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# EXISTENCE OF PERIODIC SOLUTIONS OF SECOND-ORDER NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS WITH VARIABLE DELAY

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**Abstract.** In this work, we use fixed point theorems and choose available operators to prove the existence of periodic solutions of a second-order nonlinear delay integro-differential equations (1). During the process Eq (1) is converted into an equivalent integral equation but with the same properties from which appropriate mappings are constructed. We offer existence criteria based on sufficient conditions on $f$, $p$, $q$ to conclude existence of periodic solutions.

**Keywords.** Fixed points; Periodic solution; Integro-differential equations; Variable delay.

## 1. Introduction

Ordinary and partial differential equations have long played important roles in the history of theoretical population dynamics, and they will, with no doubt, continue to serve as indispensable tools in future investigations. However, they are generally the first approximations of the considered real systems. More realistic models should include some of the past states of these systems; that is, ideally, a real system should be modeled by differential equations with time delays. Indeed, the use of delay differential equations (DDEs) in the modeling of population dynamics is currently very active, largely due to the recent rapid progress achieved in

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the understanding of the dynamics of several important classes of delay differential equations and systems. In this paper, we are interested in the analysis of qualitative theory of periodic solutions of delay differential equations. Motivated by the papers [1]–[4], [6]–[22] and the references therein, we restrict our attention to integro-differential equations as (1). The purpose of this paper is to deal with the existence of periodic solutions for the second-order nonlinear integro-differential equations with variables delay

$$x''(t)+p(t)x'(t)+q(t)x^3(t)=\int_{-\infty}^{t}D(t,s)f(s,x(s-\tau(s)))\,ds, \tag{1}$$

where

(I) $p,q\in C(\mathbb{R},\mathbb{R}^+)$ and $\tau\in C(\mathbb{R},\mathbb{R})$. $p$, $q$ and $\tau$ are all $T$-periodic continuous functions with $T>0$ is a constant.

(II) $D:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ for $t\geqslant s$, and $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous functions. where $\tau$ is a continuous scalar function, and $\tau(t)\geqslant\tau^*>0$.

(III) For all a sequence $x_n\to x$ in $C_T$ implies that $|f(t,x_n)-f(t,x)|\to 0$ uniformly for $t\in\mathbb{R}$ as $n\to\infty$.

## 2. Preliminaries

To describe the main result we use the following notation. For $T>0$, let $C_T$ be the set of all continuous scalar functions $x$, periodic in $t$ of period $T$. Then, $(C_T,\|\cdot\|)$ is a Banach space with the supremum norm

$$\|x\|=\sup_{t\in\mathbb{R}}|x(t)|=\sup_{t\in[0,T]}|x(t)|. \tag{2}$$

We let

$$C_T^J:=\{\varphi\in C_T:\|\varphi\|\leqslant J\}. \tag{3}$$

Since we are searching for the existence of periodic solutions for system (1), it is natural to assume that

$$D(t+T,s+T)=D(s,t)\ \text{ and } f(t+T,x)=f(t,x). \tag{4}$$

Also, we assume

$$\int_0^T p(u)\,du > 0 \text{ and } \int_0^T q(u)\,du > 0. \tag{5}$$

Throughout this section we assume that there exists a continuous function $F_J(t) \geqslant |f(t,x)|$ for $x \in C_T^J$ and a constant $E_1 > 0$ such that

$$\int_{-\infty}^t |D(t,s)F_J(s)|\,ds \leqslant E_1,\ E_1 \leqslant \frac{J}{\alpha T}, \tag{6}$$

with $\alpha$ to be defined later. In order to simplify notation, we define

$$\sigma = \max\{q(t),\ 0 \leqslant t \leqslant T\}, \tag{7}$$

and let

$$R_1 = \max_{t\in[0,T]} \left| \int_t^{t+T} \frac{\exp\int_t^s p(u)\,du}{\exp\left(\int_0^T p(u)\,du\right) - 1} q(s)\,ds \right|,\ Q_1 = \left(1 + \exp\int_0^T p(u)\,du\right)^2 R_1^2. \tag{8}$$

The next lemmas is crucial to our results.

**Lemma 2.1.** ([16]) *Suppose that (I), (4) and (5) hold and that*

$$\frac{R_1\left(\exp\left(\int_0^T p(u)\,du\right) - 1\right)}{Q_1 T} \geqslant 1. \tag{9}$$

*Then there are continuous $T-$periodic functions $a$ and $b$ such that $b(t) > 0$, $\int_0^T a(u)\,du > 0$ and*

$$a(t) + b(t) = p(t) \text{ and } b'(t) + b(t)a(t) = q(t) \text{ for } t \in \mathbb{R}.$$

**Lemma 2.2.** ([21]) *Denote, $A = \int_0^T p(v)\,dv$, $B = T^2 \exp\left(\frac{1}{T}\int_0^T \ln q(v)\,dv\right)^2$ and consider*

$$A^2 \geqslant 4B. \tag{10}$$

*It can be shown that*

$$\begin{aligned}
\min\left\{\int_0^T a(v)\,dv, \int_0^T b(v)\,dv\right\} &\geqslant \frac{1}{2}\left(A - \sqrt{A^2 - 4B}\right) := l, \\
\max\left\{\int_0^T a(v)\,dv, \int_0^T b(v)\,dv\right\} &\leqslant \frac{1}{2}\left(A - \sqrt{A^2 - 4B}\right) := m.
\end{aligned}$$

**Lemma 2.3.** ([21]) *Suppose the conditions of Lemma 2.1 hold and that $\phi \in C_T$. Then the equation*

$$x''(t) + p(t)x'(t) + q(t)g(x(t)) = \phi(s),$$

*has a $T-$periodic solution. Moreover, the periodic solution can be expressed by*

$$x(t) = \int_t^{t+T} G(t,s)\phi(s)\,ds,$$

*where*

$$G(t,s) = \frac{\int_t^s \left(e^{\int_t^u b(v)dv} + e^{\int_u^s a(v)dv}\right)du + \int_s^{t+T}\left(e^{\int_t^u b(v)dv} + e^{\int_u^{s+T} a(v)dv}\right)du}{\left(\exp\left(\int_0^T a(u)\,du\right) - 1\right)\left(\exp\left(\int_0^T b(u)\,du\right) - 1\right)}.$$

*Moreover, the Green's function G satisfies the following properties*

$$G(t,t+T) = G(t,t),\ G(t+T,s+T) = G(t,s), \tag{11}$$

*and*

$$\frac{T}{(e^m-1)^2} \leqslant G(t,s) \leqslant \frac{T\exp\left(\int_0^T p(u)du\right)}{\left(e^l-1\right)^2} = \alpha. \tag{12}$$

*Furthermore*

$$\frac{\partial}{\partial t}G(t,s) = -b(t)G(t,s) + K(t,s), \tag{13}$$

*with* $K(t,s) = \frac{\exp\left(\int_t^s a(v)dv\right)}{\exp\left(\int_0^T a(v)dv\right)-1}$.

The following lemma is fundamental to our results.

**Lemma 2.4.** *Assume that the hypotheses of Lemma 2.1 hold. Then, $x$ is $T-$periodic solution of (1) if and only if $x$ is a solution of the integral equation*

$$\begin{aligned} x(t) = &\int_t^{t+T} G(t,s)q(s)H(x(s))\,ds \\ &+ \int_t^{t+T} G(t,s)\int_{-\infty}^s D(s,u)f(u,x(u-\tau(u)))\,duds. \end{aligned} \tag{14}$$

**Proof.** In the proof we may assume that $x \in C_T$ and we choose $H(x) = x - x^3$ so we first rewrite (1) as

$$x''(t) + p(t)x'(t) + q(t)x(t) = q(t)H(x(t)) + \int_{-\infty}^t D(t,s)f(s,x(s-\tau(s)))\,ds.$$

Then, from Lemma 2.3 it is easy to see that there is a solution $x \in C_T$ of (1), which can be expressed by

$$x(t) = \int_t^{t+T} G(t,s)\left(q(s)H(x(s)) + \int_{-\infty}^{s} D(s,u) f(u, x(u-\tau(u)))\,du\right) ds.$$

Define operators $P_1, P_2 : C_T \to C_T$ by

$$(P_1 x)(t) := \int_t^{t+T} G(t,s)\, q(s) H(x(s))\, ds, \tag{15}$$

and

$$(P_2 x)(t) := \int_t^{t+T} G(t,s) \int_{-\infty}^{s} D(s,u) f(u, x(u-\tau(u)))\, du ds. \tag{16}$$

In view of (14), (15), (16) and the above analysis, the existence of periodic solutions for (1) is equivalent to the existence of solutions for the operator equation

$$P_1 x + P_2 x = x \text{ in } C_T. \tag{17}$$

**Definition 2.1.** ([5]) (Large Contraction) Let $(\mathscr{M}, d)$ be a metric space and consider $\mathscr{B} : \mathscr{M} \to \mathscr{M}$. Then $\mathscr{B}$ is said to be a large contraction if given $\phi, \varphi \in \mathscr{M}$ with $\phi \neq \varphi$ then $d(\mathscr{B}\phi, \mathscr{B}\varphi) \leqslant d(\phi, \varphi)$ and if for all $\varepsilon > 0$, there exists a $\delta > 1$ such that

$$[\phi, \varphi \in \mathscr{M},\, d(\phi, \varphi) \geqslant \varepsilon] \Longrightarrow d(\mathscr{B}\phi, \mathscr{B}\varphi) \leqslant \delta d(\phi, \varphi).$$

**Theorem 2.1.** ([5]) *Let $\mathscr{M}$ be a closed bounded convex nonempty subset of a Banach space $(X, \|\cdot\|)$. Suppose that $\mathscr{A}$ and $\mathscr{B}$ map $\mathscr{M}$ into $\mathscr{M}$ such that*

*(i) $x, y \in \mathscr{M}$, implies $\mathscr{A}x + \mathscr{B}y \in \mathscr{M}$,*

*(ii) $\mathscr{A}$ is compact and continuous,*

*(iii) $\mathscr{B}$ is a large contraction mapping.*

*Then there exists $z \in \mathscr{M}$ with $z = \mathscr{A}z + \mathscr{B}z$.*

**Proposition 2.1.** *If $\|\cdot\|$ is the maximum norm*

$$\mathscr{M} := \left\{ \varphi \in C_T : \|\varphi\| \leqslant \frac{1}{\sqrt{3}} \right\}.$$

*Let $\mathscr{B}$ be a mapping on $\mathscr{M}$ as follows for $\varphi \in \mathscr{M}$, $(\mathscr{B}\varphi)(t) := \varphi(t) - \varphi^3(t)$. Then $\mathscr{B}$ is a large contraction of the set $\mathscr{M}$.*

**Proof.** Indeed, let $\phi, \varphi \in \mathscr{M}$ we have for $t \in \mathbb{R}$

$$\begin{aligned} |(\mathscr{B}\varphi)(t) - (\mathscr{B}\phi)(t)| &= \left|\varphi(t) - \phi(t) - \left\{\varphi^3(t) - \phi^3(t)\right\}\right| \\ &= |(\varphi(t) - \phi(t))| \left|1 - \left\{\varphi^2(t) + \phi^2(t) + |\phi(t)\varphi(t)|\right\}\right|. \end{aligned} \tag{18}$$

On the other hand, we have

$$(\varphi(t) - \phi(t))^2 = \varphi^2(t) + \phi^2(t) - 2\phi(t)\varphi(t) \leqslant 2\left(\varphi^2(t) + \phi^2(t)\right).$$

It follows from $\varphi^2(t) + \phi^2(t) \leqslant 1$ that

$$|(\mathscr{B}\varphi)(t) - (\mathscr{B}\phi)(t)| \leqslant |\varphi(t) - \phi(t)| \left|1 - \frac{\left(\varphi^2(t) + \phi^2(t)\right)}{2}\right| \leqslant \|\varphi - \phi\|. \tag{19}$$

Now, let $\varepsilon \in (0,1)$ be given and let $\phi, \varphi \in \mathscr{M}$ with $\|\varphi - \phi\| \geqslant \varepsilon$.

Suppose that for some $t$ we have

$$\frac{\varepsilon}{2} \leqslant |\varphi(t) - \phi(t)|.$$

then

$$\varphi^2(t) + \phi^2(t) \geqslant \frac{\varepsilon^2}{8}.$$

From (19) we obtain that

$$\|\mathscr{B}\varphi - \mathscr{B}\phi\| \leqslant \|\varphi - \phi\| \left(1 - \frac{\varepsilon^2}{16}\right) \leqslant \|\varphi - \phi\|. \tag{20}$$

In order, for some $t$ we have

$$|\varphi(t) - \phi(t)| \leqslant \frac{\varepsilon}{2}.$$

So, for all $\phi, \varphi \in \mathscr{M}$

$$|(\mathscr{B}\varphi)(t) - (\mathscr{B}\phi)(t)| \leqslant \frac{1}{2}|\varphi(t) - \phi(t)| \leqslant \frac{1}{2}\|\varphi - \phi\|. \tag{21}$$

Thus from (20) and (21) we deduce that

$$\|\mathscr{B}\varphi - \mathscr{B}\phi\| \leqslant \max\left\{\frac{1}{2}, \left(1 - \frac{\varepsilon^2}{16}\right)\right\} \|\varphi - \phi\|.$$

The Proposition 2.1 is proved by letting $\delta = \max\left\{\frac{1}{2}, 1 - \frac{\varepsilon^2}{16}\right\}$.

We shall consider an example for Proposition 2.1.

**Proposition 2.2.** *For $P_1$ defined in (16), if in addition*

$$\alpha\sigma T \leqslant 1, \tag{22}$$

*then $P_1 : \mathscr{M} \to \mathscr{M}$ is a large contraction.*

**Proof.** Obviously, $P_1\varphi$ is continuous and it is easy to show that $P_1\varphi \in C_T$. So, for any $\varphi \in C_T$, we have

$$\begin{aligned} |(P_1\varphi)(t)| &= \int_t^{t+T} |G(t,s)||q(s)|\left|\varphi(s) - \varphi^3(s)\right| ds \\ &\leqslant \max_{0\leqslant t\leqslant T}\{|q(s)|\} \int_t^{t+T} |G(t,s)||\mathscr{B}\varphi(s)| ds \\ &\leqslant \alpha\sigma T\left\|\varphi - \varphi^3\right\|. \end{aligned}$$

Since, $\|\varphi\| \leqslant J$ and from (19) we have

$$|(P_1\varphi)(t)| \leqslant \alpha\sigma T\|\mathscr{B}\varphi\| \leqslant \alpha\sigma T\|\varphi\| \leqslant \alpha\sigma T J \leqslant J.$$

Thus $P_1\varphi \in \mathscr{M}$. Consequently, we have $P_1 : \mathscr{M} \to \mathscr{M}$. Now, let $\varepsilon \in (0,1)$ be given and let $\varphi, \phi \in \mathscr{M}$ with $\|\varphi - \phi\| \geqslant \varepsilon$. By (22) and from the proof of the Proposition 2.1 we have found $\delta < 1$ such that

$$|(P_1\varphi)(t) - (P_1\phi)(t)| \leqslant \alpha\sigma T\delta\|\varphi - \phi\| \leqslant \delta\|\varphi - \phi\|.$$

Then $\|P_1\varphi - P_1\phi\| \leqslant \delta\|\varphi - \phi\|$. Consequently, $P_1$ is a large contraction.

**Proposition 2.3.** *Assume the conditions of Lemmas 2.1, 2.2 hold. Suppose also that conditions (4), (6) hold. Then, $P_2 : \mathscr{M} \to \mathscr{M}$ is continuous and the image of $P_2$ is contained in a compact set, where $P_2$ is defined by (16).*

**Proof.** Let $P_2$ be defined by (16). Obviously, $P_2\varphi$ is continuous and it is easy to show that $(P_2\varphi)(t+T) = (P_2\varphi)(t)$. Observe that from (6) for $\varphi \in \mathscr{M}$ we have

$$\begin{aligned} |(P_2\varphi)(t)| &\leqslant \int_t^{t+T} |G(t,s)| \int_{-\infty}^{s} |D(s,u)||f(u,\varphi(u-\tau(u)))| du ds \\ &\leqslant \int_t^{t+T} |G(t,s)| \int_{-\infty}^{s} |D(s,u)||F_J(u)| du ds \\ &\leqslant \alpha E_1 T \leqslant J. \end{aligned} \tag{23}$$

That is, $P_2\varphi \in \mathscr{M}$. To see that $P_2$ is continuous on $C_T^J$ it suffices to show that for all sequence of points $\{\varphi_n\}_{n\geqslant 1}$ in $\mathscr{M}$ such that $\varphi_n \to \varphi \in \mathscr{M}$ as $n \to \infty$ implies that the sequence $(P_2\varphi_n)_{n\geqslant 1}$ converges to $(P_2\varphi) \in \mathscr{M}$ as $n \to \infty$ where $n$ is a positive integer. In fact that $\varphi_n \to \varphi \in \mathscr{M}$, so $P_2\varphi_n \in C_T$. Next, let $r > 0$ be given, since

$$\begin{aligned}
&\int_t^{t+T} |G(t,s)| \int_{s-r}^{s} |D(s,u)| \left| f(u, \varphi_n(u-\tau(u))) \right| du ds \\
&\leqslant \int_t^{t+T} |G(t,s)| \int_{s-r}^{s} |D(s,u)| F_J(u) du ds \\
&\leqslant \int_t^{t+T} |G(t,s)| \int_{-\infty}^{s} |D(s,u)| F_J(u) du ds \leqslant J, \qquad (24)
\end{aligned}$$

then, for all $n \geqslant 0$ and $r > 0$ we have

$$\int_t^{t+T} |G(t,s)| \int_{s-r}^{s} |D(s,u)| \left| f(u, \varphi_n(u-\tau(u))) \right| du ds \leqslant J. \qquad (25)$$

Thus from (III) and by the dominated convergence theorem we deduce that

$$\int_t^{t+T} |G(t,s)| \int_{s-r}^{s} |D(s,u)| \left| f(u, \varphi_n(u-\tau(u))) - f(u, \varphi(u-\tau(u))) \right| du ds \to 0,$$

as $n \to \infty$. That is, $P_2\varphi_n \to P_2\varphi \in \mathscr{M}$. To show that the image of $P_2$ is contained in a compact set, we calculate $\frac{d}{dt}(P_2\varphi_n)(t)$ and show that it is uniformly bounded. For that, by making use of (25), (13) and (11) we obtain by taking the derivative in, (16)

$$\begin{aligned}
\frac{d}{dt}(P_2\varphi_n)(t) &= \int_t^{t+T} \left( \frac{\partial}{\partial t} G(t,s) \right) \int_{-\infty}^{s} D(s,u) f(u, \varphi_n(u-\tau(u))) du ds \\
&= -b(t) \int_t^{t+T} G(t,s) \int_{-\infty}^{s} D(s,u) f(u, \varphi_n(u-\tau(u))) du ds \\
&+ \int_t^{t+T} K(t,s) \int_{-\infty}^{s} D(s,u) f(u, \varphi_n(u-\tau(u))) du ds. \\
&= -b(t)(P_2\varphi_n)(t) + \int_t^{t+T} K(t,s) \int_{-\infty}^{s} D(s,u) f(u, \varphi_n(u-\tau(u))) du ds.
\end{aligned}$$

where $K(t,s)$ is as given in (13). Thus the above expression yields

$$\left\| \frac{d}{dt}(P_2\varphi_n) \right\| = J \max\{b(t), 0 \leqslant t \leqslant T\} + TE_1 \frac{\exp\left(\int_0^T a(v)dv\right)}{\exp\left(\int_0^T a(v)dv\right) - 1} = E_2,$$

for some positive constant $E_2$. Thus the sequence $(P_2\varphi_n)$ is uniformly bounded and equicontinuous. Hence by Ascoli-Arzela's theorem the set $\{P_2\varphi : \varphi \in \mathscr{M}\}$ is equicontinuous and $P_2$ is continuous. So $P_2$ is compact operator on $\mathscr{M}$. Also form (23) $P_2 : \mathscr{M} \to \mathscr{M}$ we deduce that $\{P_2\varphi : \varphi \in \mathscr{M}\}$ is contained in a compact subset of $\mathscr{M}$.

## 2. Existence of periodic solutions

**Theorem 3.1.** *Let $(C_T, \|\cdot\|)$ be the Banach space of continuous $T-$periodic real valued functions and $\mathscr{M} := \left\{\varphi \in C_T : \|\varphi\| \leqslant 1/\sqrt{3}\right\}$, where $J = \frac{1}{\sqrt{3}}$. Suppose (3)–(5), (10) and (22) hold. If the condition (6) is replaced by*

$$E_1 \leqslant \frac{1}{3\alpha T} J, \tag{26}$$

*then, equation (1) has a $T-$periodic solution $x$ in the subset $\mathscr{M}$.*

**Proof.** By Proposition 2.3, the operator $P_2 : \mathscr{M} \to \mathscr{M}$ is compact and continuous. Also, from Proposition 2.2, the operator $P_1 : \mathscr{M} \to \mathscr{M}$ is a large contraction. Moreover, if $\varphi, \phi \in \mathscr{M}$, we see that$|(P_1\varphi)(t)| \leqslant \alpha\sigma T \|\mathscr{B}\varphi\| \leqslant \alpha\sigma T \|\varphi\|$

$$\begin{aligned}
\|P_1\varphi + P_2\phi\| &\leq \|P_1\varphi\| + \|P_2\phi\| \leqslant \alpha\sigma T \left\|\varphi - \varphi^3\right\| + \alpha E_1 T \\
&\leqslant \alpha\sigma T \frac{2}{3}\frac{1}{\sqrt{3}} + \alpha E_1 T \\
&\leqslant \frac{2}{3}J + \frac{1}{3}J = J,
\end{aligned}$$

because $\left\|\varphi - \varphi^3\right\| \leqslant \frac{2}{3}\frac{1}{\sqrt{3}}$. Thus $P_1\phi + P_2\varphi \in \mathscr{M}$. Clearly, all the hypotheses of the Theorem 2.1 are satisfied. Thus there exists a fixed point $x \in \mathscr{M}$ such that $P_1x + P_2x = x$. From (17) and by Lemma 2.4 this fixed point is a solution of (1). Hence the Eq (1) has a $T-$periodic solution.

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