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EXTENDING THE APPLICABILITY OF NEWTON'S METHOD FOR SECTIONS ON RIEMANNIAN MANIFOLDS USING RESTRICTED CONVERGENCE DOMAINS

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Abstract. We present a semilocal convergence analysis of Newton's method for sections on Riemannian manifolds. Using the notion of a 2- piece $L-$average Lipschitz condition introduced before in combination with our new idea of restricted convergence domains, we provide a tighter convergence analysis than in earlier studies.

Keywords. Newton's method; Riemannian manifold; Semilocal convergence; Convergence criterion.

1. Introduction

The solutions of eigenvalue problems, optimization problems with equality constraints, invariant subspaces computations ([1, 7, 13, 16–18, 21, 27, 28]) etc can rarely be found in closed form. So most solution methods for these problems are usually iterative. In particular for the preceding problems researchers compute solutions of a system of equations or they find singular points of a vector field on a Riemannian manifold.

The most popular iterative method is undoubtedly Newton's method [1, 2, 7, 13, 24]. Li and Wang in [24] provided a semi-local convergence analysis of Newton's method for sections on Riemannian manifolds. Their work extended earlier works such as [1, 2, 17] and is based on the

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concept of a 2-piece $L-$average Lipschitz condition (to be precised in Definition 3.1). In the present study we use our new idea of restricted convergence domains. That is we find a more precise location where the iterates lie leading to tighter majorizing functions. This way our semilocal convergence analysis has the following advantages over the work in [2,11,24,29,30]:

(a) Weaker sufficient convergence criteria;

(b) Tighter error estimates on the distance involved;

(c) An at least as precise information on the location of the solution.

These advantages are obtained under the same computational cost as in the preceding works.

The paper is organized as follows. Section 2 contains the necessary notions and earlier results about sections on Riemannian manifolds. In Section 3, we present the semilocal convergence analysis of Newton's methods. The special cases and applications are given in the concluding Section 4.

2. Preliminaries

In order for us to make this paper as self contained as possible we use some notions and results from [24](see also [7,11,14,16,18,25,28]).

"Let $\kappa \in \mathbb{N} \cup \{\infty, \omega\}$ and let M be a complete $m-$dimensional $C^{\kappa}-$Riemannian manifold with countable bases. Here C^{κ} is smooth or analytic when $\kappa = \infty$ or ω. Let $p \in M$ and let T_pM is the tangent space at p. The scalar product and corresponding norm on T_pM are denoted by $\langle .,. \rangle$ and $\|.\|$ respectively. The tangent bundle TM of M is defined as

$$TM := \cup_{p\in M} T_pM.$$

Recall that ([24]), a vector field X on M is a mapping from M to TM satisfying that $X(p) \in T_pM$ for each $p \in M$. Let $c : [0,1] \to M$ be a piecewise smooth curve connecting two distinct points $p, q \in M$. Then the arc-length of c is defined by $l(c) := \int_0^1 \|c'(t)\| dt$, and the Riemannian distance from p to q by $d(p,q) := \inf_c l(c)$, where the infimum is taken over all piecewise smooth curves $c : [0,1] \to M$ connecting p and q. Therefore, (M,d) is a complete metric space and the map $\exp_p : T_pM \to M$ is well- defined on T_pM ([24, p.425]). The curve $c : [0,1] \to M$ is a minimizing geodesic connecting p and q (i.e., its arc-length equals its Riemannian distance between p and q) if and only if there exists a vector $v \in T_pM$ such that $\|v\| = d(p,q)$ and $c(t) = \exp_p(tv)$ for each $t \in [0,1]$.

Let $\bigtriangledown$ denote the Levi-Civita connection on M, let $c : \mathbb{R} \to M$ be a C^κ-curve and let $P_{c,\cdot,\cdot}$ be the parallel transport on tangent bundle TM along c with respect to $\bigtriangledown$. Throughout this paper, we shall always assume that E and M are C^κ-manifolds.

Definition 2.1. ([24, Definition 2.1]) Let $\pi : E \to M$ be a $C^\kappa-$morphism. Then $\pi : E \to M$ is called a $C^\kappa-$vector bundle of rank $\hat{m}$ if the following conditions are satisfied.

(1) For each $p \in M$, $E_p := \pi^{-1}(p)$ is a real vector space of dimension $\hat{m}$.

(2) For each $p \in M$, there exist a neighborhood U of p and a $C^\kappa-$diffeomorphism $h : \pi^{-1}(U) \to U \times \mathbb{R}^{\hat{m}}$ such that, for each $q \in U$, $h(E_q) \subset \{q\} \times \mathbb{R}^{\hat{m}}$ and the mapping $h^q : E_p \to \mathbb{R}^{\hat{m}}$ defined by

$$h^q(x) = Proj \circ h(x) \text{ for each } x \in E_q \tag{2.1}$$

is a linear isomorphism, where $Proj : \{q\} \times \mathbb{R}^{\hat{m}} \to \mathbb{R}^{\hat{m}}$ is the natural projection on $\mathbb{R}^{\hat{m}}$.

Definition 2.2. ([24, Definition 2.2]) Let $\pi : E \to M$ be a $C^\kappa-$vector bundle of rank $\hat{m}$ and $\xi : E \to M$ is called a $C^\kappa-$morphism. Then $\xi : M \to E$ is called a $C^\kappa-$section of the $C^\kappa-$vector bundle $\pi : E \to M$ if $\pi \circ \xi = I_M$, where I_M denotes the identity on M.

Let $C^\kappa(M,E)$ be the set of all $C^\kappa-$sections of the $C^\kappa-$vector bundle $\pi : E \to M$. When $\kappa = \infty$ or ω, a $C^\kappa-$section ξ is called a smooth section or an analytic section, respectively. Let $C^\kappa(TM)$ be the set of all the $C^\kappa-$vector fields on M and let $C^\kappa(M)$ be the set of all $C^\kappa-$mappings from M to $\mathbb{R}$.

Definition 2.3. ([24, Definition 2.3]) Let $\pi : E \to M$ be a $C^\kappa-$vector bundle of rank $\hat{m}$. Then a mapping $D : C^\kappa(M,E) \times C^\kappa(TM) \to C^{\kappa-1}(M,E)$ is called a connection on this vector bundle if, for every $X,Y \in C^\kappa(TM)$, $\xi,\eta \in C^\kappa(M,E)$, $f \in C^\kappa(M)$ and $\lambda \in \mathbb{R}$, the following conditions are satisfied:

$$D_{X+fY}\xi = D_X\xi + fD_Y\xi,\ D_X(\xi + \lambda\eta) = D_X\xi + \lambda D_X\eta \text{ and}$$

$$D_X(f\xi) = X(f)\xi + fD_X\xi. \tag{2.2}$$

Note that, M is a $C^\kappa-$Riemannian manifold with countable bases and hence connections on the vector bundle $\pi : E \to M$ exist. For any $(\xi, X) \in C^\kappa(M,E) \times C^\kappa(TM)$, $D_x\xi$ is called the covariant derivative of ξ with respect to X. The value of $D_X\xi$ at $p \in M$ depends only on the tangent vector $v = X(p) \in T_pM$ because D is tensorial in X. Hence, the mapping $D\xi(p) : T_pM \to \pi^{-1}(p)$ given by

$$D\xi(p) := D_X\xi(p) \text{ for each } v \in T_pM \tag{2.3}$$

is well-defined and is a linear map from T_pM to $\pi^{-1}(p)$.

Definition 2.4. ([24, Definition 2.4]) Let $c : \mathbb{R} \to M$ be a $C^\kappa-$curve. For any $a, b \in \mathbb{R}$, define the mapping $P_{c,c(b),c(a)} : \pi^{-1}(c(a)) \to \pi^{-1}(c(b))$ by $P_{c,c(b),c(a)}(v) = \eta_v(c(b))$ for each $v \in \pi^{-1}(c(a))$, where η_v is the unique $C^\kappa-$section such that $D_{c'(t)}\eta_v = 0$ and $\eta_v(c(a)) = v$. Then $P_{c,\cdot,\cdot}$ is called the parallel transport on vector bundle E along c.

Throughout this paper we write $P_{q,p}$ for $P_{c,q,p}$ in the case when c is a minimizing geodesic connecting p and q.

As in [24] we shall define the higher order covariant derivatives for sections as follows. Let $k \le \kappa$ be a positive integer and let ξ be a $C^\kappa-$section. Recall that D is a connection on the vector bundle $\pi : E \to M$ and $\bigtriangledown$ is the Levi-Civita connection on M. Then the covariant derivative of order k can be inductively defined as follows.

Define the map $\mathscr{D}^1\xi = \mathscr{D}\xi : (C^\kappa(TM))^1 \to C^{\kappa-1}(M,E)$ by

$$\mathscr{D}\xi(X) = \mathscr{D}_X\xi \text{ for each } X \in C^\kappa(TM), \tag{2.4}$$

and define the map $D^k\xi : (C^\kappa(TM))^k \to C^{\kappa-k}(M,E)$ by

$$\begin{aligned}\mathscr{D}^1\xi(X_1, X_2, \cdots, X_{k-1}, X) = D_X(&\mathscr{D}^{k-1}\xi(X_1, X_2, \cdots, X_{k-1})) \\ &- \sum_{i=1}^{k-1} \mathscr{D}^{k-1}\xi(X_1, \cdots, \bigtriangledown_X X_i, \cdots, X_{k-1})\end{aligned} \tag{2.5}$$

for each $X_1, \cdots, X_{k-1}, X \in C^\kappa(TM)$. Using definition and (2.2), one can prove by mathematical induction that $\mathscr{D}^k\xi(X_1, \cdots, X_k)$ is tensorial with respect to each component X_i, i.e., k multi-linear map from $(C^\kappa(TM))^k$ to $C^{\kappa-k}(M,E)$, where the linearity refers to the structure of $C^\kappa(M)-$module. This implies that the value of $\mathscr{D}^k\xi(X_1, \cdots, X_k)$ at $p \in M$ only depends on the $k-$tuple of tangent vectors $(v_1,\cdots,v_k)=(X_1(p),\cdots,X_k(p))\in(T_pM)^k$. Consequently, for a given $p\in M$, the map $\mathscr{D}^k\xi(p):(T_pM)^k\to E_p$, defined by

$$\mathscr{D}^k\xi(p)v_i\cdots v_k:=\mathscr{D}^k\xi(X_1,\cdots,X_k)(p)\ \text{ for each }\ (v_1,\cdots,v_k)\in(T_pM)^k \tag{2.6}$$

is well-defined, where $X_i\in C^\kappa(TM)$ satisfy $X_i(p)=v_i$ for each $i=1,2,\cdots,k$. Let $p_0\in M$ be such that $D\xi(p_0)^{-1}$ exists. Then, for any piece-geodesic curve c connecting p_0 and p, $D\xi(p_0)^{-1}P_{c,p_0,p}\mathscr{D}^k\xi(p)$ is a $k-$multilinear map from $(T_pM)^k$ to $T_{p_0}M$ and as in [24], we define the norm of $D\xi(p_0)^{-1}P_{c,p_0,p}\mathscr{D}^k\xi(p)$ as

$$\|D\xi(p_0)^{-1}P_{c,p_0,p}\mathscr{D}^k\xi(p)\|=\sup\|D\xi(p_0)^{-1}P_{c,p_0,p}\mathscr{D}^k\xi(p)v_1v_2\cdots v_k\|,$$

where the supremum is taken over all $k-$tuple of vectors $(v_1,\cdots,v_k)\in(T_pM)^k$ with each $\|v_j\|_p=1$. Since $\bigtriangledown_{c'(s)}c'(s)=0$, for each geodesic $c:\mathbb{R}\to M$ on M, it follows from (2.5) that

$$\mathscr{D}^k\xi(c(s))(c'(s))^k=D_{c'(s)}(\mathscr{D}^{k-1}\xi(c(s))(c'(s))^{k-1})\ \text{for each}\ s\in\mathbb{R}. \tag{2.7}$$

We will be using the following two Lemmas from [24], extensively in proving our results.

Lemma 2.5. ([24, Lemma 2.1]) *Let $c:\mathbb{R}\to M$ be a geodesic and let $\zeta\in C^\kappa(M,E)$. Let $\{e_i\}_{i=1}^{\hat{m}}$ be a basis of $\pi^{-1}(c(0))$. Then, there exist $\hat{m}$ real-valued $C^\kappa-$functions $\{\zeta_i\}_{i=1}^{\hat{m}}$ on $\mathbb{R}$ such that*

$$\mathscr{D}^k\zeta(c(s))(c'(s))^k=\sum_{i=1}^{\hat{m}}\frac{d^k\zeta^i(s)}{ds^k}P_{c,c(s),c(0)}e_i\ \text{for each}\ k=0,1,\cdots,\kappa. \tag{2.8}$$

Lemma 2.6. ([24, Lemma 2.2]) *Let $c:\mathbb{R}\to M$ be a geodesic and let $\zeta\in C^\kappa(M,E)$. Then, for each $t\in\mathbb{R}$,*

$$P_{c,c(0),c(t)}\zeta(c(t))=\zeta(c(0))+\int_0^1P_{c,c(0),c(s)}(D\zeta(c(s))c'(s))ds. \tag{2.9}$$

Let $\xi\in C^1(M,E)$ and $p_0\in M$. Then Newton's method with initial point p_0 for ξ is defined as follows.

$$p_{n+1}=\exp_{p_n}((-D\xi(p_n))^{-1}\xi(p_n))\ \text{for each}\ \ n=0,1,\cdots.'' \tag{2.10}$$

3. Semilocal convergence analysis

Let Z be a Banach space or a Riemannian manifold. Let $B_Z(p,r)$ and $\overline{B_Z(p,r)}$ stand respectively for the open metric ball and the closed metric ball at $p \in Z$ with radius r, i.e.,

$$B_Z(p,r) := \{q \in Z | d(p,q) < r\} \text{ and } \overline{B_Z(p,r)} := \{q \in Z | d(p,q) \leq r\}.$$

We omit the subscript Z if no confusion caused.

Let $C_2(p_0,r)$ be the set of all piecewise geodesics $c : [0,T] \to M$ with $c(0) = p_0$ and $l(c) < r$ such that $c|_{[0,\tau]}$ is a minimizing geodesic and $c|_{[\tau,T]}$ is a geodesic for some $\tau \in (0,T]$. Motivated by the work of Zabrejko and Nguen [30] on Kantorovich's majorant method, Alvarez et.al. introduces in [2] a Lipschitz-type radial function $L : [0,R] \to [0,+\infty)$ for the covariant derivative of vector fields on Riemannian manifolds which satisfies for every $r \in [0,R]$ and $c \in C_2(p_0,r)$,

$$\|DX(p_0)^{-1}[P_{c,c(0),c(b)}DX(c(b)) - P_{c,c(0),c(a)}DX(c(a))]\| \quad \leq \quad L(u)l(c|_{[a,b]}),\ \forall 0 \leq a \leq b,$$

where R is a positive real number. Let L be a positive nondecreasing integrable function on $[0,R]$, where R is a positive number large enough such that $\int_0^R (R-u)L(u)du \geq R$. Let $\pi : E \to M$ be a $C^\kappa-$vector bundle with a connection D and a ξ a $C^\kappa-$ section of this vector bundle. Then the notion of Lipschitz condition in the inscribed sphere with the L average for operators from Banach spaces to Banach spaces [29] can be extended to sections on Riemannian manifold M as in the following definition [24].

Definition 3.1. [24, Definition 3.1] *Let $R > r > 0$ and let $p_0 \in M$ be such that $D\xi(p_0)^{-1}D\xi$ is said to satisfy the 2-piece $L-$average Lipschitz condition in $B(p_0,r)$, if, for any two points $p,q \in B(p_0,r)$, any geodesic c_2 connecting p,q and minimizing geodesic c_1 connecting p_0,p with $l(c_1)+l(c_2) < r$,*

$$\|D\xi(p_0)^{-1}P_{c_1,p_0,p} \circ (P_{c_2,p,q}D\xi(q)P_{c_2,q,p} - D\xi(p))\| \leq \int_{l(c_1)}^{l(c_1)+l(c_2)} L(u)du. \tag{3.1}$$

Notice that $P_{c_1,p_0,p}$ is an isometry from T_pM to $T_{p_0}M$. It then follows from (3.1) that there exists a positive nondecreasing integrable function L_0 on $[0,R_0]$ such that $R \leq R_0$ and

$$\|D\xi(p_0)^{-1}(P_{c_1,p_0,p}D\xi(p)P_{c_1,p,p_0} - D\xi(p_0))\| \leq \int_0^{l(c_1)} L_0(u)du. \tag{3.2}$$

Clearly

$$L_0(u) \leq L(u) \text{ for each } u \in [0,R] \tag{3.3}$$

holds in general and $\frac{L}{L_0}$ can be arbitrarily large [7], [13], [14]. Moreover, there exists a positive nondecreasing integrable function K on $[0,R]$ such that for any two points $p,q \in B(p_1, r-d(p_0,p_1))$, any geodesic c_2 connecting p,q with $l(c_1)+l(c_2) < r-d(p_0,p_1)$,

$$\|D\xi(p_0)^{-1}P_{c_1,p_0,p}\circ(P_{c_2,p,q}D\xi(q)P_{c_2,q,p}-D\xi(p))\| \leq \int_{l(c_1)}^{l(c_1)+l(c_2)} K(u)du. \tag{3.4}$$

Then, we have

$$K(u) \leq L(u) \text{ for each } u \in [0,R], \tag{3.5}$$

since $B(p_1, r-d(p_0,p_1)) \subseteq B(p_0,r)$.

Let $I_{T_{p_0}}$ denote the identity on $T_{p_0}M$. Notice that $P_{c_2,p,q}$ is an isometry from T_qM to T_pM.

Consider the identity

$$\begin{aligned} &D\xi(p_0)^{-1}P_{c_1,p_0,p}\circ P_{c_2,p,q}D\xi(q)P_{c_2,q,p}\circ P_{c_1,p,p_0} - I_{T_{p_0}}M \\ &= D\xi(p_0)^{-1}P_{c_1,p_0,p}(P_{c_2,p,q}D\xi(q)P_{c_2,q,p}-D\xi(p))P_{c_1,p,p_0} \\ &+ D\xi(p_0)^{-1}(P_{c_1,p_0,p}D\xi(p)P_{c_1,p,p_0}-D\xi(p_0)). \end{aligned} \tag{3.6}$$

Then, using (3.1), (3.2), (3.4) and (3.6), we get that

$$\begin{aligned} E &:= \|D\xi(p_0)^{-1}P_{c_1,p_0,p}\circ P_{c_2,p,q}D\xi(q)P_{c_2,q,p}\circ P_{c_1,p,p_0} - I_{T_{p_0}}M\| \\ &\leq \|D\xi(p_0)^{-1}P_{c_1,p_0,p}(P_{c_2,p,q}D\xi(q)P_{c_2,q,p}-D\xi(p))\|\|P_{c_1,p,p_0}\| \\ &\quad + \|D\xi(p_0)^{-1}(P_{c_1,p_0,p}D\xi(p)P_{c_1,p,p_0}-D\xi(p_0))\| \\ &\leq \int_{l(c_1)}^{l(c_1)+l(c_2)} K(u)du + \int_0^{l(c_1)} L_0(u)du \\ &\leq \int_{l(c_1)}^{l(c_1)+l(c_2)} L(u)du + \int_0^{l(c_1)} L(u)du. \end{aligned} \tag{3.7}$$

Hence, we have

$$E \leq \int_0^r L(u)du. \tag{3.8}$$

It follows that there exists a positive nondecreasing integrable function L_1 on $[0,R]$ such that

$$L_0(u) \leq L_1(u) \leq L(u) \text{ and } K(u) \leq L_1(u) \text{ for each } u \in [0,R] \tag{3.9}$$

and

$$E \leq \int_0^{l(c_1)+l(c_2)} L_1(u)du \leq \int_0^r L_1(u)du. \tag{3.10}$$

The introduction of function K is possible, since by the definition of p_1 in (2.10), this function also depends on the initial data p_0, ξ and $D\xi$. As we shall see later the iterates p_n lie in $B(p_1, r - d(p_0, p_1))$ which is a more precise location than $B(p_0, r)$ used in [2,24,29,30] leading to smaller functions (see (3.5)) which in turn provide a tighter convergence analysis with advantages as stated in the introduction of this study. Moreover, these advantages are obtained under the same computational cost, since in practice the computation of function L requires the computation of functions L_0 or K as special cases. Furthermore, K can simply replace L in all the results in [2, 11, 24, 29, 30] as it can easily be seen from the corresponding proofs.

From now on we say that $D\xi(p_0)^{-1}D\xi$ satisfies the center 2-piece L_1-average Lipschitz condition in $B(p_0, r)$ if (3.10) is satisfied. Notice that as we already showed (3.1) implies (3.10) but not necessarily vice versa.

Let $r_0 > 0$, $\bar{r_0} > 0$, $b > 0$ and $b_1 > 0$ be such that

$$\int_0^r L(u)du = 1, \int_0^{\bar{r_0}} L_1(u)du = 1, \tag{3.11}$$

$$b = \int_0^r L(u)udu, \; b_1 = \int_0^{\bar{r_0}} L_1(u)udu. \tag{3.12}$$

For $\beta > 0$, define the functions h and h_1 on $[0, R]$ by

$$h(t) = \beta - t + \int_0^t L(u)(t-u)du \tag{3.13}$$

and

$$h_1(t) = \beta - t + \int_0^t L_1(u)(t-u)du. \tag{3.14}$$

Notice that in view of (3.9), (3.11)-(3.14) we have

$$r_0 \leq \bar{r_0} \tag{3.15}$$

and

$$h_1(t) \leq h(t) \text{ for each } t \in [0, R]. \tag{3.16}$$

If $\beta \leq b$, then it follows from (3.13), (3.14) and (3.16) that function h_1 has two zeros denoted by $\bar{r_1}$ and $\bar{r_2}$ such that

$$\bar{r_1} \leq r_1 \tag{3.17}$$

and

$$\bar{r}_2 \leq r_2. \tag{3.18}$$

Next, some properties of function h are given ([24, 29]).

Proposition 3.2. *The function h is monotonically decreasing on $[0, r_0]$ and monotonically increasing on $[r_0, R]$. Moreover, if $\beta \leq b$, then h has a unique zero, respectively, in $[0, r_0]$ and $[r_0, R]$, which are denoted by r_1 and r_2.*

Hence, we arrived at the following Banach-type estimate on the norm of the inverse $D\xi(q)^{-1}$ around the point p_0.

Lemma 3.3. *Let $0 < r \leq \bar{r}_0$ and suppose that $D\xi(p_0)^{-1}D\xi$ satisfies the center 2-piece L_1-average Lipschitz condition in $B(p_0, r)$. Let $p, q \in B(p_0, r)$ and let c_1 be the minimizing geodesic connecting p_0, p and c_2 a geodesic connecting p, q satisfying $l(c_1) + l(c_2) < r$. Then, $D\xi(q)^{-1}$ exists and*

$$\begin{aligned} \|D\xi(q)^{-1}P_{c_2,q,p} \circ P_{c_1,p,p_0}D\xi(p_0)\| &\leq \frac{1}{1 - \int_0^{l(c_1)+l(c_2)} L_1(u)du} \\ &= \frac{-1}{1 - h_1'(l(c_1) + l(c_2))}. \end{aligned} \tag{3.19}$$

Proof. The proof follows from (3.10), (3.1), (3.14), the Banach Lemma on invertible operators [23] and the estimate

$$E \leq \int_0^r L_1(u)du < 1. \tag{3.20}$$

That completes the proof of the Lemma.

Remark 3.4. If (3.4) holds then a Banach-type lemma was shown in [24, p.431] but using L and h, respectively instead of L_1 and h_1 in (3.19). Notice that if (3.10) holds then in view of (3.9)

$$\frac{-1}{h_1'(l(c_1) + l(c_2))} \leq \frac{-1}{1 - h_1'(l(c_1) + l(c_2))}. \tag{3.21}$$

Inequality (3.21) is strict if $L_1 < L$. This observation leads to a tighter semilocal convergence analysis for Newton's method than in [24].

In the remainder of this section, we shall assume that $D\xi(p_0)^{-1}D\xi$ satisfies the 2-piece $L-$average Lipschitz condition in $B(p_0,r_1)$ and the center 2-piece L_1-average Lipschitz condition in $B(p_0,r_1)$. We also set

$$\beta := \|D\xi(p_0)^{-1}\xi(p_0)\| \leq b. \tag{3.22}$$

Let $\theta \in [0,1]$ and let the pair $(t,p) \in [0,r_1) \times B(p_0,r_1)$. Define

$$\hat{t}(\theta) = t - \theta h_1'(t)^{-1}h(t) \text{ and } \hat{p}(\theta) = \exp_p(-\theta D\xi(p)^{-1}\xi(p)) \tag{3.23}$$

and consider the following condition:

$$d(p_0,p) \leq t < r_1 \text{ and } \|D\xi(p)^{-1}\xi(p)\| \leq -h_1'(t)^{-1}h(t). \tag{3.24}$$

For a pair $(\hat{t},\hat{p}) \in [0,R] \times M$, we say that the pair $(\hat{t},\hat{p})$ satisfies (3.24) if (3.24) holds with $(\hat{t},\hat{p})$ in place of (t,p). The following lemma which is an extension and refinement of [12, Lemma 3.7], [2, Lemma 4.3], shows that $(\hat{t}(\theta),\hat{p}(\theta))$ retains the condition (3.24).

Lemma 3.5. *Suppose that: the pair $(t,p) \in [0,r_1) \times B(p_0,r_1)$ satisfies (3.24) and $\theta \in [0,1]$, and*

$$h_1'(t) + h_1'(t(\theta)) \leq h_1'(t(\theta)) + h_1'(t) \text{ for each } t \in [0,r_1] \tag{3.25}$$

$$h_1'(\hat{t})h_1'(\hat{t}(\theta)) \geq h_1'(\hat{t}(\theta))h_1'(\hat{t}) \text{ for each } t \in [0,r_1]. \tag{3.26}$$

Then, $t \leq \hat{t}(\theta) < r_1$ and the pair $(\hat{t}(\theta),\hat{p}(\theta))$ satisfies (3.24). Moreover, the following assertions hold:

$$\begin{aligned} \|D\xi(\hat{p}(1))^{-1}\xi(\hat{p}(1))\| &\leq \left(\frac{h_1'(\hat{t}(1)h(\hat{t}(1))}{h_1'(t)^{-1}h(t)}\right)\|D\xi(p)^{-1}\xi(p)\| \\ &\leq \left(\frac{h'(\hat{t}(1)h(\hat{t}(1))}{h'(t)^{-1}h(t)}\right)\|D\xi(p)^{-1}\xi(p)\|, \end{aligned} \tag{3.27}$$

$$\|D\xi(p_0)^{-1}P_{p_0,p}P_{c,p,\hat{p}(1)}\xi(\hat{p}(1))\| \leq \left(\frac{h(\hat{t}(1))}{h(t)}\right)\|D\xi(p_0)^{-1}P_{p_0,q}P_{\hat{c},Q,p}\xi(p)\|, \tag{3.28}$$

where c is the geodesic of M defined by $c(\lambda) := \exp_p(-\lambda\theta D\xi(p)^{-1}\xi(p))$ for each $\lambda \in [0,1]$, $q \in B(p_0,r_1)$ and $\hat{c}$ is a geodesic connecting q and p such that $d(p_0,q) + l(\hat{c}) \leq t$.

Proof. We have $t \leq \hat{t}(\theta) \leq \hat{t}(1)$, since $\hat{t}(.)$ is increasing on $[0,1]$. The function $t \mapsto h_1'(t)^{-1}h(t)$ is strictly monotonic increasing on $[0,r_1]$ and $h(r_1) = 0$, then, we have

$$\hat{t}(1) = t - h_1'(t)^{-1}h(t) \leq t - h'(t)^{-1}h(t) \leq r_1 - h'(r_1)^{-1}h(r_1) = r_1. \tag{3.29}$$

Suppose that (3.24) holds. Then

$$\theta\|D\xi(p)^{-1}\xi(p)\| \leq -\theta h'(t)^{-1}h(t). \tag{3.30}$$

It follows that

$$d(p_0,\hat{p}(\theta)) \leq d(p_0,p)+d(p,\hat{p}(\theta)) \leq t-\theta h_1'(t)^{-1}h(t) = \hat{t}(\theta) < r_1. \tag{3.31}$$

Set

$$s = -\theta h_1'(t)h(t) \text{ and } v = -\theta D\xi(p)^{-1}\xi(p). \tag{3.32}$$

Then, $c(1) = \exp_p(v) = \hat{p}(\theta)$ and by (3.29) and (3.30) we have

$$d(p_0,p)+l(c) \leq t-\theta h_1'(t)^{-1}h(t) = \hat{t}(\theta) \leq \hat{t}(1) < r_1. \tag{3.33}$$

In view of Lemma 2.6, we have that

$$P_{c,p,\hat{p}(\theta)}\xi(\hat{p}(\theta))-\xi(p) = \int_0^1 P_{c,p,c(\lambda)}D\xi(c(\lambda))c'(\lambda)d\lambda. \tag{3.34}$$

Note that $h'' = L$ and $\|v\| \leq s$. By (3.33), (3.1) is applicable, and so

$$\begin{aligned}
&\|D\xi(p_0)^{-1}P_{c_1,p_0,p}(P_{c,p,\hat{p}(\theta)}\xi(\hat{p}(\theta))-\xi(p)-D\xi(p)v)\| \\
&\leq \int_0^1\int_{d(p_0,p)}^{d(p_0,p)+\lambda\|v\|} L(u)du\|v\|d\lambda \\
&\leq \int_0^1\int_t^{t+\lambda s} h''(u)dud\lambda\,\theta\|D\xi(p)^{-1}\xi(p)\| \\
&= (h(\hat{t}(\theta))+(\theta-1)h(t))\left(\frac{\|D\xi(p)^{-1}\xi(p)\|}{-h_1'(t)^{-1}h(t)}\right)
\end{aligned} \tag{3.25}$$

thanks to (3.32) and (3.34). Since $l(c)+d(p,p_0) \leq \hat{t}(\theta) < r_1$, it follows from Lemma 3.3 that

$$\|D\xi(\hat{p}(\theta))^{-1}P_{c,\hat{p}(\theta),p}\circ P_{p,p_0}D\xi(p_0)\| \leq -h_1'(l(c)+d(p,p_0))^{-1} \leq -h_1'(\hat{t}(\theta))^{-1}. \tag{3.36}$$

In particular, taking $\theta = 1$ in (3.34) and (3.35), we have

$$\begin{aligned}
\|D\xi(p_0)^{-1}P_{p_0,p}P_{c,p,\hat{p}(1)}\xi(\hat{p}(1))\| &= \|D\xi(p_0)^{-1}P_{p_0,p}(P_{c,p,\hat{p}(1)}\xi(\hat{p}(1)) \\
&\quad -\xi(p)-D\xi(p)v)\| \\
&= h(\hat{t}(1))\frac{\|D\xi(p)^{-1}\xi(p)\|}{-h_1'(t)^{-1}h(t)}
\end{aligned} \tag{3.37}$$

and

$$\|D\xi(\hat{p}(1))^{-1}P_{c,\hat{p}(1),p}\circ P_{p,p_0}D\xi(p_0)\| \leq -h_1'(l(c)+d(p,p_0))^{-1} \leq -h_1'(\hat{t}(1))^{-1}. \tag{3.28}$$

Thus (3.27) follows from (3.36) and (3.37). Furthermore, by assumptions, $d(p_0,q)+l(\hat{c}) \leq t < r_1 \leq r_0$ and Lemma 3.3 we get that

$$\|D\xi(\hat{p}(p))^{-1}P_{\hat{c},p,q}P_{q,p_0}D\xi(p_0)\| \leq -h_1'(d(p,p_0)+l(\hat{c}))^{-1} \leq -h_1'(t)^{-1}. \tag{3.39}$$

That is $\|D\xi(p)^{-1}\xi(p)\| \leq -h'(t)^{-1}\|D\xi(p_0)^{-1}P_{p_0,q}P_{\hat{c},q,p}\xi(p)\|$. This together with (3.36) yields (3.26). Thus in view of (3.31), it remains to verify that

$$\|D\xi(\hat{p}(\theta))^{-1}\xi(\hat{p}(\theta))\| \leq -h_1'(\hat{t}(\theta))^{-1}h(\hat{t}(\theta)). \tag{3.40}$$

Note that (3.24) and (3.34) implies

$$\|D\xi(p_0)^{-1}P_{p_0,p}(P_{c,p,\hat{p}(1)}\xi(\hat{p}(\theta))-\xi(p)-D\xi(p)v)\| \leq h(\hat{t}(\theta))+(\theta-1)h(t). \tag{3.41}$$

Combining (3.35) and (3.40) we get

$$\|D\xi(p_0)^{-1}P_{c,\hat{p}(\theta),p}(P_{c,p,\hat{p}(\theta)}\xi(\hat{p}(\theta))-\xi(p)-D\xi(p)v)\| \leq \frac{h(\hat{t}(\theta))+(\theta-1)h(t)}{-h_1'(\hat{t}(\theta))}. \tag{3.42}$$

Taking $\theta=0$ in (3.35) gives that $\|D\xi(p)^{-1}P_{p,p_0}D\xi(p_0)\| \leq \frac{1}{|h'(t)|}$. Since $h''=L$ and $\|v\| \leq s$, it follows from (3.4) that

$$\begin{aligned}
E &:= \|D\xi(p_0)^{-1}P_{c,p,\hat{p}(\theta)}D\xi(\hat{p}(\theta))P_{c,\hat{p}(\theta),p}-I_{T_p}M\| \\
&\leq \|D\xi(p)^{-1}P_{p,p_0}D\xi(p_0)\|\|D\xi(p_0)^{-1}P_{p_0,p}(P_{c,p,\hat{p}(\theta)}D\xi(\hat{p}(\theta)P_{c,\hat{p}(\theta),p}-D\xi(p))\| \\
&\leq \frac{1}{|h'(t)|}\int_t^{t+s}h''(u)du \\
&= \frac{h'(\hat{t}(\theta))-h'(t)}{-h'(t)} \leq \frac{h'(\hat{t}(\theta))-h'(t)}{-h'(t)} \\
&= \frac{h'(\hat{t}(\theta))}{|h'(t)|}+1.
\end{aligned} \tag{3.43}$$

Thus the Banach Lemma is applicable to conclude that $\|D\xi(\hat{p})^{-1}P_{c,\hat{p}(\theta),p}D\xi(p)\| \leq \frac{h_1'(t)}{h_1'(t)+h'(\hat{\theta}))-h'(t)}$ because $P_{c,p,\hat{p}(\theta)}$ is an isometry; consequently, by (3.24) we have

$$\begin{aligned}
\|D\xi(\hat{p})^{-1}P_{c,\hat{p}(\theta),p}D\xi(p)\| &\leq \|D\xi(\hat{p})^{-1}P_{c,\hat{p}(\theta),p}D\xi(p)\|\|D\xi(p)^{-1}\xi(p)\| \\
&\leq -h'(\hat{t}(\theta))^{-1}h(t).
\end{aligned} \tag{3.44}$$

Therefore, combining (3.41) and (3.44), we get

$$\begin{aligned}\|D\xi(\hat{p})^{-1}\xi(\hat{p})\| &\leq \|D\xi(p_0)^{-1}P_{c,\hat{p}(\theta),p}(P_{c,p,\hat{p}(\theta)}\xi(\hat{p}(\theta))-\xi(p)-D\xi(p)v)\| \\ &\quad +(1-\theta)\|D\xi(p_0)^{-1}P_{c,\hat{p}(\theta),p}\xi(p)\| \\ &\leq \frac{h(\hat{t}(\theta))+(\theta-1)h(t)}{-h_1'(\hat{t}(\theta))}+\frac{(1-\theta)h(t)}{-h_1'(t)}\left(\frac{h_1'(t)}{h_1'(t)+h'(\hat{\theta}))-h'(t)}\right) \\ &\leq -h_1'(\hat{t}(\theta))^{-1}h(\hat{t}(\theta)).\end{aligned} \tag{3.45}$$

Therefore (3.39) is seen to hold since

$$\begin{aligned}&\frac{(\theta-1)h(t)}{-h_1'(\hat{t}(\theta))}+\frac{(1-\theta)h(t)}{-h_1'(t)}\left(\frac{h_1'(t)}{h_1'(t)+h'(\hat{t}(\theta))-h'(t)}\right) \\ &=(1-\theta)h(t)\left(\frac{h'(t)+h_1'(\hat{t}(\theta))-h'(\hat{t}(\theta))-h_1'(t)}{h_1'(\hat{t}(\theta))(h_1'(t)-h'(\hat{t}(\theta))-h'(t)}\right)\leq 0\end{aligned}$$

by (3.17). Notice also that the right-handside of (3.37) holds by (3.36). The proof is complete.

Let $\{\hat{t}_n\}$ and $\{\hat{p}_n\}$ denote the sequences generated by Newton's method, respectively, for h with initial point $\hat{t}_0=t$ and for ξ with initial point $\hat{p}_0=p$; that is

$$\hat{t}_0=t,\ \hat{t}_{n+1}=\hat{t}_n-h_1'(\hat{t}_n)^{-1}h(\hat{t}_n) \text{ for each } n=0,1,\cdots$$

and

$$\hat{p}_0=p,\ \hat{p}_{n+1}=\exp_{\hat{p}_n}(-D\xi(\hat{p}_n)^{-1}\xi(\hat{t}_n)) \text{ for each } n=0,1,\cdots.$$

In particular, in the case when $t=0$ and $p=p_0$, for simplicity, we denote the sequence $\{\hat{t}_n\}$ and $\{\hat{p}_n\}$ by $\{t_n\}$ and $\{p_n\}$, respectively. Hence

$$t_0=t,\ t_{n+1}=t_n-h_1'(t_n)^{-1}h(t_n) \text{ for each } n=0,1,\cdots \tag{3.46}$$

and

$$p_0=p,\ p_{n+1}=\exp_{p_n}(-D\xi(p_n)^{-1}\xi(p_n)) \text{ for each } n=0,1,\cdots. \tag{3.47}$$

Note that, by Lemma 3.5 and mathematical induction, if the pair $(t,p)\in[0,r_1)\times B(p_0,r_1)$ satisfies (3.24), then for each $n=0,1,\cdots$, the pair $(\hat{t}_n,\hat{p}_n)$ is well-defined and satisfies

$$d(p_0,\hat{p}_n)\leq\hat{t}_n<r_1 \text{ and } \|D\xi(\hat{p}_n)^{-1}\xi(\hat{p}_n)\|\leq -h_1'(\hat{t}_n)^{-1}h(\hat{t}_n). \tag{3.48}$$

Further we have the following proposition.

Proposition 3.6. *Suppose that the pair* $(t,p)\in[0,r_1)\times B(p_0,r_1)$ *satisfies (3.24), (3.25) and (3.26). Then the following assertions hold.*

(i) The sequence $\{\hat{t}_n\}$ *is strictly increasing and convergent to* r_1.

(ii) The sequence $\{\hat{p}_n\}$ *is well-defined, convergent to a singular point* q^* *of* ξ *in* $\overline{B(p_0,r_1)}$, *and the following assertions hold:*

$$\begin{aligned}\|D\xi(\hat{p}_n)^{-1}\xi(\hat{p}_n)\| &\le \left(\frac{h_1'(\hat{t}_n)^{-1}h(\hat{t}_n)}{h_1'(\hat{t}_{n-1})^{-1}h(\hat{t}_{n-1})}\right)\|D\xi(\hat{p}_{n-1})^{-1}\xi(\hat{p}_{n-1})\| \\ &\le \frac{h_1'(\hat{t}_n)^{-1}h(\hat{t}_n)}{h_1'(\hat{t}_{n-1})^{-1}h(\hat{t}_{n-1})}\|D\xi(\hat{p}_{n-1})^{-1}\|\|\xi(\hat{p}_{n-1})\|,\end{aligned} \tag{3.49}$$

$$d(\hat{p}_{n+1},\hat{p}_n)\le \hat{t}_{n+1}-\hat{t}_n \text{ for each } n=1,2,\cdots. \tag{3.50}$$

Moreover, let $q^*\in\overline{B(p_0,r_1)}$ *be a singular point of* ξ *satisfying* $t+d(p,q^*)=r_1$. *Then, for each* $n=0,1,\cdots,$

$$d(p_0,\hat{p}_n)=\hat{t}_n \text{ and } \hat{t}_{n+1}+d(\hat{p}_{n+1},q^*)=r_1. \tag{3.51}$$

Consequently, $d(p_0,q^*)=r_1$. *Notice that in particular the iterates remain in* $B(p_1,r_1-d(p_0,p_1))$.

Proof. Note that the function φ defined by $\varphi(t):=t-h_1'(t)^{-1}h(t)$ for each $t\in[0,r_1]$ is strictly monotonic increasing on $[0,r_1]$ because

$$\varphi'(t)=\frac{h_1'(t)(h_1'(t)-h'(t))+h_1''(t)h(t)}{h_1'(t)^2}>0,$$

since $h_1'(t)<0, h_1'(t)\le h'(t), h_1''(t)=L_1$ and $h(t)>0$ for $t\in[0,r_1]$. Thus it is easy to show by mathematical induction that

$$\hat{t}_n<\hat{t}_{n+1} \text{ and } 0\le \hat{t}_n<r_1 \text{ for each } n=0,1,\cdots. \tag{3.52}$$

Hence (i) is proved.

(ii) It is clear that the sequence $\{\hat{p}\}$ is well-defined and by (3.48), for each $n=1,2,\cdots,$

$$\|D\xi(\hat{p}_n)^{-1}\xi(\hat{p}_n)\|\le -h_1'(\hat{t}_n)^{-1}h(\hat{t}_n).$$

Hence (3.49) holds by (3.27), and

$$d(\hat{p}_{n+1},\hat{p}_n)\le\|D\xi(\hat{p}_n)^{-1}\xi(\hat{p}_n)\|\le -h_1'(\hat{t}_n)^{-1}h(\hat{t}_n)=\hat{t}_{n+1}-\hat{t}_n \tag{3.53}$$

holds for each $n=1,2,\cdots$. By (i), the proof of (ii) is complete. The rest of the proof as identical to Lemma 3.3 in [24, p.435] is omitted.

Remark 3.7.

(a) If $L_1 = L = K$, i.e., if $h_1 = h$, then the results obtained in Lemma and Proposition reduce to the corresponding ones in [24]. Otherwise, i.e., if $L_1 < L$ our results improve the error estimates. Let us show that in the case when $p = p_0$ and $t = 0$ (similarly for $\hat{t}_0 = t$ and $\hat{p}_0 = p$). It follows from the proof of Lemma that scalar sequence $\{\bar{t}_n\}$ defined by

$$\bar{t}_0 = 0, \bar{t}_1 = \beta, \bar{t}_{n+1} = \bar{t}_n - \frac{\beta_n}{h_1'(\bar{t}_n)} \text{ for each } n = 1, 2, \cdots$$

where $\beta_n = \int_0^{\bar{t}_n - \bar{t}_{n-1}} \bar{L}(\bar{t}_{n-1} + u)(\bar{t}_n - \bar{t}_{n-1} - u)du$, and

$$\bar{L} = \begin{cases} L_1, \; if \; n = 1 \\ K, \; if \; n > 1 \end{cases}$$

is also a majorizing sequence for $\{p_n\}$ which is tighter than $\{t_n\}$ and such that $\bar{t}_n \leq t_n$, and $\bar{t}_{n+1} - \bar{t}_n \leq t_{n+1} - t_n$. Clearly $\{\bar{t}_n\}$ is strictly increasing and converges to $\bar{t}^* = \lim_{n\to\infty} \bar{t}_n \leq r_1$ under the hypotheses of Lemma and can be replaced $\{t_n\}$ in the preceding results. Similarly, the corresponding sequence $\{\bar{\mu}_n\}$ in [24] is defined by

$$\bar{\mu}_0 = 0, \bar{\mu}_1 = \beta, \bar{\mu}_{n+1} = \bar{\mu}_n - \frac{\gamma_n}{h_1'(\bar{\mu}_n)} \text{ for each } n = 1, 2, \cdots$$

where $\gamma_n = \int_0^{\bar{\mu}_n - \bar{\mu}_{n-1}} L(\bar{\mu}_{n-1} + u)(\bar{\mu}_n - \bar{\mu}_{n-1} - u)du$. Then, a simple inductive argument shows that for each $n = 0, 1, 2, \cdots$, $\bar{t}_n \leq \bar{\mu}_n$, $\bar{t}_{n+1} - \bar{t}_n \leq \bar{\mu}_{n+1} - \bar{\mu}_n$ and $\bar{t}^* \leq \bar{\mu}^* = \lim_{n\to\infty} \bar{\mu}_n \leq r_1$. Notice also that if $L_1 < L$, then all preceding estimates hold as strict inequalities for each $n = 2, 3, \cdots$.

Another favorable comparison can be given between sequences $\{t_n\}$ and $\{\mu_n\}$, where $\{t_n\}$ was defined in (3.47) and $\{\mu_n\}$ is defined by

$$\mu_0 = 0, \mu_{n+1} = \mu_n - h'(\mu_n)^{-1} h(\mu_n) \text{ for each } n = 0, 1, \cdots.$$

In this case we should impose the condition

$$-\frac{h(s)}{h_1'(s)} \leq -\frac{h(t)}{h'(t)} \text{ for each } 0 \leq s \leq t \leq r_1$$

which is possible. For example in the case when L_1 and L are constant functions, the preceding condition reduces to

$$\frac{L}{2}(s^2-t^2)-(s-t)+\frac{Lst}{2}(L_1t-Ls)+st(L-L_1)+\beta(L_1s-Lt)\le 0$$

for each $0\le s\le t\le r_1$. Then, we have that $t_n\le\mu_n$, $t_{n+1}-t_n\le\mu_{n+1}-\mu_n$ and $t^*=\lim_{n\to\infty}t_n\le\mu^*=\lim_{n\to\infty}\mu_n\le r_1$.

(b) The results of section 4 in [24] concerning the convergence criterion of Newton's method and uniqueness ball of the singular point (See Theorem 4.1 and Theorem 4.2 in [24]) can be rewritten using the tighter sequences $\{\bar{t}_n\}$ (or $\{t_n\}$) instead of the old one $\{\bar{\mu}_n\}$ (or $\{\mu_n\}$). The proofs are omitted since they follow in an analogous way by simply using the new sequences $\{\bar{t}_n\}$ or $\{t_n\}$ instead of the old $\{\bar{\mu}_n\}$ or $\{\mu_n\}$ respectively.

Theorem 3.8. *Suppose that*

$$\beta:=\|D\xi(p_0)^{-1}\xi(p_0)\|\le b \tag{3.54}$$

and that $D\xi(p_0)^{-1}D\xi$ satisfies the 2-piece $L-$average Lipschitz condition, the center 2-piece L_1-average Lipschitz condition and (3.37) and (3.38) in $B(p_0,r_1)$. Let $\{p_n\}$ be the sequence generated by Newton's method (2.10) with initial point p_0. then $\{p_n\}$ is well-defined and convergent to a singular point p^* of ξ in $\overline{B(p_0,r_1)}$. Moreover, there hold

$$\begin{aligned}\|D\xi(p_0)^{-1}P_{p_0,p_{n-2}}P_{c_n,p_{n-1},p_n}\xi(p_n)\| \;&\le\; \left(\frac{t_{n+1}-t_n}{t_n-t_{n-1}}\right)\|D\xi(p_0)^{-1}\\ &\quad P_{p_0,p_{n-2}}P_{c_{n-1},p_{n-2},p_{n-1}}\xi(p_{n-1})\|,\\ &\quad \text{for each } n=2,3,\cdots,\end{aligned}$$

$$\|D\xi(p_n)^{-1}\xi(p_n)\|\le\left(\frac{t_{n+1}-t_n}{t_n-t_{n-1}}\right)\|D\xi(p_{n-1})^{-1}\xi(p_{n-1})\|,\ \text{for each } n=1,2,\cdots, \tag{3.55}$$

$$d(p_{n+1},p_n)\le t_{n+1}-t_n,\ \text{for each } n=2,3,\cdots, \tag{3.56}$$

and

$$d(p_n,p^*)\le r_1-t_n,\ \text{for each } n=2,3,\cdots, \tag{3.57}$$

where, for each n, c_n is the geodesic of M defined by

$$c_n(\lambda):=\exp_{p_{n-1}}(-\lambda D\xi(p_{n-1})^{-1}\xi(p_{n-1}))\ \text{for each } \lambda\in[0,1]. \tag{3.58}$$

Theorem 3.9. *Suppose that (3.54) holds. Let $r_1 \le r < r_2$ if $\beta < b$ and $r = r_1$ if $\beta = b$. Suppose that $D\xi(p_0)^{-1}D\xi$ satisfies the 2-piece $L-$average Lipschitz condition, the center 2-piece L_1-average Lipschitz condition and (3.37) and (3.38) in $B(p_0, r_1)$. Then, there exists a unique singular point $p^* \in \overline{B(p_0, r_1)}$ of ξ in $\overline{B(p_0, r)}$.*

Remark 3.10. Notice that tighter sequence $\{\bar{t}_n\}$ and limit point $\bar{t}^*$ can be replace by $\{t_n\}, r_1$, respectively in (3.57), (3.58) and $B(p_0, r_1)$ can be replaced by $B(p_0, \bar{t}^*)$ in Theorem 3.8 and Theorem 3.9. In this case the error bounds are improved even further as well as the information on the location of the solution, since $\bar{t}^* \le r_1$ (see also Remark 3.7 (a)). Finally, notice that (3.38) can be dropped from all preceding results, since it is only used to show the right hand side inequality in (3.39).

4. Special cases under Kantorovich's condition

In Section 3, we presented our results which improve the error estimates of the corresponding ones in [24] under the same convergence criteria. At this point we are wondering if even the convergence criteria can be weakened. We show that this is possible. Let L and L_1 be constant functions. Then, we get by (3.24) and (3.25), respectively that

$$h(t) = \frac{L}{2}t^2 - t + \beta$$

and

$$h_1(t) = \frac{L_1}{2}t^2 - t + \beta.$$

Then, (3.37) and (3.38) are reduced to

$$(L - L_1)(t - \hat{t}(\theta)) \le 0$$

respectively which hold. Therefore in this interesting case the advantages of our approach (see Remark 3.7 (a)) hold.

Moreover, according to the proof of Lemma 3.5 and the definition of sequences $\{\bar{t}_n\}$ and $\{\bar{\mu}_n\}$ we have that these sequences can be written as

$$\bar{\mu}_0 = 0,\ \bar{\mu}_1 = \beta,\ \bar{\mu}_{n+1} = \bar{\mu}_n - \frac{L(\bar{\mu}_n - \bar{\mu}_{n-1})^2}{2(1 - L\bar{\mu}_n)} \text{ for each } n = 1, 2, \cdots. \tag{4.1}$$

Notice that in this case sequence $\{\mu_n\}$ coincides with sequence $\{\bar{\mu}_n\}$. We also have that sequence $\{\bar{t}_n\}$ given by

$$\begin{aligned} \bar{t}_0 = 0, \bar{t}_1 = \beta, \bar{t}_2 = \bar{t}_1 - \frac{L_1(\bar{t}_1 - \bar{t}_0)^2}{2(1 - L_1\bar{t}_1)} \\ \bar{t}_{n+1} = \bar{t}_n - \frac{K(\bar{t}_n - \bar{t}_{n-1})^2}{2(1 - K\bar{t}_n)} \text{ for each } n = 1, 2, \cdots . \end{aligned} \tag{4.2}$$

The scalar sequence $\{\bar{\mu}_n\}$ given in (4.1) has been used as the majorizing sequence for Newton's method [7, 14, 23]. The sufficient convergence criterion for $\{\bar{\mu}_n\}$ is given by the famous for its simplicity and clarity Kantorovich criterion

$$H = L\eta \leq \frac{1}{2}. \tag{4.3}$$

The corresponding convergence criterion for majorizing sequence $\{\bar{t}_n\}$ given by us in [13] is given by

$$H_1 = \bar{L}\eta \leq \frac{1}{2}, \tag{4.4}$$

where

$$\bar{L} = \frac{1}{8}(4L_1 + \sqrt{L_1K + 8L_1^2} + \sqrt{L_1K}).$$

Notice that

$$H \leq \frac{1}{2} \Rightarrow H_1 \leq \frac{1}{2} \tag{4.5}$$

but not necessarily vice versa. Examples where (3.3) or (3.5) hold can be found in [10,11], [13]-[15].

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