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A FIXED POINT RESULT FOR *C*-KANNAN TYPE CYCLIC WEAKLY CONTRACTIONS

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Abstract. In this article, we introduce the notion of C-Kannan type cyclic weakly contractions and derive the existence of fixed points for such mappings in the framework of complete metric spaces. Our results extend and improve some fixed point theorems in the literature.

Keywords. Fixed point; Kannan type mapping; Cyclic weakly contraction mapping; Metric space.

1. Introduction

Banach's fixed point theorem for contraction mappings is one of the pivotal results in analysis, but it suffers from one major drawback i.e. in order to use the contractive condition, a self mapping T must be Lipschitz continuous, with Lipschitz constant $L < 1$. In particular, T must be continuous at all points of its domain.

A natural question is that whether we can find contractive conditions which will imply existence of fixed points in a complete metric space but will not imply continuity.

Kannan [11], [12] proved the following result, giving an affirmative answer to above question.

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Theorem 1.1. *If $T : X \to X$, where (X,d) is a complete metric space, satisfies*

$$d(Tx,Ty) \leq k[d(x,Tx)+d(y,Ty)],$$

where $0 < k < \frac{1}{2}$ and $x,y \in X$, then T has a unique fixed point.

The mappings satisfying the above inequality are called Kannan type mappings.

Alber and Guerre-Delabriere [1] introduced the concept of weakly contractive mappings and proved the existence of fixed points for single-valued weakly contractive mappings in Hilbert spaces. Thereafter, in 2001, Rhoades [16] proved the fixed point theorem which is one of the generalizations of Banach's Contraction Mapping Principle, because the weakly contractions contain contractions as a special case and Rhoades also showed that some results of [1] are true for any Banach space. In fact, weakly contractive mappings are closely related to the mappings of Boyd and Wong [4] and of Reich types [15]. Fixed point problems involving different types of contractive type inequalities have been studied by many authors (see [1]-[16] and references cited therein).

On the other hand, Kirk *et al.* [14] introduced the notion of cyclic representation and characterized the Banach Contraction Principle in the context of cyclic mappings.

In this paper, we introduce the C-Kannan type cyclic weakly contraction mapping and then derive a fixed point theorem on such class of cyclic contractions in the framework of complete metric spaces.

2. Preliminaries

Definition 2.1. (see [14]) Let X be a non-empty set and $T : X \to X$ be a self mapping. Then $X = \cup_{i=1}^{m} X_i$ is a cyclic representation of X with respect to T if

(a) X_i; $i = 1,\ldots,m$ are non-empty sets,

(b) $T(X_1) \subset X_2,\ldots, T(X_{m-1}) \subset X_m$, $T(X_m) \subset X_1$.

Definition 2.2. (see [2]) Let $F : \mathbb{R}_+^2 \to \mathbb{R}$ be a continuous mapping, it is called a C-class function if it satisfies the following conditions:

$(F_1):$ $F(s,t) \leq s$, for all $(s,t) \in \mathbb{R}_+^2$.

$(F_2):\ F(s,t)=s$ implies that $s=0$, or $t=0$,for all $(s,t)\in\mathbb{R}_+^2$.

We denote C-class functions as $\mathscr{C}$.

Example 2.3. (see [2]) *The following functions $F:[0,\infty)^2\to\mathbb{R}$ are elements of $\mathscr{C}$, for all $s,t\in[0,\infty)$:*

(1) $F(s,t)=s-t$, $F(s,t)=s\Rightarrow t=0$;

(2) $F(s,t)=ms$, $0<m<1$, $F(s,t)=s\Rightarrow s=0$;

(3) $F(s,t)=\frac{s}{(1+t)^r}$; $r\in(0,\infty)$, $F(s,t)=s\Rightarrow s=0$ *or* $t=0$;

(4) $F(s,t)=\log(t+a^s)/(1+t)$, $a>1$, $F(s,t)=s\Rightarrow s=0$ *or* $t=0$;

(5) $F(s,t)=\ln(1+a^s)/2$, $a>e$, $F(s,1)=s\Rightarrow s=0$;

(6) $F(s,t)=(s+l)^{(1/(1+t)^r)}-l$, $l>1, r\in(0,\infty)$, $F(s,t)=s\Rightarrow t=0$;

(7) $F(s,t)=s\log_{t+a}a$, $a>1$, $F(s,t)=s\Rightarrow s=0$ *or* $t=0$;

(8) $F(s,t)=s-(\frac{1+s}{2+s})(\frac{t}{1+t})$, $F(s,t)=s\Rightarrow t=0$;

(9) $F(s,t)=s\beta(s)$, $\beta:[0,\infty)\to[0,1)$,*and is continuous,* $F(s,t)=s\Rightarrow s=0$;

(10) $F(s,t)=s-\frac{t}{k+t}, F(s,t)=s\Rightarrow t=0$;

(11) $F(s,t)=s-\varphi(s), F(s,t)=s\Rightarrow s=0$,*here* $\varphi:[0,\infty)\to[0,\infty)$ *is a continuous function such that* $\varphi(t)=0\Leftrightarrow t=0$;

(12) $F(s,t)=sh(s,t), F(s,t)=s\Rightarrow s=0$,*here* $h:[0,\infty)\times[0,\infty)\to[0,\infty)$*is a continuous function such that* $h(t,s)<1$ *for all* $t,s>0$;

(13) $F(s,t)=s-(\frac{2+t}{1+t})t$, $F(s,t)=s\Rightarrow t=0$.*(8)* $F(s,t)=s-(\frac{1+s}{2+s})(\frac{t}{1+t})$, $F(s,t)=s\Rightarrow t=0$;

(14) $F(s,t)=\sqrt[n]{\ln(1+s^n)}$, $F(s,t)=s\Rightarrow s=0$;

(15) $F(s,t)=\phi(s), F(s,t)=s\Rightarrow s=0$,*here* $\phi:[0,\infty)\to[0,\infty)$ *is a upper semicontinuous function such that* $\phi(0)=0$, *and* $\phi(t)<t$ *for* $t>0$;

(16) $F(s,t)=\frac{s}{(1+s)^r}$; $r\in(0,\infty)$, $F(s,t)=s\Rightarrow s=0$;

(17) $F(s,t)=\vartheta(s)$; $\vartheta:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ *is a generalized Mizoguchi-Takahashi type function ,* $F(s,t)=s\Rightarrow s=0$;

(18) $F(s,t)=\frac{s}{\Gamma(1/2)}\int_0^\infty\frac{e^{-x}}{\sqrt{x}+t}\,dx$, *where* Γ *is the Euler Gamma function.*

Let Ψ *be the set of all continuous functions* $\psi:\mathbb{R}_+\to\mathbb{R}_+$ *satisfying the following conditions:*

(ψ_1) ψ *is continuous and strictly increasing.*

(ψ_2) $\psi(t) = 0$ *if and only of* $t = 0$.

Let Φ_u denote the class of functions $\varphi : [0,\infty) \to [0,\infty)$ which satisfy the following conditions:

(a) φ is continuous;

(b) $\varphi(t) > 0, t > 0$ and $\varphi(0) \geq 0$.

Let Ψ_u be a set of all continuous functions $\psi : [0,\infty) \times [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(ψ_1) ψ is continuous.

(ψ_2) $\psi(s,t) > 0$ if $(s,t) \neq (0,0)$ and $\psi(0,0) \geq 0$.

The following lemma of Babu and Sailaja [3] will be used in sequel.

Lemma 2.4. *Suppose* (X,d) *is a metric space. Let* $\{x_n\}$ *be a sequence in* X *such that* $d(x_n, x_{n+1}) \to 0$ *as* $n \to \infty$. *If* $\{x_n\}$ *is not a Cauchy sequence then there exist an* $\varepsilon > 0$ *and sequences of positive integers* $\{m(k)\}$ *and* $\{n(k)\}$ *with*

$m(k) > n(k) > k$ *such that* $d(x_{m(k)}, x_{n(k)}) \geq \varepsilon$, $d(x_{m(k)-1}, x_{n(k)}) < \varepsilon$ *and*

(i) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \varepsilon$;

(ii) $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon$;

(iii) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \varepsilon$.

On the lines of above lemma one can also note that $\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \varepsilon$ and $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)-1}) = \varepsilon$.

3. Main results

To begin with, we introduce the notion of C-Kannan type cyclic weakly contractions in metric spaces.

Definition 3.1. Let (X,d) be a metric space, $m \in \mathbb{N}$, $A_1, A_2 \ldots, A_m$ nonempty subsets of X and $Y = \cup_{i=1}^m A_i$. A mapping $T : Y \to Y$ is called a C-Kannan type cyclic weakly contraction if

(1) $\cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T;

(2) $\mu(d(Tx,Ty)) \leq F(\mu(\frac{1}{2}[d(x,Tx) + d(y,Ty)]), \psi(d(x,Tx), d(y,Ty)))$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1,2,\ldots,m$ where $A_{m+1} = A_1$, $F \in \mathscr{C}$, $\mu \in \Psi$ and $\psi \in \Psi_u$.

Theorem 3.2. *Let (X,d) be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ nonempty closed subsets of X and $Y = \cup_{i=1}^{m} A_i$. Suppose that T is a C-Kannan type cyclic weakly contraction. Then, T has a fixed point $z \in \cap_{i=1}^{m} A_i$.*

Proof. Let $x_0 \in X$. We can construct a sequence $x_{n+1} = Tx_n$, $n = 0,1,2,\ldots$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then the result follows. Indeed, we have $Tx_{n_0} = x_{n_0+1} = x_{n_0}$. So we assume that $x_{n+1} \neq x_n$ for any $n = 0,1,2,\ldots$. As $X = \cup_{i=1}^{m} A_i$, for any $n > 0$ there exists $i_n \in \{1,2,\ldots,m\}$ such that $x_{n-1} \in A_{i_n}$ and $x_n \in A_{i_{n+1}}$. Since T is a C-Kannan type cyclic weakly contraction, we have

$$\begin{aligned}
\mu(d(x_{n+1},x_n)) &= \mu(d(Tx_n,Tx_{n-1})) \\
&\leq F(\mu(\frac{1}{2}[d(x_n,Tx_n)+d(x_{n-1},Tx_{n-1})]),\psi(d(x_n,Tx_n),d(x_{n-1},Tx_{n-1}))) \\
&= F(\mu(\frac{1}{2}[d(x_n,x_{n+1})+d(x_{n-1},x_n)]),\psi(d(x_n,x_{n+1}),d(x_{n-1},x_n))) \\
&\leq \mu(\frac{1}{2}[d(x_n,x_{n+1})+d(x_{n-1},x_n)]).
\end{aligned} \tag{3.1}$$

Since μ is a non-decreasing function, for all $n = 1,2\ldots$, we have

$$d(x_{n+1},x_n) \leq d(x_n,x_{n-1}). \tag{3.2}$$

Thus $\{d(x_{n+1},x_n)\}$ is a monotone decreasing sequence of non-negative real numbers and so convergent. Hence there exists $r \geq 0$ such that $d(x_{n+1},x_n) \to r$. Letting $n \to \infty$ in (3.2), we obtain that $\lim d(x_{n-1},x_{n+1}) = 2r$.

Letting $n \to \infty$ in (3.1), using the continuity of μ and ψ, we obtain that $\mu(r) \leq F(\mu(r),\psi(r,r))$. This implies that, $\mu(r) = 0$, or , $\psi(r,r) = 0$ thus $r = 0$.

Thus we obtain

$$\lim_{n\to\infty} d(x_{n+1},x_n) = 0.$$

Now, we show that $\{x_n\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence. By Lemma 2.4 there exists $\varepsilon > 0$ for which we can find subsequences

$\{x_{n_k}\}$and $\{x_{m_k}\}$ of $\{x_n\}$with $n_k> m_k>k$such that

$$\begin{aligned}\varepsilon = \lim_{k\to\infty} d(x_{m(k)},x_{n(k)}) &= \lim_{k\to\infty} d(x_{m(k)},x_{n(k)+1}) \\ &= \lim_{k\to\infty} d(x_{m(k)+1},x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k)+1},x_{n(k)+1}).\end{aligned} \tag{3.3}$$

On the other hand, we have

$$\begin{aligned}&\mu(d(x_{m_{N_k}+1},x_{n_{N_k}+1})) \\ &\le F(\mu(\frac{1}{2}[d(x_{m_{N_k}},x_{m_{N_k}+1})+d(x_{n_{N_k}},x_{n_{N_k}+1})]),\psi(d(x_{m_{N_k}},x_{m_{N_k}+1}),d(x_{n_{N_k}},x_{n_{N_k}+1}))).\end{aligned} \tag{3.3}$$

When $k\to\infty$, and using Lemma 2.4, we have

$$\mu(\varepsilon)\le F(\mu(0),\psi(0,0))\le\mu(0)=0.$$

So, $\varepsilon=0$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in Y. Since Y is closed in X, then Y is also complete and there exists $x\in Y$ such that $\lim x_n = x$.

Now, we are in a position to prove that x is a fixed point of T. As $Y=\cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T, the sequence $\{x_n\}$ has infinite terms in each A_i for $i=\{1,2,\dots,m\}$. Suppose that $x\in A_i$, $Tx\in A_{i+1}$ and we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k}\in A_i$. By using the contractive condition, we obtain

$$\begin{aligned}\mu(d(x_{n_k+1},Tx)) &= \mu(d(Tx_{n_k},Tx)) \\ &\le F(\mu(\frac{1}{2}[d(x_{n_k},Tx_{n_k})+d(x,Tx)]), \\ &\quad \psi(d(x_{n_k},Tx_{n_k}),d(x,Tx))) \\ &= F(\mu(\frac{1}{2}[d(x_{n_k},x_{n_k+1})+d(x,Tx)]), \\ &\quad \psi(d(x_{n_k},x_{n_k+1}),d(x,Tx))).\end{aligned}$$

Letting $n\to\infty$ and using continuity of F,μ and lower semi-continuity of ψ, we have

$$\mu(d(x,Tx))\le F(\mu(\frac{1}{2}d(x,Tx)),\psi(0,d(x,Tx))).$$

This implies that either $\mu(d(x,Tx))=0$, or $\psi(0,d(x,Tx))=0$, $d(x,Tx)=0$. Hence x is a fixed point of T.

Now, we shall prove the uniqueness of fixed point. Suppose that x_1 and x_2 ($x_1 \neq x_2$) are two fixed points of T. Using the contractive condition and continuity of μ and ψ, we have

$$\begin{aligned}
\mu(d(x_1,x_2)) &= \mu(d(Tx_1,Tx_2)) \\
&\leq F(\mu(\frac{1}{2}[d(x_1,Tx_1)+d(x_2,Tx_2)]), \psi(d(x_1,Tx_1),d(x_2,Tx_2))) \\
&= F(\mu(\frac{1}{2}[d(x_1,x_1)+d(x_2,x_2)]), \psi(d(x_1,x_1),d(x_2,x_2))) \\
&= F(\mu(0),\psi(0,0)) \leq \mu(0) = 0,
\end{aligned}$$

which is a contradiction. Hence the result holds.

If $\mu(a) = a$, then we have the following result.

Corollary 3.3. *Let (X,d) be a complete metric space, $m \in \mathbb{N}$, $A_1,A_2,\ldots,A_m$ nonempty closed subsets of X and $Y = \cup_{i=1}^m A_i$. Suppose that $T: Y \to Y$ is an operator such that*

(1) $\cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T;

(2) $d(Tx,Ty) \leq F(\frac{1}{2}[d(x,Tx)+d(y,Ty)], \psi(d(x,Tx),d(y,Ty)))$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1,2,\ldots,m$ where $A_{m+1} = A_1$, $F \in \mathscr{C}$ and $\psi \in \Psi_u$. Then, T has a fixed point $z \in \cap_{i=1}^m A_i$.

If $F(s,t) = ks$, where $k \in [0,1)$, we have the following result.

Corollary 3.4. *Let (X,d) be a complete metric space, $m \in \mathbb{N}$, $A_1,A_2,\ldots,A_m$ nonempty closed subsets of X and $Y = \cup_{i=1}^m A_i$. Suppose that $T: Y \to Y$ is an operator such that*

(1) $\cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T;

(2) there exists $k \in [0,\frac{1}{2})$ such that $d(Tx,Ty) \leq k[d(x,Tx)+d(y,Ty)]$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1,2,\ldots,m$ where $A_{m+1} = A_1$. Then, T has a fixed point $z \in \cap_{i=1}^m A_i$.

Remark 3.5. If $F(s,t) = s - t$, in the above theorem, we have the corresponding result of Chandok [10].

Other consequences of our results for mappings involving contractions of integral type are:

Denote by Λ the set of functions $\mu: [0,\infty) \to [0,\infty)$ satisfying the following hypotheses:

(h1) μ is a Lebesgue-integrable mapping on each compact subset of $[0,\infty)$;

(h2) for any $\varepsilon > 0$, we have $\int_0^\varepsilon \mu(t) > 0$.

Corollary 3.6. *Let (X,d) be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ nonempty closed subsets of X and $Y = \cup_{i=1}^{m} A_i$. Suppose that $T : Y \to Y$ is an operator such that*

(1) $\cup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T;

(2) there exists $k \in [0, \frac{1}{2})$ such that

$$\int_0^{d(Tx,Ty)} \alpha(s)ds \leq k \int_0^{d(x,Tx)+d(y,Ty)} \alpha(s)ds$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$ where $A_{m+1} = A_1$ and $\alpha \in \Lambda$. Then, T has a fixed point $z \in \cap_{i=1}^{m} A_i$.

If we take $A_i = X$, $i = 1, 2, \ldots, m$, we obtain the following result.

Corollary 3.7. *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping such that*

$$\int_0^{d(Tx,Ty)} \alpha(s)ds \leq k \int_0^{d(x,Tx)+d(y,Ty)} \alpha(s)ds$$

for any $xy \in X$, $k \in [0, \frac{1}{2})$ and $\alpha \in \Lambda$. Then, T has a fixed point $z \in \cap_{i=1}^{m} A_i$.

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