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APPROXIMATE SOLUTIONS OF TIME-FRACTIONAL SHARMA-TASSO-OLEVER EQUATIONS VIA HOMOTOPY ANALYSIS METHODS

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Abstract. In this paper, the homotopy analysis transform method is used to solve the time-fractional Sharma-Tasso-Olever (STO) equation. This method yields an approximate analytical solution of a rapidly convergent power series with easily computable terms and produces a good approximate solution on enlarged intervals for solving the time-fractional STO equation.

Keywords. Approximate solution; Homotopy analysis method; Laplace transform; Fractional STO equation.

1. Introduction

Homotopy analysis method is based on construction of a homotopy which continuously deforms an initial value approximation to the exact solution of the given problem, it was first proposed and applied by Liao [1, 2] based on homotopy, a fundamental concept in topology and differential geometry. An auxiliary linear operator is chosen to construct the homotopy and an auxiliary parameter is used to control the region of convergence of the solution series. The coupling of homotopy analysis and Laplace transform method are not limited to any small physical parameters in the considered equation [3, 4, 5]. Therefore, this method can overcome the foregoing restrictions and limitations of perturbation techniques so that it provides us with a powerful tool to analyze strongly nonlinear problems, and the main advantage of this method is its capability of combining two powerful methods for obtaining rapid convergent series.

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Recently, fractional calculus has obtained considerable popularity and importance as generalizations of integer-order evolution equation, and used to model problems in physics, neurons, hydrology, viscoelasticity and rheology, image processing, mechanics, mechatronics, finance and control theory, see [6, 7, 8, 9, 10]. In reality, a physical phenomenon may depend on not only the time instant but also the previous time history, which can be successfully modeled by using the theory of derivatives and integrals of fractional order [11]. The STO equation has been applied to describe a wide range of physics phenomena of the evolution and interaction to nonlinear waves, such as fluid dynamics, aerodynamics, continuum mechanics, solitons and turbulence et al, it possesses an infinitely many symmetries and the bi-Hamiltonian formulation. If the Hamiltonian of conservative system is constructed using fractional derivatives, the resulting equations of motion can be nonconservative. Therefore, in many cases, the real physical processes could be modeled in a reliable manner using fractional-order differential equations rather than integer-order equations [12]. Many powerful and efficient methods have been proposed to construct the solutions for some time-fractional differential equations. Bulut and Pandir [13] applied the modified trial equation method to time-fractional STO equation by the using of the complete discrimination system for polynomial method. Golmamadian [14] constructed the exact complex solutions of nonlinear time-fractional STO equation by the direct algebraic method. Comparison of the obtained results with those of various methods, homotopy analysis method can lead to conclude that the method gives significantly important consequences, and the homotopy analysis method solution includes an auxiliary parameter which provides a convenient way of adjusting and controlling the convergence region of solution series [15, 16, 17]. The aim of this paper is to apply the homotopy analysis method combining with Laplace transform to approximate solution of the time-fractional STO equation. Since recently, many authors have paid attention to studying the solution of fractional differential equations by using various methods with combined the Laplace transform [18, 19]. We will discuss the methodology for the construction of some schemes and study their performance on test problem, the approximate solutions are very rapidly convergent. It will be concluded that the time-fractional homotopy analysis transform method is very powerful and efficient in finding approximate analytical solution as well as analytical solution of many fractional physical models.

This paper is organized as follows: Section 2 states some background material from fractional calculus. Section 3 presents the principle of the homotopy analysis transform method for the time-fractional partial differential equation. Section 4 is devoted to describe the analytical algorithm for time-fractional STO equation to derive an approximate analytical solitary wave solution. Section 5 makes some analysis for the obtained Table and Figures and discusses the present work.

2. Main results

We recall some necessary definitions for the fractional calculus (see [20, 21]) which are used throughout the remaining sections of this paper.

Definition 2.1. A real multivariable function $f(x,t)$, $t>0$ is said to be in the space C_γ, $\gamma \in \mathbb{R}$ with respect to t if there exists a real number p $(>\gamma)$, such that $f(x,t) = t^p f_1(x,t)$, where $f_1(x,t) \in C(\Omega \times T)$. Obviously, $C_\gamma \subset C_\delta$ if $\delta \leq \gamma$.

Definition 2.2. The Riemann-Liouville fractional integral of a function f is defined as

$$
\begin{aligned}
&{}_0I_t^\alpha f(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(x,\tau)d\tau, \quad \alpha > 0, \quad t \in T,\\
&{}_0I_t^0 f(x,t) = f(x,t).
\end{aligned}
$$

Definition 2.3. The Caputo fractional derivative of the order $n-1 \leq \alpha < n$ of a function $f \in C_\gamma$ $(\gamma \geq -1)$ is defined as

$$
D_t^\alpha f(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n}{\partial \tau^n} f(x,\tau)d\tau.
$$

Definition 2.4. The Laplace transform of the function $g(t)$ is defined as

$$
f(t) = L(g(t)) = \int_0^\infty e^{-st} g(t)dt,
$$

where t is the symbolic variable in $g(t)$ as determined by findsym. The inverse Laplace transform of the function $h(s)$ is defined as

$$
f(s) = L^{-1}(h(s)) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} h(s)ds, \quad t > 0,
$$

where c is a real number selected so that all singularities of $h(s)$ are to the left of the line $s = c, i$, s is a scalar symbolic object.

Definition 2.5. The Laplace transform $L(u(x,t))$ of the Riemann-Liouville fractional integral is defined as

$$L(I_t^{\alpha} u(x,t)) = s^{-\alpha} L(u(x,t)).$$

Definition 2.6. The Laplace transform $L(u(x,t))$ of the Caputo fractional derivative is defined as

$$L(D_t^{n\alpha} u(x,t)) = s^{n\alpha} L(u(x,t)) - \sum_{k=0}^{n-1} s^{n\alpha-k-1} u^{(k)}(x,0), \quad n-1 < n\alpha \leq n.$$

3. New time-fractional homotopy analysis transform methods

To illustrate the basic idea of the homotopy analysis transform method for the fractional partial differential equation, we consider the following time-fractional partial differential equation

$$D_t^{n\alpha} u(x,t) + R(x)u(x,t) + N(x)u(x,t) = 0, \quad n-1 < n\alpha \leq n, \quad x \in \mathbb{R}, \quad t > 0, \tag{1}$$

where $D_t^{n\alpha} = \frac{\partial^{n\alpha}}{\partial t^{n\alpha}}$, α is a parameter describing the order of the fractional time-derivative, $u(x,t)$ is a field variable, $R(x)$ is the linear operator in x, $N(x)$ is the general nonlinear operator in x. For simplicity we ignore all initial and boundary conditions, which can be treated in similar way. Now the methodology consists of applying Laplace transform first on both sides of equation (1), it yields

$$L(D_t^{n\alpha} u(x,t)) + L\Big(R(x)u(x,t) + N(x)u(x,t)\Big) = 0.$$

By using of the differentiation property of the Laplace transform, one has

$$L(u(x,t)) - \frac{1}{s^{n\alpha}} \sum_{k=0}^{n-1} s^{n\alpha-k-1} u^{(k)}(x,0) + \frac{1}{s^{n\alpha}} L\Big(R(x)u(x,t) + N(x)u(x,t)\Big) = 0.$$

Define the nonlinear operator

$$\eta(\phi(x,t;q)) = L(\phi(x,t;q)) - \frac{1}{s^{n\alpha}} \sum_{k=0}^{n-1} s^{n\alpha-k-1} \phi^{(k)}(x,0;q) + \frac{1}{s^{n\alpha}} L\Big(R(x)\phi(x,t;q) + N(x)\phi(x,t;q)\Big),$$

here $q \in [0,1]$ be an embedding parameter and $\phi(x,t;q)$ is the real function of x,t and q. By means of generalizing the traditional homotopy methods, we construct the zero order deformation equation as follows

$$(1-q)L(\phi(x,t;q)-u_0(x,t)) = \hbar q H(x,t)\eta(\phi(x,t;q)), \tag{2}$$

where $\hbar$ is a nonzero auxiliary parameter, $H(x,t)$ is a nonzero auxiliary function, $u_0(x,t)$ is an initial value of $u(x,t)$ and $\phi(x,t;q)$ is an unknown function. It is important that one has great freedom to choose auxiliary thing in homotopy analysis transform method. Obviously, when $q=0$ and $q=1$, there holds

$$\phi(x,t;0) = u_0(x,t), \quad \phi(x,t;1) = u(x,t),$$

respectively. Thus as q increases from 0 to 1, the solution varies from the initial value $u_0(x,t)$ to the solution $u(x,t)$. Expanding $\phi(x,t;q)$ in Taylor's series with respect to q, we have

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} q^m u_m(x,t), \tag{3}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m}\Big|_{q=0}.$$

The convergence of series solution (3) is controlled by $\hbar$. If the auxiliary linear operator, the initial value, the auxiliary parameter $\hbar$, and the auxiliary function are properly chosen, the series (3) converges at $q=1$, this is

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t), \tag{4}$$

which must be one of the solutions of original nonlinear equations. The above expression provides us with a relationship between the initial value $u_0(x,t)$ and the solution $u(x,t)$ by means of the terms (approximate solution) $u_m(x,t)$ $m = 1,2,3,\ldots$, which are still to be determined.

Defines the vectors of $u_m(x,t)$

$$\vec{u}_m(x,t) = \{u_0(x,t), u_1(x,t), \ldots, u_m(x,t)\}.$$

Differentiating the zero order deformation equation (2) m time with respect to embedding parameter q and then setting $q=0$ and finally dividing them by $m!$ we obtain the m-th order deformation equation

$$L(u_m(x,t)-\chi_m u_{m-1}(x,t)) = \hbar q H(x,t) R_m(\vec{u}_{m-1}).$$

Operating the inverse Laplace transform on both sides, yields

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar q L^{-1}\big(H(x,t)R_m(\vec{u}_{m-1})\big). \tag{5}$$

where

$$R_m(\vec{u}_{m-1}) = D_t^{n\alpha} u_{m-1}(x,t) + R(x)u_{m-1}(x,t) + N(x)u_{m-1}(x,t)$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

Theorem 3.1. *If the series (4) converges, then it must converge to the exact solution of (1).*

Proof. Let $u(x,t) = \lim\limits_{N\to\infty} U_N(x,t) = \lim\limits_{N\to\infty} \sum_{m=1}^{N} u_{m-1}(x,t)$, then we show that $u(x,t)$ is the exact solution of (1). By (5), we have

$$\begin{aligned}
&\hbar q \sum_{m=1}^{N} L^{-1}\big(H(x,t)R_m(\vec{u}_{m-1})\big) \\
&= \hbar q L^{-1}\left(\sum_{m=1}^{N} H(x,t)\Big(D_t^{n\alpha} u_{m-1}(x,t) + R(x)u_{m-1}(x,t) + N(x)u_{m-1}(x,t)\Big)\right) \\
&= \sum_{m=1}^{N} \big(u_m(x,t) - \chi_m u_{m-1}(x,t)\big) \\
&= u_N(x,t) \to 0, \text{ as } N \to \infty.
\end{aligned} \tag{6}$$

Since $\hbar \neq 0$, $q \neq 0$, we find from (6) that

$$L^{-1}\left(\sum_{m=1}^{\infty} H(x,t)\Big(D_t^{n\alpha} u_{m-1}(x,t) + R(x)u_{m-1}(x,t) + N(x)u_{m-1}(x,t)\Big)\right) = 0. \tag{7}$$

Applying L to equation (7) and using the properties calculation of Laplace transform and inverse Laplace transform, it holds

$$\sum_{m=1}^{\infty} H(x,t)\Big(D_t^{n\alpha}u_{m-1}(x,t)+R(x)u_{m-1}(x,t)+N(x)u_{m-1}(x,t)\Big)=0. \tag{8}$$

Since $H(x,t)\neq 0$, we see that (8) yields

$$\sum_{m=1}^{\infty}\Big(D_t^{n\alpha}u_{m-1}(x,t)+R(x)u_{m-1}(x,t)+N(x)u_{m-1}(x,t)\Big)=0. \tag{9}$$

The RHS of (9) is

$$\begin{aligned}
&\sum_{m=1}^{\infty} D_t^{n\alpha}u_{m-1}(x,t)+\sum_{m=1}^{\infty} R(x)u_{m-1}(x,t)+\sum_{m=1}^{\infty} N(x)u_{m-1}(x,t))\\
&\quad = D_t^{n\alpha}\sum_{m=1}^{\infty} u_{m-1}(x,t)+R(x)\sum_{m=1}^{\infty} u_{m-1}(x,t)+N(x)\sum_{m=1}^{\infty} u_{m-1}(x,t)\\
&\quad = D_t^{n\alpha}u(x,t)+R(x)u(x,t)+N(x)u(x,t)=0.
\end{aligned}$$

Also $u(x,0)=u_0(x,0)+\sum_{m=1}^{\infty}u_n(x,0)=u(x,0)$. Thus $u(x,t)$ satisfies (1) and must be the exact solution of (1). This completes the proof.

4. The time-fractional STO equation

In this section, we will apply the homotopy analysis method and Laplace transform method to the fractional STO equation. One of the fractional differential equations arising in science and engineering is STO equation with time-fractional derivative of the form

(10)

$$D_t^{\alpha}u(x,t)+3au_x^2(x,t)+3au^2(x,t)u_x(x,t)+3au(x,t)u_{xx}(x,t)+au_{xxx}(x,t)=0,\quad 0<\alpha\leq 1,$$

where $a\neq 0$ is a constant, $x\in\Omega$ is a space coordinate in the propagation direction of the field and $t\in T(=[0,t_0](t_0>0))$ is the time, the subscripts denote the partial differentiation of the function $u(x,t)$ with respect to the parameter x and/or t, D_t^{α} denotes the Caputo fractional derivative sense.

$$u(x,0)=-k\tanh(\frac{kx}{2}),$$

where k is constant. Applying the Laplace transform on both sides in equation (10) and after using the differentiation property of Laplace transform for fractional derivative, we have

$$s^{\alpha}L(u(x,t)) - s^{\alpha-1}u(x,0) + L\big(3au_x^2(x,t) + 3au^2(x,t)u_x(x,t) + 3au(x,t)u_{xx}(x,t) + au_{xxx}(x,t)\big) = 0.$$

We simplify and obtain that

$$L(u(x,t)) + \frac{k}{s}\tanh(\frac{kx}{2}) + s^{-\alpha}L\big(3au_x^2(x,t) + 3au^2(x,t)u_x(x,t) + 3au(x,t)u_{xx}(x,t) + au_{xxx}(x,t)\big) = 0.$$

Choose the linear operator as

$$\mathscr{L}(\phi(x,t;q)) = L(\phi(x,t;q)),$$

where $\phi(x,t;q)$ is defined as (3).

Define a nonlinear operator as

$$\begin{aligned}\eta(\phi(x,t;q)) = \mathscr{L}(\phi(x,t;q)) + \frac{k}{s}\tanh(\frac{kx}{2}) + s^{-\alpha}\mathscr{L}\big(&3a\phi_x^2(x,t;q) + 3a\phi^2(x,t;q)\phi_x(x,t;q)\\ &+ 3a\phi(x,t;q)\phi_{xx}(x,t;q) + a\phi_{xxx}(x,t;q)\big).\end{aligned}$$

Using above definition, we construct the zero order deformation equation with assumption $H(x,t) = 1$,

$$(1-q)\mathscr{L}(\phi(x,t;q) - u_0(x,t)) = \hbar q H(x,t)\eta(\phi(x,t;q)).$$

Obviously, when $q = 0$ and $q = 1$, there has

$$\phi(x,t;0) = u_0(x,t), \quad \phi(x,t;1) = u(x,t),$$

Thus, we obtain the m-th order deformation equation

$$L(u_m(x,t) - \chi_m u_{m-1}(x,t)) = \hbar q R_m(\vec{u}_{m-1}). \tag{11}$$

Operating the inverse Laplace transform on both sides in equation (11), we obtain

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar q L^{-1}\big(R_m(\vec{u}_{m-1})\big),$$

where

$$
\begin{aligned}
&R_m(\vec{u}_{m-1})\\
&= L(u_{m-1}(x,t)) + (1-\chi_m)\frac{k}{s}\tanh(\frac{kx}{2}) + s^{-\alpha}L\Bigg(3a\sum_{j=0}^{m-1}(u_j)_x(x,t)(u_{m-1-j})_x(x,t)\\
&\quad + 3a\sum_{j=0}^{m-1}(u_j)_x(x,t)\Big(\sum_{k=0}^{m-1-j}u_k(x,t)u_{m-1-j-k}(x,t)\Big) + 3a\sum_{j=0}^{m-1}u_j(x,t)(u_{m-1-j})_{xx}(x,t)\\
&\quad + a(u_{m-1})_{xxx}(x,t)\Bigg), \quad m \geq 1.
\end{aligned}
$$

And so, we have the solution of m-th order deformation equation (11)

(12)

$$
\begin{aligned}
&u_m(x,t)\\
&= (\chi_m+\hbar)u_{m-1}(x,t) + (1-\chi_m)\hbar k\tanh(\frac{kx}{2}) + \hbar q L^{-1}\Bigg(s^{-\alpha}L\Big(3a\sum_{j=0}^{m-1}(u_j)_x(x,t)(u_{m-1-j})_x(x,t)\\
&\quad + 3a\sum_{j=0}^{m-1}(u_j)_x(x,t)\Big(\sum_{k=0}^{m-1-j}u_k(x,t)u_{m-1-j-k}(x,t)\Big) + 3a\sum_{j=0}^{m-1}u_j(x,t)(u_{m-1-j})_{xx}(x,t)\\
&\quad + a(u_{m-1})_{xxx}(x,t)\Big)\Bigg).
\end{aligned}
$$

The zero order solitary wave solution can be taken as the initial value of the state variable, which is taken in this case as $u_0(x,t) = u(x,0) = -k\tanh(\frac{kx}{2})$, substituting this zero order as approximate solitary wave solution into (12), using the Definitions 2.5 and 2.6, it leads to the first order approximate solitary wave solution

$$
u_1(x,t) = \frac{a\hbar q k^4 t^\alpha}{2\Gamma(\alpha+1)}\Big(9\tanh^4(\frac{kx}{2}) - 11\tanh^2(\frac{kx}{2}) + 2\Big).
$$

Substituting first order approximate solitary wave solution into (12), using the Definitions 2.5 and 2.6 then leads to the second order approximate solitary wave solution in the following expression

$$
\begin{aligned}
u_2(x,t) = &\frac{a\hbar(1+\hbar)qk^4t^\alpha}{2\Gamma(\alpha+1)}\Big(9\tanh^4(\frac{kx}{2}) - 11\tanh^2(\frac{kx}{2}) + 2\Big)\\
&+ \frac{a^2\hbar^2q^2k^7t^{2\alpha}}{4\Gamma(2\alpha+1)}\Big(-837\tanh^7(\frac{kx}{2}) + 1791\tanh^5(\frac{kx}{2}) - 1169\tanh^3(\frac{kx}{2}) + 215\tanh(\frac{kx}{2})\Big).
\end{aligned}
$$

Making use of the Maple package or other software, substituting $m-1$-th order approximate solitary wave solution into (12), we obtain $u_m(x,t)$, there leads to the m-th order approximate solitary wave solution. Now, we have

$$
\begin{aligned}
& u(x,t) \\
& = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots \\
& = -k\tanh(\frac{kx}{2}) + \frac{a\hbar q k^4 t^\alpha}{2\Gamma(\alpha+1)}\Big(9\tanh^4(\frac{kx}{2}) - 11\tanh^2(\frac{kx}{2}) + 2\Big) \\
& \quad + \frac{a\hbar(1+\hbar) q k^4 t^\alpha}{2\Gamma(\alpha+1)}\Big(9\tanh^4(\frac{kx}{2}) - 11\tanh^2(\frac{kx}{2}) + 2\Big) \\
& \quad + \frac{a^2\hbar^2 q^2 k^7 t^{2\alpha}}{4\Gamma(2\alpha+1)}\Big(-837\tanh^7(\frac{kx}{2}) + 1791\tanh^5(\frac{kx}{2}) - 1169\tanh^3(\frac{kx}{2}) + 215\tanh(\frac{kx}{2})\Big) \\
& \quad + \cdots .
\end{aligned}
$$

In view of the Theorem 3.1, as m tends to infinity, the iteration series leads to the exact solitary wave solution of the time-fractional STO equation

$$u(x,t) = -k\tanh\Big(\frac{k}{2}\big(x - \frac{ak^2}{\Gamma(1+\alpha)}t^\alpha\big)\Big),$$

which is the same exact solution in [22] via the simplest equation method.

5. Discussion

The aim of present work is to develop an effective and new coupling method of homotopy analysis method and Laplace transform method for the time-fractional STO equation. Taking $a=1$, $k=2$, $\hbar=-1.5$, 3-dimensional surface representation of the approximate solution $u(x,t)$ for the time-fractional STO equation with space x and time t for different values of the order α is presented respectively in Figure 1, the solution u is still a single soliton wave solution for all values of the order α. It shows that the balancing scenario between nonlinearity and dispersion is still valid. Figure 2 presents the solitary wave solution of the order at $\alpha=1$, this figure shows that the good agreement with the approximate solution and solitary wave solution. By computing the absolute error $|u(x,t)-U_3(x,t)|$, Figure 3 the curves depicted the behavior of the approximate solution $u(x,t)$ due to the variation of the order α. This behavior indicates that

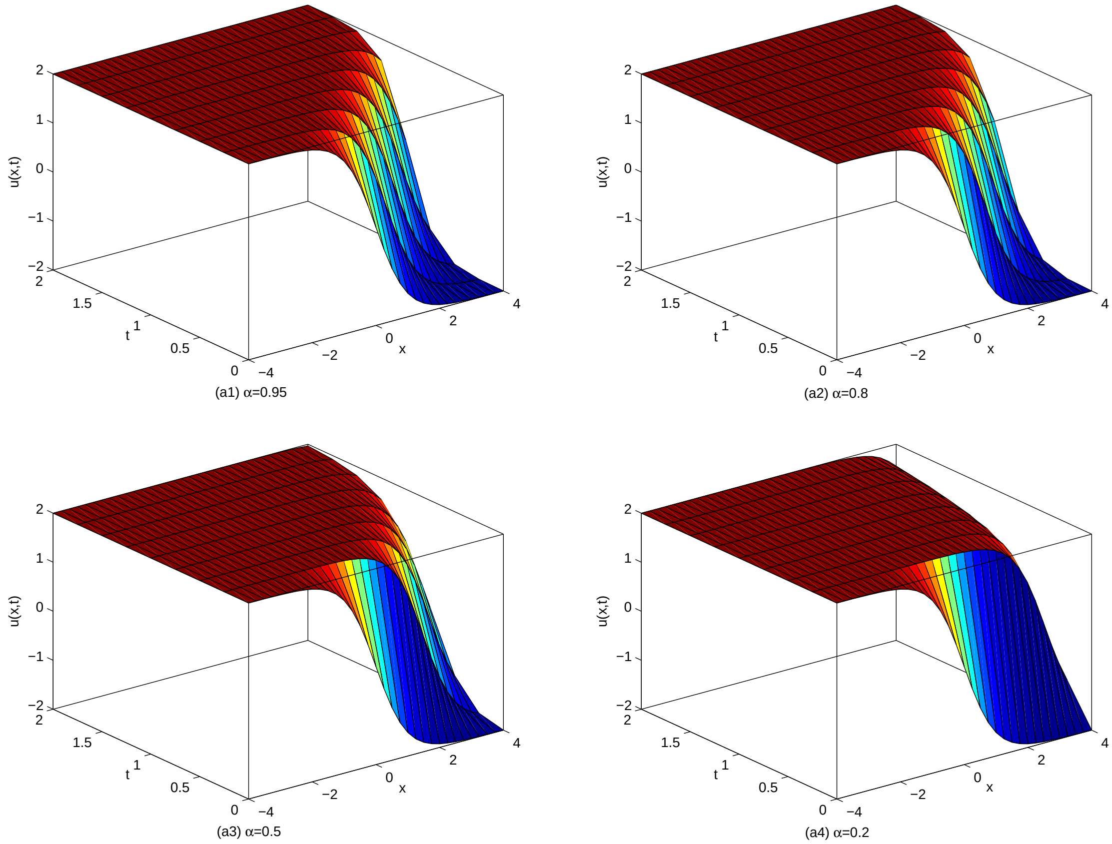

Figure 1 The surfaces of the approximate solutions $u(x,t)$

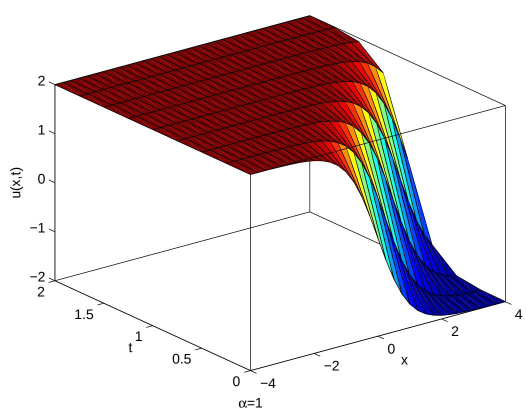

Figure 2 The surfaces of the exact solution $u(x,t)$

the increasing of the value t and x, increases $u(1,t)$ of the solitary wave solution and decreases $u(x,0.5)$ of the solitary wave solution, respectively. That is, the order α can be used to modify the shape of the solitary wave without change of the nonlinearity and the dispersion effects in

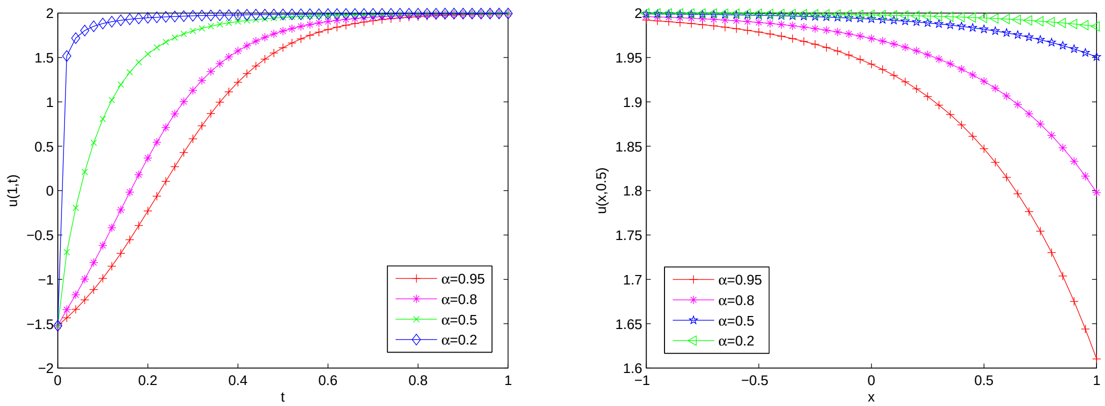

Figure 3 The curves of the approximate solution $u(x,t)$ at $x=1$ and at $t=0.5$, respectively

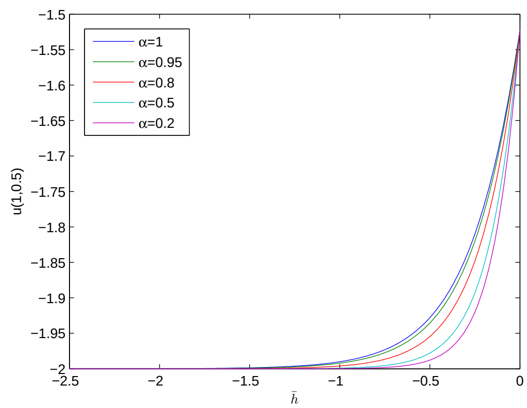

Figure 4 The curves of $\hbar$ for different value of α at point $(1,0.5)$

the medium. Figure 4 shows the $\hbar$-curve obtained from the 3th-order homotopy analysis transform method approximate solution of time-fractional STO equation at $x=1$, $t=0.5$. In our study, it is obvious that the acceptable range of auxiliary parameter is $\hbar<-1$. We still have freedom to choose the auxiliary parameter according to $\hbar$ curve. The valid regions of convergence correspond to the line segments nearly parallel to the horizontal axis. Table 1 shows that the new coupling method of homotopy analysis method and Laplace transform method obtained nearly identical to the known exact solution and the approximate analytical solution obtained increases very rapidly with the increases in x and t by proposed method. Meanwhile, the homotopy analysis transform method is capable of reducing the volume of the computational work as compared to the classical methods with high accuracy of the numerical result and will considerably benefit mathematicians and scientists working in the field of partial differential equations. Different from the other numerical methods are given low degree of accuracy for large values

TABLE 1. Absolute error in different value α at $t = 0.5$

x	$\hbar = -1.5$				
	$\alpha = 1$	$\alpha = 0.95$	$\alpha = 0.8$	$\alpha = 0.5$	$\|u - U_3\|$ $(\alpha = 0.5)$
12.0	-1.9999999917	-1.9999999896	-1.9999999790	-1.9999999106	8.11121299e-08
10.0	-1.9999995498	-1.9999994356	-1.9999988553	-1.9999951212	4.42856616e-06
8.0	-1.9999754233	-1.9999691886	-1.9999375049	-1.9997336490	2.41774255e-04
6.0	-1.9986585994	-1.9983184517	-1.9965907420	-1.9855094471	1.31491523e-02
4.0	-1.9280551601	-1.9102136391	-1.8219926388	-1.3374816304	5.90573529e-01
2.0	0.0000000000	0.2251276590	0.8709753567	1.6621106564	1.66211066e-00
0.0	1.9280551601	1.9424035993	1.9713951856	1.9932517402	6.51965800e-02
-2.0	1.9986585994	1.9989299597	1.9994723805	1.9998761962	1.21759679e-03
-4.0	1.9999754233	1.9999803963	1.9999903350	1.9999977323	2.23090851e-05
-6.0	1.9999995498	1.9999996409	1.9999998229	1.9999999584	4.08607840e-07
-8.0	1.9999999917	1.9999999934	1.9999999967	1.9999999992	7.48391015e-09
-10.0	1.9999999998	1.9999999998	1.9999999999	1.9999999999	1.37080125e-10

of x and t. Therefore, the homotopy analysis transform method better handle with the nonlinear problems without any assumption and restriction.

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