J. Nonlinear Funct. Anal. 2016 (2016), Article ID 38

MATHRES
Journal of Nonlinear Functional Analysis
Available online at http://jnfa.mathres.org

POSITIVE SOLUTIONS FOR NONLINEAR SINGULAR DIFFERENCE EQUATIONS WITH PERIODIC BOUNDARY CONDITIONS

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Abstract. We study the existence of positive solutions for second order nonlinear repulsive singular difference equations with periodic boundary conditions. Our nonlinearity may be singular in its dependent variable. The proof of the main result relies on a nonlinear alternative principle of Leray-Schauder.

Keywords. Positive solution; Singular difference equation; Leray-Schauder alternative principle.

1. Introduction

In this paper, for fixed positive integer N, we establish the existence positive solutions for the following nonlinear difference equations

$$-\Delta[p(n-1)\Delta x(n-1)]+q(n)x(n)=f(n,x(n))+e(n), \tag{1}$$

with the boundary conditions

$$x(0)=x(N), \quad p(0)\Delta x(0)=p(N)\Delta x(N), \tag{2}$$

where $\{x(n)\}_{t=0}^{N+1}$ is a desired solution, $f(n,x)$ is a function defined for all n in $[1,N]$ and all real numbers x. We call boundary conditions (2) the periodic boundary conditions which are important representatives of nonseparated boundary conditions. For convenience, we denote by $\mathbb{N},\mathbb{Z}$ and $\mathbb{R}$ the sets of all natural, integers numbers and real numbers, respectively. For $a,b\in\mathbb{Z}$,

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Received June 26, 2016

let $\mathbb{Z}(a)=\{a,a+1,\dots\},\mathbb{Z}[a,b]=\{a,a+1,\dots,b\}$ when $a\le b$. As usual, Δ denotes the forward difference operator defined by

$$\Delta x(n)=x(n+1)-x(n).$$

In particular, the nonlinearity f may have a repulsive singularity at $x=0$, from the physical explanation, which means that

$$\lim_{x\to 0^+} f(n,x)=+\infty, \text{ uniformly in } n\in\mathbb{Z}[1,N].$$

Such repulsive singularity appears in many problems of applications such as the Brillouin focusing systems and nonlinear elasticity .

Equation (1) can be considered as a discrete analogue of the following singular second order differential equation

$$-(p(t)y')'+q(t)y=f(t,y)+e(t).$$

During the last few decades, the study of the existence of periodic solutions for singular differential equations have deserved the attention of many researchers [10, 12, 13, 14, 16, 17, 18, 19]. Some classical tools have been used to study singular differential equations in the literature, including the degree theory [12, 19, 20], the method of upper and lower solutions [3, 16], Schauder's fixed point theorem [7], some fixed point theorems in cones for completely continuous operators [6, 11] and a nonlinear Leray-Schauder alternative principle [5, 9].

For the existence of periodic solutions of difference equations, some results have been obtained using the variational methods or the topological methods [1, 4, 8, 21]. For example, by critical point theorem, Guo and Yu [8] studied the existence of periodic solutions for the following difference equation

$$\Delta[p(n)\Delta x(n-1)]+q(n)x(n)=f(n,x(n)),$$

where the nonlinearity f is of sublinear growth or superlinear growth at infinity. Based on the method of upper and lower solutions, Atici and Cabada [1] studied the existence of periodic solutions for difference equation

$$-\Delta^2 x(n-1)+q(n)x(n)=f(n,x(n)).$$

In this paper, we establish the existence of positive periodic solutions of (1), proof of the existence of positive solutions is based on an application of a nonlinear alternative of Leray-Schauder, which has been used by many authors [5, 9]. Here we emphasize that the new results are applicable to the case of a strong singularity as well as the case of a weak singularity, and that e does not need to be positive.

The rest of this paper is organized as follows. In Section 2, some preliminary results are given, including a famous nonlinear alternative of Leray-Schauder type. In Section 3, the main results are stated and proved.

2. Preliminaries

Let us denote $\{\varphi(n)\}_{n=0}^{N+1}$ and $\{\psi(n)\}_{n=0}^{N+1}$ by the solutions of the corresponding homogeneous equations

$$-\Delta[p(n-1)\Delta x(n-1)]+q(n)x(n)=0, \quad n\in\mathbb{Z}[1,N],$$

satisfying the initial conditions

$$\varphi(0)=\varphi(1)=0; \quad \psi(0)=0, p(0)\psi(1)=1,$$

respectively.

Let

$$D=\varphi(N)+p(N)\Delta\psi(N)-2. \tag{3}$$

Throughout this paper, we always assume that

(H) $p(n)>0, q(n)\geq 0$, $q(\cdot)\not\equiv 0, n\in\mathbb{Z}[0,N]$.

Lemma 2.1. [2] *If* (H) *holds, then* $D>0$.

Lemma 2.2. [2] *Assume* (H) *holds. For the solution of the problem*

$$\begin{cases} -\Delta[p(n-1)\Delta x(n-1)]+q(n)x(n)=h(n), & n\in\mathbb{Z}[1,N], \\ x(0)=x(N), \quad p(0)\Delta x(0)=p(N)\Delta x(N), \end{cases} \tag{4}$$

the formula

$$x(n)=\sum_{s=1}^{N}G(n,s)h(s),\quad n\in\mathbb{Z}[0,N],$$

holds, where

$$\begin{aligned}G(n,s)\ =\ &\frac{\psi(N)}{D}\varphi(n)\varphi(s)-\frac{p(N)\Delta\varphi(N)}{D}\psi(n)\psi(s)\\ &+\begin{cases}\frac{p(N)\Delta\psi(N)-1}{D}\varphi(n)\psi(s)-\frac{\varphi(N)-1}{D}\varphi(s)\psi(n), & 0\le s\le n\le N+1,\\ \frac{p(N)\Delta\psi(N)-1}{D}\varphi(s)\psi(n)-\frac{\varphi(N)-1}{D}\varphi(n)\psi(s), & 0\le n\le s\le N+1,\end{cases}\end{aligned}$$

is the Green's function, the number D is defined by (3).

Lemma 2.3. [2] *Under condition* (H)*, the Green's function* $G(n,s)$ *of the boundary value problem (4) is positive, i.e.,* $G(n,s)>0$ *for* $n,s\in\mathbb{Z}[0,N]$.

We denote

$$A=\min_{n,s\in\mathbb{Z}[1,N]}G(n,s),\quad B=\max_{n,s\in\mathbb{Z}[1,N]}G(n,s),\quad \sigma=A/B. \tag{5}$$

Thus $B>A>0$ *and* $0<\sigma<1$.

Remark 2.4. If $p(t)=1,q(t)=\alpha>0$, then the Green's function $G(n,s)$ of the boundary value problem (4) has the form

$$G(n,s)=\begin{cases}\frac{\beta^{n-s}+\beta^{s-n+N}}{(\beta-\beta^{-1})(\beta^{n}-1)}, & 0\le s\le n\le N+1,\\ \frac{\beta^{s-n}+\beta^{n-s+N}}{(\beta-\beta^{-1})(\beta^{n}-1)}, & 0\le n\le s\le N+1,\end{cases}$$

where $\beta=\frac{\alpha+2+\sqrt{\alpha(\alpha+2)}}{2}$ and a direct calculation shows that

$$A=\frac{2\beta^{N/2}}{(\beta-\beta^{-1})(\beta^{N}-1)},B=\frac{1+\beta^{N}}{(\beta-\beta^{-1})(\beta^{N}-1)},\sigma=\frac{2\beta^{N/2}}{1+\beta^{N}}<1,$$

if N is even.

Let

$$X=\{x:\mathbb{Z}[0,N]\to\mathbb{R}\mid x(0)=x(N),\quad p(0)\Delta x(0)=p(N)\Delta x(N)\}.$$

Then X is a Banach space with the norm

$$\|u\|=\max_{n\in\mathbb{Z}[1,N]}|u(n)|.$$

Define the operator $\mathscr{A}: X \to X$ by

$$(\mathscr{A}x)(n) = \sum_{s=1}^{N} G(n,s) f(s,x(s)), \quad n \in \mathbb{Z}[0,N].$$

3. Main results

In this section, we state and prove the new existence results for (1). In order to prove our main results, the following nonlinear alternative of Leray-Schauder is need, which can be found in [15].

Lemma 3.1. *Assume Ω is a relatively compact subset of a convex set E in a normed space X. Let $\mathscr{A}: \overline{\Omega} \to E$ be a compact map with $0 \in \Omega$. Then one of the following two conclusions holds:*

(i) *T has at least one fixed point in $\overline{\Omega}$.*

(ii) *There exist $u \in \partial\Omega$ and $0 < \lambda < 1$ such that $u = \lambda \mathscr{A} u$.*

Let us define

$$\gamma(n) = \sum_{s=1}^{N} G(n,s) e(s),$$

which corresponds to the unique solution of the difference equation

$$-\Delta[p(n-1)\Delta u(n-1)] + q(n)u(n) = e(n). \tag{6}$$

Now we present our main existence result of positive solution to problem (1).

Theorem 3.2. *Suppose that condition* (H) *holds and $\gamma_* \geq 0$. Furthermore, we assume that*

(H_1) *For each constant $L > 0$, there exists a function $\phi_L(n) > 0$ for all $n \in \mathbb{Z}[1,N]$ such that $f(n,x) \geq \phi_L(n)$ for all $(n,x) \in \mathbb{Z}[1,N] \times (0,L]$;*

(H_2) *There exist continuous, non-negative functions $g(x), h(x)$ and $k(n)$ such that*

$$0 \leq f(n,x) \leq \{g(x) + h(x)\}k(n) \quad \text{for all } (n,x) \in \mathbb{Z}[1,N] \times (0,+\infty),$$

and $g(x) > 0$ is non-increasing and $h(x)/g(x)$ is non-decreasing in x;

(H_3) *There exists a positive number r such that*

$$\frac{r}{g(\sigma r + \gamma_*)\left\{1 + \frac{h(r+\gamma_*)}{g_i(r+\gamma_*)}\right\}} > \|K\|,$$

here

$$K(n)=\sum_{s=1}^{N}G(n,s)k(s),\quad \gamma_*=\min_n\gamma(n),\quad \gamma^*=\max_n\gamma(n)$$

Then (1)-(2) has at least one positive periodic solution $x(n)>\gamma(n)$ for all $n\in\mathbb{Z}[0,N]$ and $0<\|x-\gamma\|<r$.

Proof. We first show that

$$-\Delta[p(n-1)\Delta x(n-1)]+q(n)x(n)=f(n,x(n)+\gamma(n)), \tag{7}$$

together with (2) has a positive solution x satisfying $x(n)+\gamma(n)$ for $n\in\mathbb{Z}[0,N]$ and $0<\|x\|<r$. If this is true, it is easy to see that $u(n)=x(n)+\gamma(n)$ will be a positive solution of (1)-(2) with $0<|u-\gamma|<r$ since

$$\begin{aligned}&-\Delta[p(n-1)\Delta u(n-1)]+q(n)u(n)\\ =&-\Delta[p(n-1)\Delta(x(n-1)+\gamma(n-1))]+q(n)(x(n)+\gamma(n))\\ =&\ f(n,x(n)+\gamma(n))+e(n)\\ =&\ f(n,u(n))+e(n).\end{aligned}$$

Since (H_3) holds, let $J_0=\{j_0,j_0+1,\cdots\}$, we can choose $j_0\in\{1,2,\cdots\}$ such that $\frac{1}{j_0}<\sigma r+\gamma_*$ and

$$g(\sigma r+\gamma_*)\left\{1+\frac{h(r+\gamma^*)}{g(r+\gamma^*)}\right\}\|K\|+\frac{1}{j_0}<r.$$

Fix $j\in J_0$ and consider the family of systems

$$-\Delta[p(n-1)\Delta x(n-1)]+q(n)x(n)=\lambda f_j(n,x(n))+\frac{q(n)}{j},\quad n\in\mathbb{Z}[1,N],$$

where $\lambda\in[0,1]$,

$$f_j(n,x)=\begin{cases}f_j(n,x) & \text{if } x\geq 1/j,\\ f_j(\frac{1}{j}) & \text{if } x\leq 1/j.\end{cases}$$

Problem (7)-(2) is equivalent to the following fixed point problem

$$x(n)=\lambda\sum_{s=1}^{N}G(n,s)f_j(s,x(s)+\gamma(s))+\frac{1}{j}=\lambda\left(\mathscr{A}_jx,\right)(n)+\frac{1}{j} \tag{8}$$

here we used the fact

$$\sum_{s=1}^{N}G(n,s)q(s)\equiv 1.$$

We claim that any fixed point x of (8) for any $\lambda \in [0,1]$ must satisfy $\|x\| \neq r$. Otherwise, assume that x is a fixed point of (8) for some $\lambda \in [0,1]$ such that $\|x\| = r$. Note that

$$\begin{aligned} x(n) - \frac{1}{j} &= \lambda \sum_{s=1}^{N} G(n,s) f_j(s, x(s) + \gamma(s)) ds \\ &\geq \lambda A \sum_{s=1}^{N} f(s, x(s) + \gamma(s)) ds \\ &= \sigma B \lambda \sum_{s=1}^{N} f(s, x(s) + \gamma(s)) ds \\ &\geq \sigma \max_{n} \left\{ \lambda \sum_{s=1}^{N} G(n,s) f(s, x(s) + \gamma(s)) ds \right\} \\ &= \sigma \left\| x - \frac{1}{j} \right\|. \end{aligned}$$

Hence, for all $n \in \mathbb{Z}[1,N]$, we have

$$x(n) \geq \sigma \|x - \frac{1}{j}\| + \frac{1}{j} \geq \sigma(\|x\| - \frac{1}{j}) + \frac{1}{j} \geq \sigma r. \tag{9}$$

Therefore,

$$x(n) + \gamma(n) \geq \sigma r + \gamma_* > \frac{1}{j}.$$

Using (8), we have from condition (H_2), for all t,

$$\begin{aligned} x(n) &= \lambda \sum_{s=1}^{N} G(n,s) f_j(s, x(s) + \gamma(s)) + \frac{1}{j} \\ &= \lambda \sum_{s=1}^{N} G(n,s) f(s, x(s) + \gamma(s)) + \frac{1}{j} \\ &\leq \sum_{s=1}^{N} G(n,s) f(s, x(s) + \gamma(s)) + \frac{1}{j} \\ &\leq \sum_{s=1}^{N} G(n,s) k(s) g(x(s) + \gamma(s)) \left\{ 1 + \frac{h(x(s) + \gamma(s))}{g(x(s) + \gamma(s))} \right\} \\ &\leq g(\sigma r + \gamma_*) \left\{ 1 + \frac{h(r + \gamma^*)}{g(r + \gamma^*)} \right\} \|K\| + \frac{1}{j_0}. \end{aligned}$$

Therefore,

$$r = \|x\| \leq g(\sigma r) \left\{ 1 + \frac{h(r)}{g(r)} \right\} \|K\| + \frac{1}{j_0}.$$

This is a contradiction to the choice of j_0 and the claim is proved. From this claim, the nonlinear alternative of Leray-Schauder guarantees that

$$x(n) = (\mathscr{A}^n x)(n) + \frac{1}{j}$$

has a fixed point, denoted by $x_j(n)$, in $B_r = \{x \in X : \|x\| < r\}$, i.e.,

$$-\Delta[p(n-1)\Delta x(n-1)] + q(n)x(n) = f_j(n, x(n) + \gamma(n)) + \frac{q(n)}{j}, \tag{10}$$

has a periodic solution x_j with $\|x_j\| < r$.

Next we claim that these solutions $x_j(n) + \gamma(n)$ have a uniform positive lower bound, that is, there exists a constant $\delta > 0$, independent of $j \in J_0$, such that

$$\min_{n \in \mathbb{Z}[1,\mathbb{N}]} \{x_j(n) + \gamma(n)\} \geq \delta \tag{11}$$

for all $j \in J_0$. To see this, we know from (H_1) that there exists a continuous function $\phi_r \succ 0$ such that $f(n,x) \geq \phi_{r+\gamma^*}(n)$ for all $(n,x) \in \mathbb{Z}[1,\mathbb{N}] \times (0, r+\gamma^*]$. Now let $x_{r+\gamma^*}(n)$ be the unique solution to the problem (6), then we have

$$x_{r+\gamma^*}(n) = \sum_{s=1}^{N} G(n,s)\phi_{r+\gamma^*}(s) \geq \Phi_*,$$

here

$$\Phi_* = \inf_n \Phi(n), \quad \Phi(n) = \sum_{s=1}^{T} G(n,s)\phi_{r+\gamma^*}(s).$$

Next, we show that (11) holds for $\delta = \Phi_* + \gamma_* > 0$. To see this, since $x_j(n) + \gamma(n) \leq r + \gamma^*$ and $x_j(n) + \gamma_* \geq \frac{1}{j}$, we have

$$\begin{aligned} x_j(n) + \gamma(n) &= \sum_{s=1}^{N} G(n,s) f_j(s, x(s) + \gamma(s)) + \gamma(n) + \frac{1}{j} \\ &\geq \sum_{s=1}^{N} G(n,s)\phi_{r+\gamma^*} + \frac{1}{j} \\ &\geq \Phi_* + \gamma_* := \delta. \end{aligned}$$

The fact $\|x_j(n)\| < r$ and (11) show that $\{x_j\}_{j \in J_0}$ is a bounded family on $\mathbb{Z}[1,N]$. Moreover, we have

$$x_j(0) = x_j(N), \quad p(0)\Delta x_j(0) = p(N)\Delta x_j(N),$$

which implies that

$$x_j(N+1) = \frac{p(0)}{p(N)}\Delta x_j(0) + x_j(N).$$

Thus the Arzela–Ascoli Theorem guarantees that $\{x_j\}_{j\in J_0}$ has a subsequence, $\{x_{j_k}\}_{k\in N}$ converging uniformly on $\mathbb{Z}[1,N]$ to a function $x \in \mathbb{Z}[1,N]$. $x(n)$ satisfies $\delta \le x(n)+\gamma(n) < r+\gamma^*$ for all $n \in \mathbb{Z}[1,N]$. Moreover, x_{j_k} satisfies the integral equation

$$x_{j_k}(n) = \sum_{s=1}^{N} G(n,s) f(s, x_{j_k}(s)) + \frac{1}{j_k}, \quad i = 1,\dots,n.$$

Letting $k \to \infty$, we arrive at

$$x(n) = \sum_{s=1}^{N} G(n,s) f(s, x(s)+\gamma(s)).$$

Therefore, x is a positive periodic solution of (1) and satisfies $0 < \|x\| \le r$.

Corollary 3.3. *Suppose that $\alpha > 0, \beta \ge 0$ and $\mu \in R$ is a given positive parameter. Let the nonlinearity in (1) be*

$$f(n,x) = \frac{1}{x^\alpha} + \mu x^\beta,$$

(i) *if $\beta < 1$, then (1) has at least one positive periodic solution for each $\mu > 0$;*

(ii) *if $\beta \ge 1$, then (1) has at least one positive periodic solution for each $0 < \mu < \mu_1$, where μ_1 is some positive constant;*

Proof. We will apply Theorem 3.2. To this end, assumption (H_1) is fulfilled by $\phi_L = L^{-\alpha}$. If we take

$$g(x) = x^{-\alpha}, \quad h(x) = \mu x^\beta, \quad k(n) = 1,$$

then (H_2) is satisfied. Let $\omega(n) = \sum_{s=1}^{N} G(n,s)$ Then the existence condition (H_3) becomes

$$\mu < \frac{r(\sigma r + \gamma_*)^\alpha - \|\omega\|}{\|\omega\|(r+\gamma^*)^{\alpha+\beta}}$$

for some $r > 0$. So (1) has at least one positive periodic solution for

$$0 < \mu < \mu_1 := \sup_{r>0} \frac{r(\sigma r + \gamma_*)^\alpha - \|\omega\|}{\|\omega\|(r+\gamma^*)^{\alpha+\beta}}.$$

Note that $\mu_1 = \infty$ if $\beta < 1$ and $\mu_1 < \infty$ if $\beta \ge 1$. We have (i) and (ii). This completes the proof.

Acknowledgements

The authors are grateful to the reviewers for useful suggestions which improved the contents of this paper. This work was supported by the National Natural Science Foundation of China (Grant No.11461016), the Scientific Research Foundation of Hainan University (KYQD1544), Hainan Natural Science Foundation (Grant No.20167246).

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