quad i=1,2.$$

It is well known that the existence of the solution to the system (3.2) is equivalent to the existence of fixed point of the operator F. So we shall seek a fixed point of F in our cone P.

Lemma 3.2. *If* (v_1, v_2) *with* $(w_1, w_2) \leq (v_1, v_2)$ *is a positive solution of the system* (3.2) *then* $(v_1 - w_1, v_2 - w_2)$ *is a positive solution of the SSS* (1.1).

Proof. Suppose that (v_1, v_2) with $(w_1, w_2) \leq (v_1, v_2)$ is a positive solution of system (3.2), then from (3.2) and the definition of $[.]^*$, we have

$$\begin{cases} -[p(t)v_i^{\triangle}(t)]^{\nabla} + q(t)v_i(t) = f_i(t, v_1(t) - w_1(t), v_2(t) - w_2(t)) + h_{i_+}(t),\ t \in (a,b), i = 1,2, \\ \alpha v_i(a) - \beta v_i^{[\triangle]}(a) = \sum_{k=1}^{m-2} \alpha_k v_i(\xi_k), \quad \gamma v_i(b) + \delta v_i^{[\triangle]}(b) = \sum_{k=1}^{m-2} \beta_k v_i(\xi_k), \quad i = 1,2. \end{cases} \tag{3.3}$$

Let $u_i(t) = v_i(t) - w_i(t), i = 1,2$, then $v_i(t) = u_i(t) + w_i(t)$, $v_i^{\triangle}(t) = (u_i + w_i)^{\triangle}(t) = u_i^{\triangle}(t) + w_i^{\triangle}(t)$ and $[p(t)v_i^{\triangle}(t)]^{\nabla} = [p(t)(u_i^{\triangle}(t) + w_i^{\triangle}(t))]^{\nabla} = [p(t)u_i^{\triangle}(t)]^{\nabla} + [p(t)w_i^{\triangle}(t)]^{\nabla}$, thus (3.3) becomes

$$\begin{cases} -[p(t)u_i^{\triangle}(t)]^{\nabla} + q(t)u_i(t) = f_i(t, u_1(t), u_2(t)) + h_{i_+}(t) - h_{i_-}(t), \quad t \in (a,b), \quad i = 1,2, \\ \alpha u_i(a) - \beta u_i^{[\triangle]}(a) = \sum_{k=1}^{m-2} \alpha_k u_i(\xi_k), \quad \gamma u_i(b) + \delta u_i^{[\triangle]}(b) = \sum_{k=1}^{m-2} \beta_k u_i(\xi_k), \quad i = 1,2. \end{cases} \tag{3.4}$$

Notice that $h_i(t) = h_{i_+}(t) - h_{i_-}(t), i = 1,2$ and (3.4), we know that $(u_1, u_2) = (v_1 - w_1, v_2 - w_2)$ is a positive solution of the SSS (1.1). This completes the proof.

Lemma 3.3. *Assume that $(H_1) - (H_6)$ hold. Then $F : P \to P$ is well defined. Moreover, $F : P \to P$ is a completely continuous operator.*

Proof. For any fixed $(u_1, u_2) \in P$, choose $0 < d_1, d_2 < 1$ such that $d_1\|u_1\| < 1$ and $d_2\|u_2\| < 1$, then for $t \in [a,b]$, we get

$$[u_i(t) - w_i(t)]^* \leq u_i(t) \leq \|u_i\| < \frac{1}{d_i} \leq \max\{\frac{1}{d_1}, \frac{1}{d_2}\} = d, \quad i = 1,2.$$

Thus, we get

$$0 \leq [u_i(t) - w_i(t)]^* \leq d, \quad i = 1,2,$$

from which, the assumption (H_6), the properties of ϕ_1, ϕ_2 and Lemma 2.6, for any $t \in [a,b]$, we have

$$\begin{aligned} F_i(u_1,u_2)(t) = &\int_a^b G(t,s)[f_i(s, [u_1(s) - w_1(s)]^*, [u_2(s) - w_2(s)]^*) + h_{i_+}(s)]\nabla s + A(f_i + h_{i_+})\phi_1(t) \\ &+ B(f_i + h_{i_+})\phi_2(t) \\ \leq &\int_a^b G(s,s)(N_i + N_i h_{i_+}(s))\nabla s + A\phi_1(t)\int_a^b G(s,s)(N_i + N_i h_{i_+}(s))\nabla s \\ &+ B\phi_2(t)\int_a^b G(s,s)(N_i + N_i h_{i_+}(s))\nabla s \\ \leq &N_i \int_a^b G(s,s)(1 + h_{i_+}(s))\nabla s + AN_i\phi_1(b)\int_a^b G(s,s)(1 + h_{i_+}(s))\nabla s \end{aligned}$$

$$+BN_i\phi_2(a)\int_a^b G(s,s)(1+h_{i_+}(s))\nabla s$$
$$= N_i(1+A\phi_1(b)+B\phi_2(a))\int_a^b G(s,s)(1+h_{i_+}(s))\nabla s<\infty,\quad i=1,2,$$

where $N_i = \max_{t\in[a,b],u_1,u_2\in[0,d]} f_i(t,u_1,u_2)+1,\quad i=1,2$. Thus $F:P\to E$ is well defined.

Now we shall prove that $F(P)\subseteq P$. For any $(u_1,u_2)\in P$, let $(v_1(t),v_2(t)) = F(u_1,u_2)(t)$. Then for $t\in[a,b]$, we get

$$v_i(t) = \int_a^b G(t,s)[f_i(s,[u_1(s)-w_1(s)]^*,[u_2(s)-w_2(s)]^*)+h_{i_+}(s)]\nabla s + A(f_i+h_{i_+})\phi_1(t)$$
$$+B(f_i+h_{i_+})\phi_2(t)$$
$$\le \int_a^b G(s,s)[f_i(s,[u_1(s)-w_1(s)]^*,[u_2(s)-w_2(s)]^*)+h_{i_+}(s)]\nabla s + A(f_i+h_{i_+})\phi_1(b)$$
$$+B(f_i+h_{i_+})\phi_2(a)$$

and so

$$\|v_i\| \le \int_a^b G(s,s)[f_i(s,[u_1(s)-w_1(s)]^*,[u_2(s)-w_2(s)]^*)+h_{i_+}(s)]\nabla s + A(f_i+h_{i_+})\phi_1(b)$$
$$+B(f_i+h_{i_+})\phi_2(a).$$

From Lemma 2.3, for $t\in[a,b]$, we obtain

$$v_i(t) = \int_a^b G(t,s)[f_i(s,[u_1(s)-w_1(s)]^*,[u_2(s)-w_2(s)]^*)+h_{i_+}(s)]\nabla s + A(f_i+h_{i_+})\phi_1(t)$$
$$+B(f_i+h_{i_+})\phi_2(t)$$
$$\ge g(t)\int_a^b G(s,s)[f_i(s,[u_1(s)-w_1(s)]^*,[u_2(s)-w_2(s)]^*)+h_{i_+}(s)]\nabla s + A(f_i+h_{i_+})\frac{\phi_1(t)}{\phi_1(b)}\phi_1(b)$$
$$+B(f_i+h_{i_+})\frac{\phi_2(t)}{\phi_2(a)}\phi_2(a)$$
$$\ge g(t)\int_a^b G(s,s)[f_i(s,[u_1(s)-w_1(s)]^*,[u_2(s)-w_2(s)]^*)+h_{i_+}(s)]\nabla s + A(f_i+h_{i_+})g(t)\phi_1(b)$$
$$+B(f_i+h_{i_+})g(t)\phi_2(a)$$
$$= g(t)[\int_a^b G(s,s)[f_i(s,[u_1(s)-w_1(s)]^*,[u_2(s)-w_2(s)]^*)+h_{i_+}(s)]\nabla s + A(f_i+h_{i_+})\phi_1(b)$$
$$+B(f_i+h_{i_+})\phi_2(a)]$$
$$\ge g(t)\|v_i\|,\quad i=1,2.$$

This yields that $F(P)\subseteq P$.

Let $D \subset P$ be any bounded set. Then there exists a constant $M > 0$ such that $\|u_i\| \le M$, $i = 1,2$ for any $(u_1,u_2) \in D$. Furthermore for any $(u_1,u_2) \in D$ and $t \in [a,b]$, we find

$0 \le [u_i(t) - w_i(t)]^* \le u_i(t) \le \|u_i\| \le M, \quad i = 1,2.$

Thus, by (H_6) and Lemma 2.6, for any $t \in [a,b]$, we have

$$\begin{aligned}
F_i(u_1,u_2)(t) = &\int_a^b G(t,s)[f_i(s,[u_1(s)-w_1(s)]^*,[u_2(s)-w_2(s)]^*)+h_{i_+}(s)]\nabla s + A(f_i+h_{i_+})\phi_1(t)\\
&+B(f_i+h_{i_+})\phi_2(t)\\
\le &\int_a^b G(s,s)(M_i+M_ih_{i_+}(s))\nabla s + A\phi_1(t)\int_a^b G(s,s)(M_i+M_ih_{i_+}(s))\nabla s\\
&+B\phi_2(t)\int_a^b G(s,s)(M_i+M_ih_{i_+}(s))\nabla s\\
\le &M_i\int_a^b G(s,s)(1+h_{i_+}(s))\nabla s + AM_i\phi_1(b)\int_a^b G(s,s)(1+h_{i_+}(s))\nabla s\\
&+BM_i\phi_2(a)\int_a^b G(s,s)(1+h_{i_+}(s))\nabla s\\
= &M_i(1+A\phi_1(b)+B\phi_2(a))\int_a^b G(s,s)(1+h_{i_+}(s))\nabla s < \infty, \quad i = 1,2,
\end{aligned}$$

where $M_i = \max\limits_{t\in[a,b],u_1,u_2\in[0,M]} f_i(t,u_1,u_2)+1, \quad i = 1,2$. Therefore $F(D)$ is uniformly bounded.

Similarly, we can easily find $F(D)$ is equicontinuous on $[a,b]$. Thus from the Ascoli-Arzela Theorem, we know that $F(D)$ is a relatively compact set.

Finally, from the continuity of $f_i, i = 1,2$, it is not difficult to check that $F : P \to P$ is continuous. Hence $F : P \to P$ is a completely continuous operator.

Now, we want to give the main result of this paper. To prove the main theorem, we need the following assumptions for the functions f_i, i= 1, 2.

(H_7) There exist $t_1,t_2 \in (a,b)$ such that

$$\lim_{u,v\to\infty}\min_{t\in[t_1,t_2]}\frac{f_i(t,u,v)}{|u|+|v|} = \infty$$

uniformly on $[t_1,t_2]$,

(H_8) There exist $t_1,t_2 \in (a,b)$ such that

$$\lim_{u,v\to0^+}\min_{t\in[t_1,t_2]}\frac{f_i(t,u,v)}{|u|+|v|} = \infty$$

uniformly on $[t_1,t_2]$.

Theorem 3.4. *Let $(H_1)-(H_8)$ hold. For each r satisfying*

$r > \max\{2C_1, 2C_2, K_i(1+A\phi_1(b)+B\phi_2(a))\int_a^b G(s,s)(1+h_{i_+}(s))\nabla s\}$,

where

$K_i = \max\{f_i(t,u_1,u_2)+1 : (t,u_1,u_2) \in [a,b]\times[0,r]\times[0,r]\}$ *and* C_i *is given in* (3.1)

then the SSS (1.1) *has two positive solutions* $(\tilde{u_1},\tilde{u_2})$ *and* $(\hat{u_1},\hat{u_2})$ *such that*

$0 < \|\hat{u_i}\| < r < \|\tilde{u_i}\|, \quad i = 1,2.$

Proof. Assume that there exist $\lambda_0 \geq 1$ and $(\tilde{u_1},\tilde{u_2}) \in \partial P_r$ such that $F(\tilde{u_1},\tilde{u_2}) = \lambda_0(\tilde{u_1},\tilde{u_2})$ where $P_r = \{(u_1,u_2)\in P : \|u_1\| < r, \|u_2\| < r\}$. Then $\frac{1}{\lambda_0}(F_1(\tilde{u_1},\tilde{u_2}),F_2(\tilde{u_1},\tilde{u_2})) = (\tilde{u_1},\tilde{u_2})$ and $0 < \frac{1}{\lambda_0} \leq 1$. Moreover for $t \in [a,b]$, we obtain

$0 \leq [\tilde{u_i}(t)-w_i(t)]^* \leq \tilde{u_i}(t) \leq \|\tilde{u_i}\| = r, \quad i = 1,2.$

Now, using Lemma 2.1 and 2.6 and the properties of the operators A, B, for $t \in [a,b]$, we get

$$
\begin{aligned}
\tilde{u_i}(t) &= \frac{1}{\lambda_0}\{\int_a^b G(t,s)[f_i(s,[\tilde{u_1}(s)-w_1(s)]^*,[\tilde{u_2}(s)-w_2(s)]^*)+h_{i_+}(s)]\nabla s + A(f_i+h_{i_+})\phi_1(t) \\
&\quad + B(f_i+h_{i_+})\phi_2(t)\} \\
&\leq \int_a^b G(s,s)[f_i(s,[\tilde{u_1}(s)-w_1(s)]^*,[\tilde{u_2}(s)-w_2(s)]^*)+h_{i_+}(s)]\nabla s + A(f_i+h_{i_+})\phi_1(t) \\
&\quad + B(f_i+h_{i_+})\phi_2(t) \\
&\leq \int_a^b G(s,s)[f_i(s,[\tilde{u_1}(s)-w_1(s)]^*,[\tilde{u_2}(s)-w_2(s)]^*)+h_{i_+}(s)]\nabla s \\
&\quad + A\phi_1(t)\int_a^b G(s,s)(K_i+K_ih_{i_+}(s))\nabla s + B\phi_2(t)\int_a^b G(s,s)(K_i+K_ih_{i_+}(s))\nabla s \\
&\leq K_i\int_a^b G(s,s)(1+h_{i_+}(s))\nabla s + A\phi_1(b)K_i\int_a^b G(s,s)(1+h_{i_+}(s))\nabla s \\
&\quad + B\phi_2(a)K_i\int_a^b G(s,s)(1+h_{i_+}(s))\nabla s \\
&= K_i(1+A\phi_1(b)+B\phi_2(a))\int_a^b G(s,s)(1+h_{i_+}(s))\nabla s, \quad i = 1,2.
\end{aligned}
$$

Thus, we get

$$r \leq K_i(1+A\phi_1(b)+B\phi_2(a))\int_a^b G(s,s)(1+h_{i_+}(s))\nabla s, \quad i = 1,2,$$

and so

$$\frac{r}{K_i(1+A\phi_1(b)+B\phi_2(a))} \leq \int_a^b G(s,s)(1+h_{i_+}(s))\nabla s, \quad i = 1,2.$$

This is a contradiction. Then by Theorem 3.1, we have

$$i(F,P_r,P) = 1. \tag{3.5}$$

On the other hand, let us choose the constant K such that

$$K > (\inf_{t\in[t_1,t_2]} g(t) \max_{t\in[a,b]} \int_{t_1}^{t_2} G(t,s)\nabla s)^{-1}.$$

In view of (H_7), there exists $N > 0$ such that

$$f_i(t,u_1,u_2) \geq K(u_1+u_2), \quad u_1 \geq N, u_2 \geq N \text{ and } t \in [t_1,t_2], \quad i = 1,2.$$

Now, set

$$R = r + 2N(\inf_{t\in[t_1,t_2]} g(t))^{-1}. \tag{3.6}$$

Now,we show that $F(u_1,u_2) \nleq (u_1,u_2)$ for any $(u_1,u_2) \in \partial P_R$. In fact, otherwise, there exists $(\hat{u_1},\hat{u_2}) \in \partial P_R$ such that $(\hat{u_1},\hat{u_2}) \geq F(\hat{u_1},\hat{u_2}) = (F_1(\hat{u_1},\hat{u_2}),F_2(\hat{u_1},\hat{u_2}))$.

From (3.1) and the fact that $(\hat{u_1},\hat{u_2}) \in \partial P_R$, for $t \in [a,b]$, we get

$$w_i(t) \leq C_i g(t) \leq C_i \frac{\hat{u}_i(t)}{R}, \quad i = 1,2, \tag{3.7}$$

and noting that $R > r > \max\{2C_1,2C_2\}$, from (3.6) and (3.7), for all $t \in [t_1,t_2]$, we obtain

$$\hat{u}_i(t) - w_i(t) \geq (1-\frac{C_i}{R})\hat{u}_i(t) \geq \frac{1}{2}\hat{u}_i(t) \geq \frac{1}{2}g(t)R \geq \frac{R}{2}\inf_{t\in[t_1,t_2]} g(t) > N > 0, \quad i = 1,2.$$

Considering this, for $t \in [a,b]$, we get

$$\begin{aligned}
\hat{u}_i(t) \geq & \int_a^b G(t,s)[f_i(s,[\hat{u_1}(s)-w_1(s)]^*,[\hat{u_2}(s)-w_2(s)]^*)+h_{i_+}(s)]\nabla s + A(f_i+h_{i_+})\phi_1(t) \\
& +B(f_i+h_{i_+})\phi_2(t) \\
\geq & \int_{t_1}^{t_2} G(t,s)f_i(s,[\hat{u_1}(s)-w_1(s)]^*,[\hat{u_2}(s)-w_2(s)]^*)\nabla s \\
= & \int_{t_1}^{t_2} G(t,s)f_i(s,\hat{u_1}(s)-w_1(s),\hat{u_2}(s)-w_2(s))\nabla s \\
\geq & \int_{t_1}^{t_2} G(t,s)K(\hat{u_1}(s)-w_1(s)+\hat{u_2}(s)-w_2(s))\nabla s \\
\geq & \int_{t_1}^{t_2} G(t,s)KR\inf_{t\in[t_1,t_2]} g(t)\nabla s
\end{aligned}$$

and so

$$R \geq KR\inf_{t\in[t_1,t_2]} g(t) \max_{t\in[a,b]} \int_{t_1}^{t_2} G(t,s)\nabla s.$$

That is,

$$K \leq (\inf_{t\in[t_1,t_2]} g(t) \max_{t\in[a,b]} \int_{t_1}^{t_2} G(t,s)\nabla s)^{-1}.$$

This contradicts the K that we choose. So from Theorem 3.1, we get

$$i(F,P_R,P) = 0. \tag{3.8}$$

Next, let L be a positive real number such that

$$L > (\inf_{t\in[t_1,t_2]} g(t) \max_{t\in[a,b]} \int_{t_1}^{t_2} G(t,s)\nabla s)^{-1}.$$

In view of (H_8), there exists $c(\max\{C_1,C_2\} < c < r)$ such that

$$f_i(t,u_1,u_2) \geq L(u_1+u_2), \quad u_i \in [0,c], t \in [t_1,t_2], \quad i = 1,2.$$

We show that $F(u_1,u_2) \nleq (u_1,u_2)$ for any $(u_1,u_2) \in \partial P_c$. In fact, otherwise, there exists $(\check{u}_1,\check{u}_2) \in \partial P_c$ such that $(\check{u}_1,\check{u}_2) \geq F(\check{u}_1,\check{u}_2) = (F_1(\check{u}_1,\check{u}_2),F_2(\check{u}_1,\check{u}_2))$. Moreover

$$0 \leq [\check{u}_i(t)-w_i(t)]^* \leq \check{u}_i(t) \leq \|\check{u}_i\| = c, \quad i = 1,2,$$

From (3.1) and the fact that $(\check{u}_1,\check{u}_2) \in \partial P_c$, for $t \in [a,b]$, we get

$$w_i(t) \leq C_i g(t) \leq C_i \frac{\check{u}_i(t)}{c}, \quad i = 1,2,$$

this implies, for all $t \in [t_1,t_2]$,

$$\check{u}_i(t)-w_i(t) \geq (1-\frac{C_i}{c})\check{u}_i(t) \geq \frac{1}{2}\check{u}_i(t) \geq \frac{1}{2}g(t)c \geq \frac{c}{2}\inf_{t\in[t_1,t_2]} g(t) > 0, \quad i = 1,2. \tag{3.9}$$

Considering (3.9), we get

$$\begin{aligned}
\check{u}_i(t) \geq F_i(\check{u}_1,\check{u}_2)(t) &= \int_a^b G(t,s)[f_i(s,[\check{u}_1(s)-w_1(s)]^*,[\check{u}_2(s)-w_2(s)]^*)+h_{i_+}(s)]\nabla s \\
&\quad +A(f_i+h_{i_+})\phi_1(t)+B(f_i+h_{i_+})\phi_2(t) \\
&\geq \int_{t_1}^{t_2} G(t,s)f_i(s,[\check{u}_1(s)-w_1(s)]^*,[\check{u}_2(s)-w_2(s)]^*)\nabla s \\
&= \int_{t_1}^{t_2} G(t,s)f_i(s,\check{u}_1(s)-w_1(s),\check{u}_2(s)-w_2(s))\nabla s \\
&\geq \int_{t_1}^{t_2} G(t,s)L(\check{u}_1(s)-w_1(s)+\check{u}_2(s)-w_2(s))\nabla s \\
&\geq \int_{t_1}^{t_2} G(t,s)Lc \inf_{t\in[t_1,t_2]} g(t)\nabla s,
\end{aligned}$$

and so

$$c \geq Lc \inf_{t\in[t_1,t_2]} g(t) \max_{t\in[a,b]} \int_{t_1}^{t_2} G(t,s)\nabla s.$$

Thus we obtain

$$L \leq (\inf_{t\in[t_1,t_2]} g(t) \max_{t\in[a,b]} \int_{t_1}^{t_2} G(t,s)\nabla s)^{-1},$$

which is in contradiction to the choose of the constant L. Hence applying Theorem 3.1, we obtain

$$i(F,P_c,P) = 0. \tag{3.10}$$

Therefore by $(3.5),(3.8)$ and (3.10), $c < r < R$, we have

$$i(F, P_R \backslash \overline{P_r}, P) = -1, \quad i(F, P_r \backslash \overline{P_c}, P) = 1.$$

Then F has a fixed point $(\tilde{u_1}, \tilde{u_2})$ in $P_R \backslash \overline{P_r}$ and a fixed point $(\hat{u_1}, \hat{u_2})$ in $P_r \backslash \overline{P_c}$. Obviously,

$$c < \|\hat{u_i}\| < r < \|\tilde{u_i}\| < R, \quad i = 1,2.$$

Finally we will show that $\hat{u_i}(t) \geq w_i(t)$ and $\tilde{u_i}(t) \geq w_i(t), \quad i = 1,2.$

By (3.1), we get

$$\hat{u_i}(t) \geq g(t)\|\hat{u_i}\| > cg(t) > 2C_i g(t) \geq 2w_i(t), \quad i = 1,2,$$

$$\tilde{u_i}(t) \geq g(t)\|\tilde{u_i}\| > rg(t) > 2C_i g(t) \geq 2w_i(t), \quad i = 1,2.$$

By Lemma 3.1, we know that $(\hat{u_1} - w_1, \hat{u_2} - w_2)$ and $(\tilde{u_1} - w_1, \tilde{u_2} - w_2), \quad i = 1,2$ are the positive solutions of the SSS (1.1).

Example 3.5. *Let* $\mathrm{T} = \{2^k : k \in Z\} \cup \{0\}$. *Consider the following SSS,*

$$\begin{cases} -u_i^{\triangle\nabla}(t) = f_i(t, u_1(t), u_2(t)) + h_i(t), \quad t \in (0,1), \quad i = 1,2, \\ u_i(0) - u_i^{\triangle}(0) = u_i(1) + u_i^{\triangle}(1) = 0, \quad i = 1,2. \end{cases}$$

Let $f_1(t, u_1, u_2) = t^2(1-t)u_1^{\frac{3}{2}} u_2^2 + \sqrt{u_1}$, $h_1(t) = -t$, $f_2(t, u_1, u_2) = \frac{4}{250} t(1-t) u_1^{\frac{3}{2}} + \frac{1}{10^2}\sqrt{u_1 + u_2}$, $h_2(t) = -t^2$.

Clearly f_1 *and* f_2 *satisfy the conditions* (H_7) *and* (H_8). *We easily calculate the followings;*

$$\int_0^1 G(s,s) h_{1_-}(s) \nabla s = \int_0^1 G(s,s) s \nabla s = \frac{1}{3}\int_0^1 (1+s)(2-s) s \nabla s = \frac{16}{35},$$

$$\int_0^1 G(s,s) h_{2_-}(s) \nabla s = \int_0^1 G(s,s) s^2 \nabla s = \frac{1}{3}\int_0^1 (1+s)(2-s) s^2 \nabla s = \frac{3776}{9765},$$

$$\int_0^1 h_{1_-}(s)\nabla s = \int_0^1 s \nabla s = \frac{2}{3}, \ \int_0^1 h_{2_-}(s)\nabla s = \int_0^1 s^2 \nabla s = \frac{4}{7},$$

$$\int_0^1 G(s,s)(1 + h_{i_+}(s))\nabla s = \frac{1}{3}\int_0^1 (1+s)(2-s)\nabla s = \frac{44}{63} \textit{ for } i = 1,2,$$

$$C_1 = \frac{1}{3}\int_0^1 h_{1_-}(s)\nabla s \phi_1(1)\phi_2(0) = \frac{8}{9}, \ C_2 = \frac{1}{3}\int_0^1 h_{2_-}(s)\nabla s \phi_1(1)\phi_2(0) = \frac{16}{21},$$

$K_1 = \max\{t^2(1-t)u_1^{\frac{3}{2}} u_2^2 + \sqrt{u_1} + 1)\} = \frac{1}{12} r^{\frac{7}{2}} + r^{\frac{1}{2}} + 1$ *for* $(t, u_1, u_2) \in [0,1] \times [0,r] \times [0,r]$

and

$K_2 = \max\{\frac{4}{250} t(1-t) u_1^{\frac{3}{2}} + \frac{1}{10^2}\sqrt{u_1 + u_2} + 1\} = \frac{1}{250} r^{\frac{3}{2}} + \frac{1}{10^2}(2r)^{\frac{1}{2}} + 1$ *for* $(t, u_1, u_2) \in [0,1] \times [0,r] \times [0,r]$.

If we choose $r = \frac{17}{9}$, *we have* $r > \max\{\frac{16}{9}, \frac{32}{21}, K_1 \frac{44}{63}, K_2 \frac{44}{63}\}$.

Then, by Theorem (3.2)*, the dynamic system has two positive solutions* $(\tilde{u_1}, \tilde{u_2})$ *and* $(\hat{u_1}, \hat{u_2})$ *such that* $0 < \|\hat{u}_i\| < \dfrac{17}{9} < \|\tilde{u}_i\|, \quad i = 1, 2.$

References

[1] R. Anderson, P. J. Y. Wong, Positive solutions for second-order semipositone problems on time scales, Comput. Math. Appl. 58 (2009), 281-291.

[2] D. R. Anderson, C. Zhai, Positive solutions to semi-positone second-order three-point problems on time scales, Appl. Math. Comput. 215 (2010), 3713-3720.

[3] R. Aris, Introduction to the Analysis of Chemical Reactors, Prentice-Hall, Englewood Cliffs, NJ, 1965.

[4] N.A. Asif, R.A. Khan, Positive solutions to singular system with four-point coupled boundary conditions, J. Math. Anal. Appl. 386 (2012), 848-861.

[5] M. Bohner, A. Peterson, Dynamic Equations on Time Scales, An Introduction with Applications, Birkhauser, Boston, Cambridge, MA, 2001.

[6] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Scales, Birkh*ä*user, Boston, Cambridge, MA, 2003.

[7] E. Cetin, S. G. Topal, Existence of multiple positive solutions for the system of higher order boundary value problems on time scales, Math. Comput. Modelling 52 (2010), 1-11.

[8] J. Henderson, R. Luca, Existence and multiplicity for positive solutions of a system of higher-order multipoint boundary value problems, Nonlinear Differ. Equ. Appl. 20 (2013), 1035-1054.

[9] M. Joao, L. Sebastian, U. Pedro, Local superlinearity for elliptic system involving parameters, J. Diff. Equat. 211 (2005), 1-19.

[10] R. Ma, Multiple nonnegative solutions of second-order systems of boundary value problems, Nonlinear Anal. 42 (2000), 1003-1010.

[11] W.T. Li, H.R. Sun, Multiple positive solutions for nonlinear dynamical systems on a measure chain, J. Comput. Appl. Math. 162 (2004), 421-430.

[12] Y. Liu, B. Yan, Multiple solutions of singular boundary value problems for differential systems, J. Math. Anal. Appl. 287 (2003), 540-556.

[13] L. Liu, X. Zhang, Y. Wu, On existence of positive solutions of a two-point boundary value problem for a nonlinear singular semipositone system, Appl. Math. Comput. 192 (2007), 223-232.

[14] K.R. Prasad, A.K. Rao and B. Bharathi, Positive solutions for system of 2n-th order Sturm Liouville boundary value problems on time scales, Proc. Indian Acad. Sci. (Math. Sci.) 124 (2014), 67-79

[15] H. Wang, Multiplicity of positive radical solutions for an elliptic systems on an annulus, Nonlinear Anal. 42 (2002), 803-811.

[16] H. Wang, On the number of positive solutions of nonlinear systems, J. Math. Anal. Appl. 281 (2003), 287-306.

[17] Y. Yang, F. Meng, Positive solutions for the singular semipositone boundary value problem on time scales, Math. Comput. Modelling 52 (2010), 481-489.

[18] I. Yaslan Karaca, Multiple positive solutions for dynamic m-point boundary value problems, Dyn. Sys. Appl. 17 (2008), 25-42.

[19] Y. Zhou, Y. Xu, Positive solutions of three-point boundary value problems for systems of nonlinear second order ordinary differential equations, J. Math. Anal. Appl. 320 (2006), 78-590.

[20] J. Zhao, H. Lian, W. Ge, Existence of positive solutions for nonlinear m -point boundary value problems on time scales, Bound. Value Probl. 2012 (2012), Article ID 4.