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OPTIMALITY CONDITIONS FOR WEAK EFFICIENT SOLUTIONS OF VECTOR EQUILIBRIUM PROBLEMS WITH CONSTRAINTS

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Abstract. In this paper, we present a necessary as well as sufficient optimality condition for the weak efficient solution to the vector equilibrium problem with constraints VEPC in terms of contingent epiderivatives. First, we study some important properties of the contingent epiderivative of a single-valued mapping in Banach spaces. Second, we establish a Fritz John type necessary optimality condition for VEPC in terms of contingent epiderivatives. Under assumptions on quasiconvexity of scalar functions, a Fritz John type necessary optimality condition becomes a Fritz John type sufficient optimality condition.

Keywords. Contingent epiderivative; Quasiconvex function; Stable function; Steady function; Weak efficient solution.

1. Introduction

The vector equilibrium problems generalize many well known problems in the optimization theory as vector complementarity problems, vector saddle point problems, vector optimization problems and vector variational inequality problems, which have been widely investigated by many authors (see, e.g., Jimenez-Novo [3-4], Aubin-Frankowska [5], Jourani [6], Luc [7], Gong [8], Jahn-Rauh [9], Luu-Hang [10], Morgan-Romaniello [14], Long-Huang-Peng [15], Tran Van Su [16], Bianchi-Hadjisavvas-Schaible [17], etc and the references therein). B. Jimenez *et al.* [3, 4] derived first-order optimality conditions in vector optimization involving stable functions in terms of contingent derivatives. Jahn-Rauh [9] provided first-order necessary and sufficient optimality conditions for vector equilibrium problems without constraints in terms of

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contingent epiderivatives. L.R. Marin and M. Sama *et al.* [1,2] investigated the existences of contingent epiderivatives and its applications to the vector optimization problem. Most recently, Do Van Luu and Dinh Dieu Hang [10] have investigated the optimality conditions for vector variational inequalities and Tran Van Su [16] provided the second-order optimality conditions for vector equilibrium problem. We see that, in most of these papers, optimality conditions for the weak efficient solution to the vector equilibrium problems with constraints in terms of contingent epiderivatives have not been fully discovered yet. Furthermore, these studies do not aim at obtaining optimality conditions for vector equilibrium problems with constraints in terms of contingent epiderivatives with single-valued stable functions. Our study now is, therefore, to establish this for weak efficient solution to the vector equilibrium problems with constraints in terms of contingent epiderivatives.

The remainder of this paper is organized as follows. After some preliminaries and definitions, the relationship between the contingent epiderivative and the contingent derivative of a single-valued mapping in Banach spaces is a well-presented analysis in Section 3. A Fritz John type necessary optimality condition as well as a Fritz John type sufficient optimality condition for the weak efficient solutions to the VEPC is analyzed in Section 4.

2. Preliminaries

Let X,Y,Z be Banach spaces and let Q and S be cones in Y and Z, respectively. Denote by Y^* topological dual space of Y. The dual cone of Q is defined by

$$Q^+ = \{\xi \in Y^* \mid <\xi, q> \geq 0 \quad \forall q \in Q\},$$

where $< .,. >$ instead of the coupling between Y and Y^* and the dual cone of S is defined similarly. Let us denote by $A \subset B$ instead of A is a subset of B. Let $A \subset X$, by $int(A), cl(A)$ and $cone(A) = \{ta \mid t \geq 0,\ a \in A\}$ indicates the interior of A, the closure of A, and the cone hull of A, repectively. For each $y \in Y,\ \delta > 0,\ B(y,\delta)$ stands for the open ball with center at y and radius $\delta > 0$. Let $F : X \longrightarrow 2^Y$ be a set-valued mapping from X into 2^Y, where 2^Y instead of the family of all the subsets of Y. The effective domain, the graph and the epigraph of a set-valued mapping F are given respectively as

$$dom(F) = \{x \in X \mid F(x) \neq \emptyset\},$$

$$graph(F) = \{(x,y) \in X \times Y \,|\, y \in F(x)\},$$

$$epi(F) = \{(x,y) \in X \times Y \,:\, x \in dom(F),\ y \in F(x) + Q\}.$$

Let the mappings $f : X \longrightarrow Y$, $g : X \longrightarrow Z$ and define the mapping $(f,g) : X \longrightarrow Y \times Z$ by $(f,g)(x) = (f(x), g(x))$ $(\forall x \in X)$. Denote by $F = f + Q$, means that $F(x) = f(x) + Q \quad (\forall x \in X)$ and in this case F is called a profile mapping. Let us provide next the definitions about the contingent cones, which will be used in this paper

Definition 2.1. [4, 11] *Let M be a subset of Y and let $\bar{z} \in cl(M)$.*

(i) The contingent cone $T(M,\bar{z})$ of M at $\bar{z}$ is defined as

$$T(M,\bar{z}) = \{y \in Y : \exists t_n \longrightarrow 0^+, \exists y_n \longrightarrow y \text{ such that } \bar{z} + t_n y_n \in M\ \forall\, n\}.$$

(ii) The adjacent cone $A(M,\bar{z})$ of M at $\bar{z}$ is defined as

$$A(M,\bar{z}) = \{y \in Y : \forall t_n \longrightarrow 0^+, \exists\, y_n \longrightarrow y \text{ such that } \bar{z} + t_n y_n \in M \ \ \forall\, n\}.$$

(iii) The interior tangent cone $IT(M,\bar{z})$ of M at $\bar{z}$ is defined as

$$IT(M,\bar{z}) = \{y \in Y :\, \exists\, \delta > 0 \text{ such that } \bar{z} + t\,u \in M\ \forall\, t \in (0,\delta],\ \forall u \in B(y,\delta)\}.$$

(iv) The normal cone $N(M,\bar{z})$ to M at $\bar{z}$ is defined as

$$N(M,\bar{z}) = \{\xi \in Y^* \ :< \xi, z > \leq 0 \quad \forall z \in T(M,\bar{z})\}.$$

Definition 2.2. [1,2,3,4,7,9] *Let $f : X \longrightarrow Y$ be a single-valued map and let $\bar{x} \in X$.*

(i) The contingent derivative of f (resp., $f+Q$) at point $\bar{x}$ is the set-valued map $D_c f(\bar{x})$ (resp., $D_c(f+Q)(\bar{x})$) from X to 2^Y defined as

$$graph\Big(D_c f(\bar{x})\Big) = T(graph(f), (\bar{x}, f(\bar{x})))$$

$$(resp.,\ graph\Big(D_c(f+Q)(\bar{x})\Big) = T(graph(f+Q), (\bar{x}, f(\bar{x}))).$$

(ii) The contingent epiderivative of f at $\bar{x}$ is the single-valued map $\underline{D}f(\bar{x})$ from X into Y defined as

$$epi\Big(\underline{D}f(\bar{x})\Big) = T(epi\,(f), (\bar{x}, f(\bar{x}))).$$

Definition 2.3. [4] *A function f is said to be stable at $\bar{x}$ if there exist a neighborhood U of $\bar{x}$ and $L > 0$ such that*

$$||f(x) - f(\bar{x})|| \leq L||x - \bar{x}|| \quad \forall\, x \in U.$$

If $||f(x)-f(x')|| \leq L||x-x'|| \quad \forall\, x,x' \in U$, *then we say that* f *is Lipschitz around* $\overline{x}$. *If for each* $\overline{x} \in X$ *there exists a neighborhood* U *of* $\overline{x}$ *such that f is Lipschitz around* $\overline{x}$, *we will say that f is locally Lipschitz on X.*

Definition 2.4. [4] *A function* $f : X \longrightarrow Y$ *is said to be steady at* $\overline{x}$ *in the direction* $v \in X$ *if*

$$\lim_{(t,u)\longrightarrow(0^+,v)} \frac{f(\overline{x}+tu)-f(\overline{x}+tv)}{t} = 0.$$

It is said that f is steady at $\overline{x}$ *if f is steady at* $\overline{x}$ *in all the directions.*

Following [4], f is steady at $\overline{x}$ in the direction 0 if and only if f is stable at $\overline{x}$. Thus, if f is steady at $\overline{x}$ then f is stable at $\overline{x}$. If f is Frechet differentiable at $\overline{x}$ then its Frechet derivative at $\overline{x}$ is denoted by $\triangledown h(\overline{x})$. It is easy to see that

$$D_c f(\overline{x})v = \{\triangledown h(\overline{x})v\}$$

for all $v \in X$.

Definition 2.5. [1,7] *(i)* $\quad y \in A$ *is an ideal minimal point of A with respect to Q if* $a \geq y$ *for any* $a \in A$. *The set of all ideal points is denoted by* $IMin(A)$ *or* $IMin(A|Q)$. *It is easy to see that*

$$IMin(A) = \{a \in A \,:\, A \subset \{a\} + Q\}.$$

(ii) $\quad y \in A$ *is an efficient point of A with respect to Q if* $a \geq y$ *for some* $y \in A$ *then* $y \geq a$. *The set of all efficient points is denoted by* $Min(A)$ *or* $Min(A|Q)$ *and it is given as*

$$Min(A) = \{a \in Y \,:\, A \cap (\{a\} - Q) = \{a\}\}.$$

The notions $IMax(A), Max(A)$ are defined dually. Notice that in this definition 2.5, we denote by $x \geq y$ indicates $x \in y+Q$. Following Luc, T. D [7, Prop., 2.2] that, if $IMin(A) \neq \emptyset$ then $IMin(A) = Min(A)$ and it is a point whenever Q is pointed.

Definition 2.6. [4] *The Hadamard derivative of* $f : X \longrightarrow Y$ *at* $\overline{x}$ *in the direction* $v \in X$ *is*

$$df(\overline{x},v) = \lim_{(t,u)\longrightarrow(0^+,v)} \frac{f(\overline{x}+tu)-f(\overline{x})}{t}.$$

f is said to be Hadamard differentiable at $\overline{x}$ *in the direction v if* $df(\overline{x},v)$ *exists.*

It is easy to see that

$$D_c f(\overline{x})v = \{df(\overline{x},v)\} = \{\triangledown h(\overline{x})v\}.$$

A function which Lipschitz around $\overline{x}$ or Hadamard differentiable at $\overline{x}$ then steady at that point.

Finally, let us denote by $t_n \longrightarrow 0^+$ instead of a sequence of positive numbers with limit 0 and by $(t_n, x_n, y_n) \longrightarrow (0^+, x, y)$ indicates $t_n \longrightarrow 0^+$, $x_n \longrightarrow x$, $y_n \longrightarrow y$ as $n \longrightarrow \infty$.

3. Contingent epiderivatives

Let X, Y be Banach spaces with Y be partially ordered by a pointed closed convex cone Q, $\bar{x} \in X$ and the mapping $f : X \longrightarrow Y$. For each $v \in X$, denote by $L(f,v)$ $(resp., L_Q(f,v))$ instead of the following set

$$\{z \in Y \ : \ D_c f(\bar{x})v \subset \{z\} + Q\}$$
$$\Big(resp., \quad \{z \in Y \ : \ D_c(f+Q)(\bar{x})v \subset \{z\} + Q\}\Big).$$

It is not difficult to check that for all $v \in X$, $L_Q(f,v) \subset L(f,v)$ and making use of Definition 2.5 (i), we get $L_Q(f,v) \cap D_c(f+Q)(\bar{x})v = \{\underline{D}f(\bar{x})v\}$.

Proposition 3.1. *For all $v \in X$, we have the following assertions hold*

(i) If the contingent epiderivative $\underline{D}f(\bar{x}) : X \longrightarrow Y$ exists then $L_Q(f,v) \neq \emptyset$ and $L(f,v) \neq \emptyset$. Moreover, $\underline{D}f(\bar{x})v \in L_Q(f,v)$.

(ii) If $L_Q(f,v) \neq \emptyset$ then $\underline{D}f(\bar{x})$ exists if and only if $IMax(L_Q(f,v)) \in D_cF(\bar{x})v$ for all $v \in dom(D_cF(\bar{x}))$ and $F = f + Q$.

Proof. For the case (i): It is immediate from Theorem 3.1 [1].

For the case (ii): It follows from Theorem 5.6 [1, p. 755] and the claim follows. ■

Theorem 3.1. *Assume that the contingent epiderivative of f at $\bar{x}$ exists in all directions and $IMin(D_c f(\bar{x})v) \neq \emptyset \quad (\forall v \in X)$. Then the following conditions are equivalent*

(i) $\{\underline{D}f(\bar{x})v\} = IMin(D_c f(\bar{x})v) \quad (\forall v \in X)$.

(ii) $L(f,v) = L_Q(f,v) \quad (\forall v \in X)$.

(iii) $D_c(f+Q)(\bar{x})v \subset D_c f(\bar{x})v + Q \quad (\forall v \in X)$.

(iv) $D_c(f+Q)(\bar{x})v + Q = D_c f(\bar{x})v + Q \quad (\forall v \in X)$.

Proof. The implication $i \Longrightarrow ii$: If (i) holds then

$$\{\underline{D}f(\bar{x})v\} = IMin(D_c f(\bar{x})v) \quad (\forall v \in X).$$

Let us prove that

$$L(f,v) \subset L_Q(f,v) \quad (\forall v \in X). \tag{3.1}$$

In fact, for every $v \in X$ and for every $z \in L(f,v)$, we have $D_c f(\bar{x})v \subset \{z\} + Q$, which is equivalent to

$$D_c f(\bar{x})v + Q \subset \{z\} + Q. \tag{3.2}$$

If the inclusion of $D_c(f+Q)(\bar{x})v \subset \{z\} + Q$ does not true, then there exists $a \in D_c(f+Q)(\bar{x})v$ but $a \notin z + Q$. In other words, in view of Proposition 3.1 (ii) $\underline{D}f(\bar{x})v \in L_Q(f,v)$ and a consequence is

$$a \in D_c(f+Q)(\bar{x})v \subset \{\underline{D}f(\bar{x})v\} + Q. \tag{3.3}$$

Making use of Definition 2.5 (i) we get

$$\underline{D}f(\bar{x})v \in D_c f(\bar{x})v \subset \{\underline{D}f(\bar{x})v\} + Q. \tag{3.4}$$

Because of $\underline{D}f(\bar{x})v$ is an ideal minimal point of $D_c f(\bar{x})v$ with respect to Q. Combining (3.3)-(3.4), yields that $a \in D_c f(\bar{x})v + Q$. Therefore

$$D_c f(\bar{x})v + Q \not\subset \{z\} + Q,$$

which contradicting (3.2). From there we conclude that $z \in L_Q(f,v)$, it means that (3.1) holds. So, the implication $i \Longrightarrow ii$ holds.

The implication $ii \Longrightarrow i$: Suppose that we have (ii), means that $L(f,v) = L_Q(f,v) \quad (\forall v \in X)$. Thus,

$$IMax(L(f,v)) = IMax(L_Q(f,v)) \quad (\forall v \in X).$$

Note that the contingent epiderivative $\underline{D}f(\bar{x})$ exists in all directions by assumption, hence $IMin\big(D_c(f+Q)(\bar{x})v \,|\, Q\big) \neq \emptyset \; \forall v \in X$ (see Marin and Sama [1]). Besides, $IMin\big(D_c f(\bar{x})v \,|\, Q\big) \neq \emptyset \; \forall \; v \in X$. By taking into account Corollary 5.3 [1, p. 754] it infers that (i) is true. So, the case $(i) \Longleftrightarrow (ii)$ is provided.

The implication $(i) \Longrightarrow (iii)$: Suppose (iii) does not true, then there exists $v_0 \in X$ such that $D_c(f+Q)(\bar{x})v_0 \not\subset D_c f(\bar{x})v_0 + Q$, means that, there is $z \in D_c(f+Q)(\bar{x})v_0$ but

$$z \notin D_c f(\bar{x})v_0 + Q. \tag{3.5}$$

By virtue of Theorem 3.1 [1] we get $z \in \underline{D}f(\bar{x})v_0 + Q$. In other words, by the definitions of ideal minimal point $\underline{D}f(\bar{x})v_0$ it follows that $\underline{D}f(\bar{x})v_0 \in D_c f(\bar{x})v_0$ and consequently $z \in D_c f(\bar{x})v_0 + Q$, which conflicts with (3.5). So, $(i) \Longrightarrow (iii)$ is proved finish.

The implication $(iii) \Longrightarrow (i)$: From the fact

$$D_c(f+Q)(\bar{x})v \subset D_c f(\bar{x})v + Q \quad (\forall v \in X)$$

we deduce that $L(f,v) = L_Q(f,v) \quad (\forall v \in X)$ and the rest proof is inferred from the implication $(ii) \Longrightarrow (i)$. This shows that the equivalence of (i) and (iii) is true.

Finally, we prove that $(iii) \Longleftrightarrow (iv)$. The implication $(iv) \Longrightarrow (iii)$ is obvious. We see the implication $(iii) \Longrightarrow (iv)$. From (iii) and the fact that $Q+Q=Q$ we have

$$D_c(f+Q)(\bar{x})v + Q \subset D_c f(\bar{x})v + Q \quad (\forall v \in X).$$

As $graph(f) \subset graph(f+Q)$, thus

$$D_c f(\bar{x})v + Q \subset D_c(f+Q)(\bar{x})v + Q \quad (\forall v \in X).$$

So,

$$D_c f(\bar{x})v + Q = D_c(f+Q)(\bar{x})v + Q \quad (\forall v \in X)$$

and the claim follows. So, the theorem is proved complete. ■

Theorem 3.2. *Assume that the contingent epiderivative of f at $\bar{x}$ exists in all directions and $IMin(D_c f(\bar{x})v) \neq \emptyset \quad (\forall v \in X)$. Then we have*

$$\{\underline{D}f(\bar{x})v\} = IMin(D_c f(\bar{x})v) \quad (\forall v \in X)$$

if one of the following conditions is fulfilled

(i) Cone Q has a compact base, say B.

(ii) The set $\{\, e \in Q \mid ||e|| = 1 \,\}$ is compact.

Proof. Let us first assume that (i) holds. By fixing $v \in X$ and taking $z = \underline{D}f(\bar{x})v$, which implies that $(v,z) \in T(epi(f),(\bar{x}, f(\bar{x})))$. By the definition of contingent cone $T(epi(f),(\bar{x}, f(\bar{x})))$ there exist $(t_n, v_n, z_n) \longrightarrow (0^+, v, z)$ and $q_n \in Q$ such that $f(\bar{x}) + t_n(z_n - q_n) = f(\bar{x} + t_n v_n)$ $(\forall n \geq 1)$. By hypotheses, B is a compact base of cone Q and as $q_n \in Q$, it follows that there exist some $c_n > 0$ and $b_n \in B$ such that $q_n = c_n b_n$ $(\forall n \geq 1)$. As Q has a compact base B, without loss of generality, we may assume that $b_n \longrightarrow b \in B$. We next prove that $q_n \longrightarrow 0$. Suppose contrary, that $q_n \not\longrightarrow 0$ then $c_n \not\longrightarrow 0$. As $c_n > 0$ thus there is $\varepsilon > 0$ such that $c_n \geq \varepsilon$ $(\forall n \geq 1)$. It follows that $t_n(1-\varepsilon c_n^{-1})q_n \in Q$ $(\forall n \geq 1)$ and moreover

$$\begin{aligned} f(\bar{x}) + t_n(z_n - \varepsilon c_n^{-1} q_n) &= f(\bar{x}) + t_n(z_n - q_n) + t_n(1-\varepsilon c_n^{-1})q_n \\ &\in f(\bar{x} + t_n v_n) + Q \ (\forall\, n \geq 1). \end{aligned}$$

As $z_n - \varepsilon c_n^{-1} q_n = z_n - \varepsilon b_n \longrightarrow z - \varepsilon b$, thus by the definition of contingent derivative of $f+Q$ at $(\bar{x}, v)$ and by the definition of profile mapping $f+Q$, it follows that $z - \varepsilon b \in D_c(f+Q)(\bar{x})v$. By virtue of Theorem 3.1 in [1] and Proposition 2.2 in [7], we deduce that

$$Min(D_c(f+Q)(\bar{x})v) = IMin(D_c(f+Q)(\bar{x})v) = \{\underline{D}f(\bar{x})v\}. \tag{3.6}$$

Therefore

$$D_c(f+Q)(\bar{x})v \cap (\underline{D}f(\bar{x})v - Q) = \{\underline{D}f(\bar{x})v\}.$$

It is clear that

$$z - \varepsilon b \in D_c(f+Q)(\bar{x})v \cap (\{\underline{D}f(\bar{x})v\} - Q),$$

which is equivalent to $z - \varepsilon b = z \Longleftrightarrow \varepsilon b = 0$ implies $b = 0$ and this conflicts with the fact that $\varepsilon b \neq 0$. So, $q_n \longrightarrow 0$ and hence $z \in D_c f(\bar{x})v$. Consequently

$$\underline{D}f(\bar{x})v \in D_c f(\bar{x})v \subset D_c(f+Q)(\bar{x})v \subset \underline{D}f(\bar{x})v + Q.$$

Adapting the definition of ideal minimal point we deduce that

$$\{\underline{D}f(\bar{x})v\} = IMin(D_c f(\bar{x})v) \quad (\forall\, v \in X)$$

and the claim follows.

Let us second assume that (ii) holds. By preceding proof, one gets

$$\{\underline{D}f(\bar{x})v\} = Min(D_c(f+Q)(\bar{x})v) \quad (\forall\, v \in X).$$

By virtue of theorem 4.1 in [1, p. 751], we deduce that

$$Min(D_c(f+Q)(\bar{x})v) \subset D_c f(\bar{x})v,$$

which implies that $\underline{D}f(\bar{x})v \in D_c f(\bar{x})v$ $(\forall\, v \in X)$.

Obviously, the inclusion of

$$D_c f(\bar{x})v \subset D_c(f+Q)(\bar{x})v \ (\forall\, v \in X)$$

holds and a consequence is

$$D_c f(\overline{x})v \subset \{\underline{D}f(\overline{x})v\} + Q \ (\forall v \in X),$$

because $0 \in Q$ and $D_c(f+Q)(\overline{x})v \subset \{\underline{D}f(\overline{x})v\} + Q \ (\forall v \in X)$. From there we conclude and the theorem is proved complete. ■

Corollary 3.1. *Assume that $dim(Y) < +\infty$ and the contingent epiderivative of f at $\overline{x}$ exists in all directions and moreover $IMin(D_c f(\overline{x})v) \neq \emptyset \quad (\forall v \in X)$. Then we have*

(i) $\{\underline{D}f(\overline{x})v\} = IMin(D_c f(\overline{x})v) \quad (\forall v \in X)$.

(ii) $L(f,v) \cap D_c f(\overline{x})v = \{\underline{D}f(\overline{x})v\} \quad (\forall v \in X)$.

Proof. As $dim(Y) < +\infty$ and Q is a pointed closed convex cone in Y, thus Q has compact base (see in Luc [7, Remark 1.6]). From here, the proof of case (i) is inferred from Theorem 3.2. Analogously, the case (ii) is also inferred from the case (i) and the claim follows. ■

4. Optimality conditions

In this subsection, unless some specialized states, let X, Y, Z and W be finite-dimensional Banach spaces, C be a nonempty subset in X, $Q \subset Y$ be a closed pointed convex convex in Y and $S \subset Z$ be a convex cone in Z and moreover $intQ \neq \emptyset$, $intS \neq \emptyset$. Let the vector bifunction $F : X \times X \longrightarrow Y$ with $F(x,x) = 0 \ (\forall x \in X)$ and the objective functions $g : X \longrightarrow Z$, $h : X \longrightarrow W$. Let us consider the following vector equilibrium problem with constraints (write short, VEPC): Finding a vector $\overline{x} \in K$ such that

$$F(\overline{x},x) \notin -intQ \quad (\forall x \in K).$$

$K = \{x \in C \ | \ g(x) \in -S, \ h(x) = 0\}$ is called the feasible set of VEPC. A vector $\overline{x}$ solved VEPC will be called a weak efficient solution of VEPC.

For each $\overline{x} \in K$ (vector $\overline{x}$ is called a feasible point for VEPC), denote by

$$F_{\overline{x}} = F(\overline{x}, \ . \ ) : X \longrightarrow Y, \quad F_{\overline{x}}(K) = \bigcup_{x \in K} F(\overline{x},x).$$

By definition we have $\overline{x} \in K$ solved VEPC if and only if $F_{\overline{x}}(K) \cap (-intQ) = \emptyset$.

Theorem 4.1. *Let $\overline{x}$ be a feasible point of VEPC and assume, in addition, that $F_{\overline{x}}$ and g be steady at $\overline{x}$; h be continuous on a neighborhood of $\overline{x}$ and Frechet differentiable at $\overline{x}$ with the* *system* $\{\nabla h_k(\overline{x})\}_{k=1,2,\dots,dim(W)}$ *linearly independent. Suppose that the contingent epiderivatives of* $F_{\overline{x}}$ *and* g *at* $\overline{x}$ *exist in all the directions* $v \in X$. *If* $\overline{x}$ *is a weak efficient solution of VEPC then* **either** *there exist* $(\lambda,\eta) \in Y^* \times Z^*$ *with* $(\lambda,\eta) \neq (0,0)$ *and there exists* $\gamma \in W^*$ *satisfying*

$$\lambda \in Q^+,\ \eta \in N(-S, g(\overline{x})), \tag{4.1}$$

$$<\lambda, \underline{D}F_{\overline{x}}(\overline{x})v> + <\eta, \underline{D}g(\overline{x})v> + <\gamma, \nabla h(\overline{x})v> \geq 0 \quad \forall\, v \in X, \tag{4.2}$$

or *there exists* $\overline{v} \in ker\, \nabla h(\overline{x})$ *such that*

$$(\underline{D}F_{\overline{x}}(\overline{x})\overline{v},\ \underline{D}g(\overline{x})\overline{v}) \notin D_c(F_{\overline{x}}, g)(\overline{x})\overline{v}.$$

If, moreover, condition (S) : *For every* $\eta \in N(-S, g(\overline{x})) \setminus \{0\}$, *there exists at least one direction* $\overline{v} \in ker\nabla h(\overline{x})$ *such that* $<\eta, z_g> < 0$ $(\forall\, z_g \in D_c g(\overline{x})\overline{v})$ *holds, then* $\lambda \neq 0$ *(Kuhn Tucker type optimality condition).*

Proof. Assume that the assertions of Theorem 4.1 are false. Denote by

$$T = \{(\lambda,\eta) \in Y^* \times Z^* \mid \lambda \in Q^+, \eta \in N(-S, g(\overline{x})) \text{ such that } (\lambda,\eta) \neq (0,0)\}.$$

Then we can find a direction $v_0 \in X$ such that $\forall\big((\lambda,\eta),\gamma\big) \in T \times W^*$,

$$<\lambda, \underline{D}F_{\overline{x}}(\overline{x})v_0> + <\eta, \underline{D}g(\overline{x})v_0> + <\gamma, \nabla h(\overline{x})v_0> < 0 \tag{4.3}$$

and furthermore

$$(\underline{D}F_{\overline{x}}(\overline{x})v,\ \underline{D}g(\overline{x})v) \in D_c(F_{\overline{x}}, g)(\overline{x})\, v \quad (\forall\, v \in ker\, \nabla h(\overline{x})). \tag{4.4}$$

Making use of (4.3) in the case $\gamma = 0$, we get

$$<\lambda, \underline{D}F_{\overline{x}}(\overline{x})v_0> + <\eta, \underline{D}g(\overline{x})v_0> < 0 \quad \forall (\lambda,\eta) \in T. \tag{4.5}$$

Let us prove that

$$\nabla h(\overline{x})v_0 = 0. \tag{4.6}$$

In fact, if (4.6) is false, then there exists $\overline{\gamma} \in W^*$ such that

$$<\overline{\gamma},\ \nabla h(\overline{x})v_0> > 0. \tag{4.7}$$

In other words, it is clear that

$$\forall\, \alpha \in \mathbb{R}^+ \setminus \{0\}, \quad <\alpha\overline{\gamma},\ \nabla h(\overline{x})v_0> = \alpha\ <\overline{\gamma},\ \nabla h(\overline{x})v_0> > 0.$$

We take the pair $(\lambda,\eta)\in T$ is arbitrary. Making use of (4.3) and $\alpha\overline{\gamma}\in W^*$, we get

$$\alpha<\overline{\gamma},\nabla h(\overline{x})v_0><-\big(<\lambda,\underline{D}F_{\overline{x}}(\overline{x})v_0>+<\eta,\underline{D}g(\overline{x})v_0>\big).$$

By dividing both sides of obtained inequality by $\alpha>0$, we obtain as follows

$$<\overline{\gamma},\nabla h(\overline{x})v_0><-\frac{1}{\alpha}\big(<\lambda,\underline{D}F_{\overline{x}}(\overline{x})v_0>+<\eta,\underline{D}g(\overline{x})v_0>\big).$$

By letting $\alpha\longrightarrow+\infty$, $<\overline{\gamma},\nabla h(\overline{x})v_0>\leq 0$, which conflicts with (4.7). So, $v_0\in ker\nabla h(\overline{x})$. On the other hand, it follows from (4.4) that

$$(\underline{D}F_{\overline{x}}(\overline{x})v_0,\ \underline{D}g(\overline{x})v_0)\in D_c(F_{\overline{x}},g)(\overline{x})v_0.$$

By a similar argument as in the proof of Lemma 6.3 [4], we get

$$(\underline{D}F_{\overline{x}}(\overline{x})v_0,\underline{D}g(\overline{x})v_0)\notin(-intQ)\cap IT(-S,g(\overline{x})).$$

It is well-known that $IT(-S,g(\overline{x}))$ and $-intQ$ are open sets and hence $(-intQ)\cap IT(-S,g(\overline{x}))$ is an open set. From here, it is not difficult to check that there exist $(\overline{\lambda},\overline{\eta})\in T$ such that

$$<\overline{\lambda},\underline{D}F_{\overline{x}}(\overline{x})v_0>+<\overline{\eta},\underline{D}g(\overline{x})v_0)>\geq 0,$$

which contradicting condition(4.5) and the conclusion follows.

If the condition (S) is fulfilled then $\lambda\neq 0$. In fact, if it were not so, then $\lambda=0$, which yields that $\eta\in N(-S,g(\overline{x}))\setminus\{0\}$. Furthermore, one can take an $\overline{v}\in ker\nabla h(\overline{x})$ such that

$$<\eta,z_g><0\ \ \forall z_g\in D_cg(\overline{x})\overline{v}.$$

Because of $graph(g)\subset graph(g+S)$, and in view of [7, Theorem 3.1] we deduce that

$$D_cg(\overline{x})\overline{v}\subset\underline{D}g(\overline{x})\overline{v}+S.$$

Therefore $\forall z_g\in D_cg(\overline{x})\overline{v}\Longrightarrow z_g-\underline{D}g(\overline{x})\overline{v}\in S$. Consequently,

$$0><\eta,z_g>\geq<\eta,\underline{D}g(\overline{x})\overline{v}>\quad\forall z_g\in D_cg(\overline{x})\overline{v},$$

which contradicts (4.2), as was to be shown. ■

Remark 4.1. From the fact that $N(-S,g(\overline{x}))=-(T(-S,g(\overline{x}))^+)$, it follows that

$$\eta\in N(-S,g(\overline{x}))\Longleftrightarrow\eta\in S^+,<\eta,g(\overline{x})>=0.$$

Corollary 4.1. *Let $\overline{x}$, $F_{\overline{x}}$, g and h be given as in Theorem 4.1. Assume, in addition, that there is at least one of the functions $F_{\overline{x}}$ and g is Hadamard differentiable at $\overline{x}$ and the cones Q and S has compact bases. Then, if $\overline{x}$ is a weak efficient solution of VEPC then the Fritz John type optimality condition holds, means that there exist $(\lambda,\eta,\gamma)\in Y^*\times Z^*\times W^*$ with $(\lambda,\eta)\neq(0,0)$ satisfying the conditions (4.1) and (4.2). If, in addition, the condition (S) in Theorem 4.1 is fulfilled, then the Kuhn Tucker type optimality condition holds, means that $\lambda\neq 0$.*

Proof. As Q and S has compact bases, thus by direct applying Theorem 3.2 it follows that $\underline{D}k(\overline{x})v\in D_c k(\overline{x})v \quad \forall v\in X$, where k belongs to $\{F_{\overline{x}},\ g\}$. On the other hand, making use of the assumptions and Proposition 3.3 (ii) [4, p. 452] we get

$$(\underline{D}F_{\overline{x}}(\overline{x})v,\ \underline{D}g(\overline{x})v)\in D_c(F_{\overline{x}},g)(\overline{x})v\ \forall v\in X,$$

and the claim follows. ■

Corollary 4.2. *Let $\overline{x}$, $F_{\overline{x}}$, g and h be given as in Theorem 4.1. Suppose, furthermore, that there is at least one of the functions $F_{\overline{x}}$ and g is Hadamard differentiable at $\overline{x}$ and the cone S is pointed closed convex. Then, if $\overline{x}$ is a weak efficient solution of VEPC then the Fritz John type optimality condition is valid. If, in addition, the condition (S) in Theorem 4.1 holds, then the Kuhn Tucker type optimality condition is valid.*

Proof. In view of the proof of Corollary 3.1 in section 3, the cones Q and S has compact bases. By using Corollary 4.1 we conclude. The proof is complete. ■

We recall [10] that:

► *An extended-real-valued function l, defined on a set $C\subset X$, is said to be quasiconvex at $\overline{x}\in C$ with respect to C if and only if, for each $x\in C$,*

$$l(x)\leq l(\overline{x}) \quad\Longrightarrow\quad \forall\, t\in(0,1), \qquad l(tx+(1-t)\overline{x})\leq l(\overline{x}).$$

l is said to be quasiconvex on C if and only if it is quasiconvex at each $x\in C$.

Theorem 4.2. *Let X be a Banach space, Y,Z and W be finite-dimensional and let $\overline{x}\in C$, $F_{\overline{x}},g$ be steady at $\overline{x}$ and h be Frechet differentiable at $\overline{x}$. Suppose, furthermore, that*

(i) The contingent epiderivatives of $F_{\overline{x}}$ and g at $\overline{x}$ exist in all directions.

(ii) There exist $(\lambda,\eta,\gamma)\in Y^*\times Z^*\times W^*$ *with* $\lambda\neq 0$ *satisfying (4.1) and the following inequality*

$$<\lambda,\underline{D}F_{\overline{x}}(\overline{x})v>+<\eta,\underline{D}g(\overline{x})v>+<\gamma,\nabla h(\overline{x})v>>0 \quad \forall\, v\in X\setminus\{0\}.$$

(iii) The scalar functions $\lambda_0 F_{\overline{x}}$*,* $\eta_0 g$ *and* $\gamma_0 h$ *are quasiconvex at* $\overline{x}$ *with respect to C, and C is convex.*

Then vector $\overline{x}$ *is a weak efficient solution to the VEPC.*

Proof. Contrary with the conclusion in Theorem 4.2, let us may assume that there exists element $x\in C\setminus\{\overline{x}\}$ such that $F_{\overline{x}}(x)\in -intQ$, $g(x)\in -S$ and $h(x)=0$. Let us consider the mapping $F:X\longrightarrow Y\times Z\times W$ be given by

$$F(x)=(F_{\overline{x}}(x),g(x),h(x)) \quad \forall\, x\in X.$$

Then $F=(F_{\overline{x}},g,h)$ is steady at $\overline{x}$ as $F_{\overline{x}},g$ and h are steady at that point. So, F is stable at $\overline{x}$. Without loss of generality, suppose that, for all sequences $(t_n)_{n\geq 1}\subset(0,1)$ such that $t_n\longrightarrow 0^+$ we get

$$\lim_{n\longrightarrow\infty}\frac{F(\overline{x}+t_n(x-\overline{x}))-F(\overline{x})}{t_n}=(a,b,c)\in D_cF(\overline{x})(x-\overline{x}). \tag{4.8}$$

It is not difficult to see that $<\lambda,F_{\overline{x}}(x)><<\lambda,F_{\overline{x}}(\overline{x})>$ and by making use of the definition of quasiconvexity of $\lambda_0 F_{\overline{x}}$ at $\overline{x}$ with respect to C, for n sufficiently large,

$$<\lambda,F_{\overline{x}}(\overline{x}+t_n(x-\overline{x}))><<\lambda,F_{\overline{x}}(\overline{x})>.$$

Consequently

$$<\lambda,a>=\lim_{n\longrightarrow\infty}\frac{\lambda_0 F_{\overline{x}}(\overline{x}+t_n(x-\overline{x}))-\lambda_0 F_{\overline{x}}(\overline{x})}{t_n}\leq 0. \tag{4.9}$$

By virtue of Proposition 3.3 [5, p. 452] we deduce that

$$a\in D_cF_{\overline{x}}(\overline{x})(x-\overline{x})\subset D_c(F_{\overline{x}}+Q)(\overline{x})(x-\overline{x}).$$

On the other hand, by hypotheses

$$D_c(F_{\overline{x}}+Q)(\overline{x})(x-\overline{x})=\underline{D}F_{\overline{x}}(\overline{x})(x-\overline{x})+Q,$$

which implies that $a-\underline{D}F_{\overline{x}}(\overline{x})(x-\overline{x})\in Q$. Since $\lambda\in Q^+$ hence

$$<\lambda,\underline{D}F_{\overline{x}}(\overline{x})(x-\overline{x})>\leq<\lambda,a>\leq 0. \tag{4.10}$$

In the same way as above, we also obtain the following inequalities

$$<\eta,\underline{D}g(\overline{x})(x-\overline{x})>\leq<\eta,b>\leq 0, \tag{4.11}$$

$$< \gamma, \nabla h(\bar{x})(x-\bar{x}) > \leq < \eta, c > \leq 0. \tag{4.12}$$

Combining (4.10)-(4.12), yields that

$$< \lambda, \underline{D}F_{\bar{x}}(\bar{x})(x-\bar{x}) > + < \eta, \underline{D}g(\bar{x})(x-\bar{x}) > + < \gamma, \nabla h(\bar{x})(x-\bar{x}) > \leq 0,$$

which contradictics the hypotheses and the claim follows. ■

Remark 4.2. Notice that all the results in our paper are still valid if, either the functions $F_{\bar{x}}$ and g Lipschitz around $\bar{x}$, or the functions $F_{\bar{x}}$ and g Hadamard differentiable at $\bar{x}$ and its contingent epiderivatives at $(\bar{x}, v)$ are replaced respectively by its Hadamard derivatives at $(\bar{x}, v)$.

To close this paper, we give an example to illustrate as follows.

Example 4.1. *Let X be a real Banach space, $Y = Z = \mathbb{R}$, $C = cl\, B_X(0,1) = \{x \in X : ||x|| \leq 1\}$, $\bar{x} = 0$, $Q = S = \mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$ and $F_{\bar{x}}, g : X \longrightarrow \mathbb{R}$ be defined respectively as*

$F_{\bar{x}}(x) = ||x|| - ||\bar{x}||,\ x \in X;$

$g(x) = ||x||^2 - 1,\ x \in X.$

The mappings $F_{\bar{x}}, g$ satisfying all the assumptions of Theorem 4.2 and condition (i) is inferred from Corollary 1 [9, p. 194]. Making use of Theorem 6 [9] and Theorem 3.2 in section 3, for every $v \in X$ we get

$\underline{D}F_{\bar{x}}(\bar{x})v = max\{< \xi, v > \mid \xi \in X^*,\ ||\xi||_{X^*} \leq 1\},$

$\underline{D}g(\bar{x})v = IMin(D_c g(\bar{x})v \mid \mathbb{R}_+) = 0.$

By taking $\lambda = 1 \in Q^+$ and $\eta = 0 \in N(-S, g(\bar{x}))$ then the left-hand side of (4.2) is equal to

$L(v) = max\{< \xi, v > \mid \xi \in X^*,\ ||\xi|| \leq 1\}\ \forall v \in X \setminus \{0\}.$

Let us see condition (ii), which is equivalent to $L(v) > 0,\ \forall v \in X \setminus \{0\}$.
In fact, for $\forall v \in X \setminus \{0\}$, by virtue of a corollary of Hahn-Banach Theorem there exists $\xi_0 \in X^*$ such that $< \xi_0, v > = ||v|| > 0$ and $||\xi_0||_{X^*} = 1$. So, condition (ii) is valid. Finally, for $\lambda = 1$, $\lambda_0 F_{\bar{x}}(x) = ||x||\ \forall x \in X$ is a convex function and therefore it is also quasiconvex at $\bar{x}$ with respect to C, where C is obviously convex. So, condition (iii) is also valid. From here, we conclude that $\bar{x}$ is a weak efficient solution to the VEPC. ■

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