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APPROXIMATING FIXED POINTS OF TWO FINITE FAMILIES OF *I*-ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS

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Abstract. Let E be a uniformly convex Banach space and let K be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^N, \{S_i\}_{i=1}^N : K \to E$ be two finite families of I-asymptotically quasi-nonexpansive mappings. It is proved that a iteration sequence converges strongly to a common fixed point of $\{T_i\}_{i=1}^N$, $\{S_i\}_{i=1}^N$ under certain conditions. The results presented in this paper improve and extend the corresponding results in the existing literature.

Keywords. Non-self I-asymptotically quasi-nonexpansive mappings; Common fixed point; Uniformly convex Banach space; Iterative process.

1. Introduction

Let K be a nonempty subset of a real normed linear space E. Let T be a self mapping of K.

(1) T is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in K$.

(2) T is said to be asymptotically nonexpansive if there exists a real sequence $\{k_n\} \subset [0, \infty)$, with $\lim_{n\to\infty} k_n = 0$ such that $\|T^n x - T^n y\| \le (1 + k_n)\|x - y\|$ for all $x, y \in K$.

(3) T is said to be quasi-nonexpansive if $\|Tx - p\| \le \|x - p\|$ for all $x \in K$ and $p \in F(T)$ where $F(T) = \{x \in K : Tx = x\}$ is the set of all fixed points of T.

(4) T is said to be asymptotically quasi-nonexpansive if there exists a real sequence $\{k_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that $\|T^n x - p\| \le (1 + k_n)\|x - p\|$ for all $x \in K$ and $p \in F(T)$.

(5) T is said to be uniformly L-Lipschitzian, if there exists a constant $L > 0$ such that $\|T^n x - T^n y\| \le L\|x - y\|$ for all $x, y \in K$.

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Remark 1.1. From the above definitions, it is easy to see that if $F(T)$ is nonempty, a nonexpansive mapping must be quasi-nonexpansive, and an asymtotically nonexpansive mapping must be asmyptotically quasi-nonexpansive. It is obvious that, an asymptotically nonexpansive mapping is also uniformly L-Lipschitzian with $L = \sup\{k_n : n \geq 1\}$. However, the converses of these claims are not true in general.

Being an important generalization of nonexpansive mappings, the concept of asymptotically nonexpansive self mapping was proposed by Goebel and Kirk [4] in 1972. In [2], it was proved that if E is uniformly convex, and K is a bounded closed and convex subset of E, then every asymptotically nonexpansive self mapping has a fixed point.

In 2003, Chidume, Ofoedu and Zegeye [3] further generalized the concept of asymptotically nonexpansive self mapping, and proposed the concept of non-self asymptotically nonexpansive mapping, which is defined as follows:

Let K be a nonempty subset of a real normed linear space E and let $P : E \to K$ be a nonexpansive retraction of E onto K.

(1) Mapping $T : K \to E$ is called non-self asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [0,\infty)$ with $k_n \to 0$ as $n \to \infty$ such that for any positive integer n and all $x, y \in K$

$$\left\| T(PT)^{n-1} x - T(PT)^{n-1} y \right\| \leq (1 + k_n) \|x - y\|.$$

(2) Mapping $T : K \to E$ is said to be a non-self asymptotically quasi-nonexpansive mapping if $F(T) = \{x \in K : Tx = x\} \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [0,\infty)$ with $k_n \to 0$ as $n \to \infty$ such that for any positive integer n and all $x \in K$ and $p \in F(T)$

$$\left\| T(PT)^{n-1} x - p \right\| \leq (1 + k_n) \|x - p\|.$$

In [3], Chidume, Ofoedu and Zegeye obtained the strong convergence theorem of fixed points of a non-self asymptotically nonexpansive mapping. Since then, many authors ([1, 6, 7, 9, 16, 17], etc.) also obtained some convergence theorems for such non-self mappings in uniformly convex Banach spaces.

In the last decades many papers have been published on the approximation of fixed points for certain classes of the following I-nonexpansive mappings in various spaces ([5, 10, 15, 19]).

Let K be a nonempty subset of a real normed linear space E, and $T, I : K \to K$ be mappings. Denote by $\mathbb{F} = F(T) \cap F(I) = \{x \in K : Tx = Ix = x\}$, the set of common fixed points of the mappings T and I.

(1) T is called I-nonexpansive on K if $\|Tx - Ty\| \leq \|Ix - Iy\|$ for all $x, y \in K$.

(2) T is called I-asymptotically nonexpansive on K if there exists a sequence $\{k_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that $\|T^n x - T^n y\| \leq (1 + k_n)\|I^n x - I^n y\|$ for all $x, y \in K$ and $n = 1, 2, \ldots$.

(3) T is called I-quasi-nonexpansive on K if $\|Tx - p\| \leq \|Ix - p\|$ for all $x \in K$, $p \in \mathbb{F}$ and $n = 1, 2, \ldots$.

(4) T is called I-asymptotically quasi-nonexpansive on K if there exists a sequence $\{k_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that $\|T^n x - p\| \leq (1 + k_n)\|I^n x - p\|$ for all $x \in K$, $p \in \mathbb{F}$ and $n = 1, 2, \ldots$.

Remark 1.2. From above definitions, it is easy to see that if $\mathbb{F}$ is nonempty, an I-nonexpansive mapping must be I-quasi-nonexpansive, and an I-asymptotically nonexpansive mapping must be I-asymptotically quasi-nonexpansive. But the converse does not hold.

For non-self mappings $T, I : K \to E$, and the nonexpansive retraction P from E onto K,

(1) T is called non-self I-asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \leq (1 + k_n)\left\|I(PI)^{n-1}x - I(PI)^{n-1}y\right\|$$

for all $x, y \in K$, and $n = 1, 2, \ldots$.

(2) T is called non-self I-asymptotically quasi-nonexpansive if $\mathbb{F} = F(T) \cap F(I) \neq \emptyset$, there exists a sequence $k_n \subset [0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that

$$\left\|T(PT)^{n-1}x - p\right\| \leq (1 + k_n)\left\|I(PI)^{n-1}x - p\right\|$$

for all $x \in K$, $p \in \mathbb{F}$ and $n = 1, 2, \ldots$.

(3) T is called non-self I-uniformly L-Lipschitzian if there exists $L > 0$ such that

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \leq L\left\|I(PI)^{n-1}x - I(PI)^{n-1}y\right\|$$

for all $x, y \in K$ and $n = 1, 2, \ldots$.

Remark 1.3. If I is a identity mapping, then non-self I-asymptotically nonexpansive mapping and non-self I-asymptotically quasi-nonexpansive mapping reduce to non-self asymptotically nonexpansive mapping and non-self asymptotically quasi-nonexpansive mapping, proposed by Chidume et al. [3], respectively.

Recently, some authors ([10, 19, 20, 21]) obtained strong convergence theorems of fixed points for I-nonexpansive mappings and I-asymptotically quasi-nonexpansive mappings in Banach space. After, Yao and Wang [18] extended of the iterative process, defined in [14], to nonself I-asymptotically quasi-nonexpansive defined on a uniformly convex Banach space.

Inspired by above works, we introduce, in this paper, an iteration process for two finite families of non-self I-asymptotically quasi-nonexpansive mappings and an initial point $x_1 \in K$ as follows:

$$\begin{aligned}
x_2 &= P\left((1-\alpha_1-\beta_1)I_1(PI_1)x_1+\alpha_1 S_1(PS_1)x_1+\beta_1 T_1(PT_1)x_1\right)\\
x_3 &= P\left((1-\alpha_2-\beta_2)I_2(PI_2)x_2+\alpha_2 S_2(PS_2)x_2+\beta_2 T_2(PT_2)x_2\right)\\
&\vdots\\
x_{N+1} &= P\left((1-\alpha_N-\beta_N)I_N(PI_N)x_N+\alpha_N S_N(PS_N)x_N+\beta_N T_N(PT_N)x_N\right)\\
x_{N+2} &= P\left((1-\alpha_{N+1}-\beta_{N+1})I_1(PI_1)^2x_{N+1}+\alpha_{N+1} S_1(PS_1)^2x_{N+1}+\beta_{N+1} T_1(PT_1)^2x_{N+1}\right)\\
&\vdots\\
x_{2N+1} &= P\left((1-\alpha_{2N}-\beta_{2N})I_{2N}(PI_{2N})^2x_{2N}+\alpha_{2N} S_{2N}(PS_{2N})^2x_{2N}+\beta_{2N} T_{2N}(PT_{2N})^2x_{2N}\right)\\
x_{2N+2} &= P\left((1-\alpha_{2N+1}-\beta_{2N+1})I_2(PI_2)^3x_{2N+1}+\alpha_{2N+1} S_2(PS_2)^3x_{2N+1}+\beta_{2N+1} T_2(PT_2)^3x_{2N+1}\right)\\
&\vdots
\end{aligned}$$

which can be written in compact form as:

(1)

$$x_{n+1}=P\left((1-\alpha_n-\beta_n)I_{i(n)}\left(PI_{i(n)}\right)^{k(n)-1}x_n+\alpha_n S_{i(n)}\left(PS_{i(n)}\right)^{k(n)-1}x_n+\beta_n T_{i(n)}\left(PT_{i(n)}\right)^{k(n)-1}x_n\right),$$

where $T_n = T_{n \ mod \ N}, S_n = S_{n \ mod \ N}$ for integer $n \geq 1$, with the mod function taking values in the set $\{1,2,...,N\}$. In other words, if $n=(k-2)N+i$, $i=i(n)\in\{1,2,...,N\}$, $k=k(n)\geq 1$ is a positive integer and $k(n) \to \infty$ as $n \to \infty$, then we set $T_n = T_i, S_n = S_i$ and $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$. In the sequel, we designate the set $\{1,2,...,N\}$ by J.

The iteration (1) can be expressed in the following form:

$$(2) \qquad \begin{cases} x_1 \in K, \\ x_{n+1} = P\left((1-\alpha_n-\beta_n) I_n (PI_n)^{n-1} x_n + \alpha_n S_n (PS_n)^{n-1} x_n + \beta_n T_n (PT_n)^{n-1} x_n\right), \end{cases}$$

where P is a nonexpansive retraction from E onto K, $n \geq 1$ and $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$.

Let $T_i, S_i : K \to E$ be non-self I_i-asymptotically quasi-nonexpansive mappings with sequences $\{r_n^i\}, \{t_n^i\} \subset [0,\infty)$ such that

$$\left\| T_i (PT_i)^{n-1} x - p \right\| \leq \left(1 + r_n^i\right) \left\| I_i (PI_i)^{n-1} x - p \right\|,$$

$$\left\| S_i (PS_i)^{n-1} x - p \right\| \leq \left(1 + t_n^i\right) \left\| I_i (PI_i)^{n-1} x - p \right\|.$$

Also, let $I_i : K \to E$ be a non-self asymptotically nonexpansive mapping with $\{l_n^i\} \subset [0,\infty)$ and P be a nonexpansive retraction from E onto K. Throughout this paper, we assume that $r_n = \max\left\{r_n^{(1)}, r_n^{(2)}, ..., r_n^{(N)}\right\}$, $t_n = \max\left\{t_n^{(1)}, t_n^{(2)}, ..., t_n^{(N)}\right\}$ and $l_n = \max\left\{l_n^{(1)}, l_n^{(2)}, ..., l_n^{(N)}\right\}$ for each positive integer n and $h_n = \max_{n\in\mathbb{N}}\{r_n, t_n, l_n\}$ and $F := \cap_{i=1}^{N} \left(F(T_i) \cap F(S_i) \cap F(I_i)\right) \neq \emptyset$.

In this paper, we prove that the sequence $\{x_n\}$ defined by (2) converges strongly to a common fixed point of two finite families of non-self I-asymptotically quasi-nonexpansive mappings in uniformly convex Banach spaces, and obtained the suffecient and necessary conditions that this iteration process converges strongly to a common fixed point such mappings under some mild conditions.

The purpose of this paper is to study convergence of the sequence in (2) to a common fixed point of two finite families of non-self I-asymptotically quasi-nonexpansive mappings in uniformly convex Banach spaces. Our work is a significant generalization of the corresponding results in the literature.

2. Preliminaries

A subset of E is said to be a retract if there exists a continuous mapping $P : E \to K$ such that $Px = x$ for all $x \in K$. A mapping $P : E \to K$ is called a retraction if $P^2 = P$. It is well-known that every closed convex subset of uniformly convex Banach space E is a retract.

If mapping P is a retraction, then $Px = x$ for all $x \in R(P)$, range of P.

Definition 2.1. Let K be a nonempty closed convex subset of a real uniformly convex Banach space E and let $T : K \to E$ be a non-self mapping. Then the mapping T is said to be

(1) demiclosed at y if whenever $\{x_n\} \subset K$ such that $\{x_n\}$ converges weakly to $x \in K$ and $Tx_n \to y$ then $Tx = y$.

(2) semi-compact if for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence say $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to some p in K.

(3) completely continuous if the sequence $\{x_n\}$ in K converges weakly to x_0 implies that $\{Tx_n\}$ converges strongly to Tx_0.

For approximating fixed points of nonexpansive mappings, Senter and Dotson [12] introduced Condition (A). Later on, Maiti and Ghosh [8], Tan and Xu [13] studied Condition (A) and point out that Condition (A) is weaker than the requirement of semi-compactness on mapping.

Definition 2.2. A mapping $T : K \to K$ is said to satisfy Condition (A) if there exists a nondecreasing function $f : [0, +\infty) \to [0, +\infty)$ with $f(0) = 0$, $f(r) > 0$, for all $r \in (0, +\infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in K$, where $d(x, F(T)) = \inf\{d(x, p) : p \in F(T)\}$, $F(T)$ is the fixed point set of T.

For obtaining strong convergence of common fixed points of two finite families of I-asymptotically quasi-nonexpansive mappings, we introduced the following Condition (B).

Definition 2.3. The mappings $\{T_i, S_i, I_i : i \in U\}$ are said to satisfy Condition (B) if there exists a nondecreasing function $f : [0, +\infty) \to [0, +\infty)$ with $f(0) = 0$, $f(r) > 0$, for all $r \in (0, +\infty)$ such that

$$\max_{1 \leq i \leq N} \left\{ \frac{1}{3} \left(\|x - T_i x\| + \|x - S_i x\| + \|x - I_i x\| \right) \right\} \geq f(d(x, F))$$

for all $x \in K$, where $F := \cap_{i=1}^{N} (F(T_i) \cap F(S_i) \cap F(I_i)) \neq \emptyset$ and $d(x, F) = \inf\{d(x, p) : p \in F\}$.

In order to prove the main results, we also need the following lemmas.

Lemma 2.4. [13] *Let* $\{a_n\},\{b_n\}$ *and* $\{c_n\}$ *be sequences of nonnegative real numbers satisfying*

$$c_{n+1} \leq (1+a_n)\,c_n + b_n, \;\; \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} a_n < \infty$ *and* $\sum_{n=1}^{\infty} b_n < \infty$*, then* $\lim_{n\to\infty} c_n$ *exists.*

Lemma 2.5. [11] *Let* E *be a uniformly convex Banach space and let* a,b *be two constants with* $0 < a < b < 1$*. Suppose that* $\{t_n\} \subset [a,b]$ *is a real sequence and* $\{x_n\},\{y_n\}$ *are two sequences in* E*. Then the conditions*

$$\lim_{n\to\infty} \|t_n x_n + (1-t_n)\,y_n\| = d, \;\; \limsup_{n\to\infty} \|x_n\| \leq d, \;\; \limsup_{n\to\infty} \|y_n\| \leq d$$

imply that $\lim_{n\to\infty} \|x_n - y_n\| = 0$*, where* $d \geq 0$ *is a constant.*

Lemma 2.6. [3] *Let* E *be a uniformly convex Banach space,* K *a nonempty closed subset of* E*, and let* $T : K \to E$ *be an asymptotically nonexpansive mapping with a sequence* $\{k_n\} \subset [0,\infty)$ with $k_n \to 0$ as $n \to \infty$*. Then* $B - T$ *is demiclosed at zero, where* B denotes the mapping $B : K \to K$ defined by $Bx = x$, respectively. *That is, for each sequence* $\{x_n\}$ *in* K, if $\{x_n\}$ converges weakly to $q \in K$ and $\{(B-T)\,x_n\}$ converges strongly to 0, then $Tq = q$.

3. Main results

The following lemma plays an important role in this paper.

Lemma 3.1. *Let* E *be a real Banach space,* K *be a nonempty closed convex subset of* E*,* $T_i, S_i : K \to E$ *be non-self* I_i*-asymptotically quasi-nonexpansive mappings and* $I_i : K \to E$ *be a non-self asymptotically nonexpansive mapping with respect to* P *with sequence* $h_n \subset [0,\infty)$ *such that* $\sum_{n=1}^{\infty} h_n < \infty$*. Suppose that* $\{x_n\}$ *is generated by* (2)*, where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are two sequences in* $(0,1)$ *satisfying* $\alpha_n + \beta_n \leq 1$ *for all* $n \geq 1$*. Then*

(1) $\lim_{n\to\infty} \|x_n - p\|$ *exists for each* $p \in F$*.*

(2) The iterative sequence $\{x_n\}$ *generated by* (2) *converges strongly to a common fixed point in* F *if and only if* $\liminf_{n\to\infty} d\,(x_n, F) = 0$*.*

Proof. Using fact that P is nonexpansive retraction, for any $p \in F \neq \emptyset$ and (1), we have

$$\begin{aligned}
\|x_{n+1}-p\| &\leq (1-\alpha_n-\beta_n)\left\|I_n(PI_n)^{n-1}x_n-p\right\|+\alpha_n\left\|S_n(PS_n)^{n-1}x_n-p\right\| \\
&\quad +\beta_n\left\|T_n(PT_n)^{n-1}-p\right\| \\
&\leq (1-\alpha_n-\beta_n)(1+h_n)\|x_n-p\|+\alpha_n(1+h_n)\left\|I_n(PI_n)^{n-1}x_n-p\right\| \\
&\quad +\beta_n(1+h_n)\left\|I_n(PI_n)^{n-1}x_n-p\right\| \\
&\leq (1-\alpha_n-\beta_n)(1+h_n)\|x_n-p\|+\alpha_n(1+h_n)(1+h_n)\|x_n-p\| \\
&\quad +\beta_n(1+h_n)(1+h_n)\|x_n-p\| \\
&\leq [(1-\alpha_n-\beta_n)(1+h_n)+\alpha_n(1+h_n)+\beta_n(1+h_n)]\|x_n-p\| \\
&\leq \left(1+2h_n+h_n^2\right)\|x_n-p\|, \qquad (3)
\end{aligned}$$

where $n=(k-2)N+i$, $i=i(n)\in\{1,2,...,N\}$, $k=k(n)\geq 1$. Since $\{h_n\}$ is a nonincreasing bounded sequence and $\sum_{n=1}^{\infty}h_n<\infty$ implies that $\sum_{n=1}^{\infty}h_n^2<\infty$. Then $\sum_{n=1}^{\infty}\left(2h_n+h_n^2\right)<\infty$. It now follows from Lemma 2.4 that $\lim_{n\to\infty}\|x_n-p\|$ exists for all $p\in F$. It follows from (3) that

$$d(x_{n+1},F)\leq\left(1+2h_n+h_n^2\right)d(x_n,F).$$

From Lemma 2.4, we obtain $\lim_{n\to\infty}d(x_n,F)$ exists. Furthermore, since $\liminf_{n\to\infty}d(x_n,F)=0$, then $\lim_{n\to\infty}d(x_n,F)=0$.

Next we prove that $\{x_n\}$ is a Cauchy sequence in K. For any given $\varepsilon>0$, since $\lim_{n\to\infty}d(x_n,F)=0$, there exists natural number N_1 such that $d(x_n,F)<\frac{\varepsilon}{3}$ when $n\geq N_1$. Thus, there exists $p^*\in F$ such that for above ε there exists positive integer $N_2\geq N_1$ such that $\|x_n-p^*\|<\frac{\varepsilon}{2}$ as $n\geq N_2$. Thus, for arbitrarily chosen $n,m\geq N_2$, we have

$$\begin{aligned}
\|x_n-x_m\| &\leq \|x_n-p^*\|+\|x_m-p^*\| \\
&< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset K of a Banach space E and so it converges to a point q in K. And $\lim_{n\to\infty}d(x_n,F)=0$ gives that $d(q,F)=0$. By the routine proof, we find F is a closed subset of K. Thus $q\in F$. This completes the proof.

Before proving our main results, we would like to remark as follows.

Let $T_i : K \to E$ be I_i-uniformly L_i-Lipschitzian non-self I_i-asymptotically quasi-nonexpansive mappings, $S_i : K \to E$ be I_i-uniformly $L_i^{'}$-Lipschitzian non-self I_i-asymptotically quasi-nonexpansive mappings and $I_i : K \to E$ be a uniformly Γ_i-Lipschitzian non-self asymptotically nonexpansive mapping. Throughout this paper we take $L = \max_{1\le i\le N}\left\{L_i, L_i^{'}\right\}$ and $\Gamma = \max_{1\le i\le N}\{\Gamma_i\}$.

Now, we are in a position to give the main results in this paper.

Theorem 3.2. *Let E be a uniformly convex Banach space, K be a nonempty closed convex subset of E, $T_i, S_i : K \to E$ be I_i-uniformly L-Lipschitzian non-self I_i-asymptotically quasi-nonexpansive mappings and let $I_i : K \to E$ be a be uniformly Γ-Lipschitzian non-self asymptotically nonexpansive mapping with respect to P with sequence $h_n \subset [0,\infty)$ such that $\sum_{n=1}^{\infty} h_n < \infty$. Suppose that for any given $x_1 \in K$, the sequence $\{x_n\}$ is generated by (1) satisfying*

$$\left\|x_n - T_n (PT_n)^{n-1} x_n\right\| \le \left\|T_n (PT_n)^{n-1} x_n - S_n (PS_n)^{n-1} x_n\right\|,$$

where $0 < a \le \alpha_n, \beta_n \le b < 1$ for some $a, b \in (0,1)$ and $\alpha_n + \beta_n < 1$ for all $n \ge 1$. Then

$$\lim_{n\to\infty} \|x_n - I_i x_n\| = \lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - T_i x_n\| = 0$$

for each $i \in J$.

Proof. From Lemma 3.1, we know that $\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in F$. We suppose that $\lim_{n\to\infty} \|x_n - p\| = d$. Then $\lim_{n\to\infty} \|x_{n+1} - p\| = d$, that is,

$$\begin{aligned}
(4)\lim_{n\to\infty} \|x_{n+1} - p\| &= \lim_{n\to\infty} \Big\|P\Big((1-\alpha_n-\beta_n) I_n (PI_n)^{n-1} x_n + \alpha_n S_n (PS_n)^{n-1} x_n \\
&\quad +\beta_n T_n (PT_n)^{n-1} x_n\Big) - p\Big\| \\
&\le \lim_{n\to\infty} \Big\|(1-\alpha_n-\beta_n)\Big(I_n (PI_n)^{n-1} x_n - p\Big) + \alpha_n \Big(S_n (PS_n)^{n-1} x_n - p\Big) \\
&\quad +\beta_n \Big(T_n (PT_n)^{n-1} x_n - p\Big)\Big\| \\
&= \lim_{n\to\infty} \Big\|(1-\alpha_n-\beta_n)\Big(I_n (PI_n)^{n-1} x_n - p\Big) + (\alpha_n+\beta_n)\Big[\frac{\alpha_n}{\alpha_n+\beta_n} \\
&\quad \Big(S_n (PS_n)^{n-1} x_n - p\Big) + \frac{\beta_n}{\alpha_n+\beta_n}\Big(T_n (PT_n)^{n-1} x_n - p\Big)\Big]\Big\| \\
&= d.
\end{aligned}$$

Since I_i is a uniformly L_i-Lipschitzian asymptotically nonexpansive mapping, we have

$$\left\| I_n (PI_n)^{n-1} x_n - p \right\| \leq (1+h_n) \|x_n - p\| .$$

Taking lim sup on both sides, we obtain

$$\limsup_{n\to\infty} \left\| I_n (PI_n)^{n-1} x_n - p \right\| \leq \limsup_{n\to\infty} (1+h_n) \|x_n - p\| = d \tag{5}$$

and

$$\begin{aligned} &\limsup_{n\to\infty} \left\| \frac{\alpha_n}{\alpha_n+\beta_n} \left(S_n (PS_n)^{n-1} x_n - p \right) + \frac{\beta_n}{\alpha_n+\beta_n} \left(T_n (PT_n)^{n-1} x_n - p \right) \right\| \\ \leq\ & \limsup_{n\to\infty} \left(\frac{\alpha_n}{\alpha_n+\beta_n} (1+h_n) \left\| I_n (PI_n)^{n-1} x_n - p \right\| + \frac{\beta_n}{\alpha_n+\beta_n} (1+h_n) \left\| I_n (PI_n)^{n-1} x_n - p \right\| \right) \\ \leq\ & \limsup_{n\to\infty} \left(\frac{\alpha_n}{\alpha_n+\beta_n} (1+h_n)^2 \|x_n - p\| + \frac{\beta_n}{\alpha_n+\beta_n} (1+h_n)^2 \|x_n - p\| \right) \\ =\ & \limsup_{n\to\infty} (1+h_n)^2 \|x_n - p\| \left[\frac{\alpha_n}{\alpha_n+\beta_n} + \frac{\beta_n}{\alpha_n+\beta_n} \right] \\ =\ & \limsup_{n\to\infty} (1+h_n)^2 \|x_n - p\| \\ =\ & d. \end{aligned} \tag{6}$$

It follows from $(4), (5), (6)$ and Lemma 2.5 that

$$\begin{aligned} & \lim_{n\to\infty} \left\| \left(I_n (PI_n)^{n-1} x_n - p \right) - \left[\frac{\alpha_n}{\alpha_n+\beta_n} \left(S_n (PS_n)^{n-1} x_n - p \right) \right.\right. \\ & \left.\left. + \frac{\beta_n}{\alpha_n+\beta_n} \left(T_n (PT_n)^{n-1} x_n - p \right) \right] \right\| \\ =\ & \lim_{n\to\infty} \left(\frac{1}{\alpha_n+\beta_n} \right) \left\| (\alpha_n+\beta_n) \left(I_n (PI_n)^{n-1} x_n - p \right) - \alpha_n \left(S_n (PS_n)^{n-1} x_n - p \right) \right. \\ & \left. + \beta_n \left(T_n (PT_n)^{n-1} x_n - p \right) \right\| \\ =\ & \lim_{n\to\infty} \left(\frac{1}{\alpha_n+\beta_n} \right) \left\| x_{n+1} - I_n (PI_n)^{n-1} x_n \right\| \\ =\ & 0. \end{aligned}$$

Since $0 < a \leq \alpha_n, \beta_n \leq b < 1$, we have

$$\lim_{n\to\infty} \left\| x_{n+1} - I_n (PI_n)^{n-1} x_n \right\| = 0. \tag{7}$$

In a similar way, we have

$$\lim_{n\to\infty}\left\|x_{n+1}-S_n(PS_n)^{n-1}x_n\right\|=0 \tag{8}$$

and

$$\lim_{n\to\infty}\left\|x_{n+1}-T_n(PT_n)^{n-1}x_n\right\|=0. \tag{9}$$

On the other hand, we have

$$\left\|S_n(PS_n)^{n-1}x_n-T_n(PT_n)^{n-1}x_n\right\|\leq\left\|S_n(PS_n)^{n-1}x_n-x_{n+1}\right\|+\left\|x_{n+1}-T_n(PT_n)^{n-1}x_n\right\|.$$

Therefore, by using (8) and (9), we obtain

$$\lim_{n\to\infty}\left\|S_n(PS_n)^{n-1}x_n-T_n(PT_n)^{n-1}x_n\right\|=0. \tag{10}$$

It follows from $(7),(8),(9)$ that

$$\lim_{n\to\infty}\left\|S_n(PS_n)^{n-1}x_n-I_n(PI_n)^{n-1}x_n\right\|=0 \tag{11}$$

and

$$\lim_{n\to\infty}\left\|T_n(PT_n)^{n-1}x_n-I_n(PI_n)^{n-1}x_n\right\|=0. \tag{12}$$

From (10) and the condition $\left\|x_n-T_n(PT_n)^{n-1}x_n\right\|\leq\left\|T_n(PT_n)^{n-1}x_n-S_n(PS_n)^{n-1}x_n\right\|$, we obtain

$$\lim_{n\to\infty}\left\|x_n-T_n(PT_n)^{n-1}x_n\right\|=0. \tag{13}$$

Moreover,

$$\left\|x_n-S_n(PS_n)^{n-1}x_n\right\|\leq\left\|x_n-T_n(PT_n)^{n-1}x_n\right\|+\left\|S_n(PS_n)^{n-1}x_n-T_n(PT_n)^{n-1}x_n\right\|.$$

Hence by using (10) and (13), we have

$$\lim_{n\to\infty}\left\|x_n-S_n(PS_n)^{n-1}x_n\right\|=0. \tag{14}$$

In a similar way, we have

$$\lim_{n\to\infty}\left\|x_n-I_n(PI_n)^{n-1}x_n\right\|=0. \tag{15}$$

Since

$$\begin{aligned}\|x_{n+1}-x_n\| &\leq (1-\alpha_n-\beta_n)\left\|I_n(PI_n)^{n-1}x_n-x_n\right\|+\alpha_n\left\|S_n(PS_n)^{n-1}x_n-x_n\right\| \\ &\quad +\beta_n\left\|T_n(PT_n)^{n-1}x_n-x_n\right\|,\end{aligned}$$

we find from $(13),(14)$ and (15) that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{16}$$

By induction, for each positive integer k, we also have

$$\lim_{n\to\infty}\|x_{n+k}-x_n\|=0. \tag{17}$$

When $n>N$, we have

$$\begin{aligned}\|x_n-I_nx_n\| &\leq \left\|x_n-I_n(PI_n)^{n-1}x_n\right\|+\left\|I_n(PI_n)^{n-1}x_n-I_nx_n\right\| \\ &\leq \left\|x_n-I_n(PI_n)^{n-1}x_n\right\|+\Gamma\left\|I_n(PI_n)^{n-2}x_n-x_n\right\| \\ &\leq \left\|x_n-I_n(PI_n)^{n-1}x_n\right\|+\Gamma\left\{\left\|I_{n-N}(PI_{n-N})^{n-2}x_{n-N}-x_{n-N}\right\|\right. \\ &\quad \left.+\|x_{n-N}-x_n\|+\left\|I_n(PI_n)^{n-2}x_n-I_{n-N}(PI_{n-N})^{n-2}x_{n-N}\right\|\right\} \\ &\leq \left\|x_n-I_n(PI_n)^{n-1}x_n\right\|+\Gamma\left\{\left\|I_{n-N}(PI_{n-N})^{n-2}x_{n-N}-x_{n-N}\right\|\right. \\ &\quad \left.+\|x_{n-N}-x_n\|+\Gamma\|x_n-x_{n-N}\|\right\}.\end{aligned}$$

From (15) and (17), we have

$$\lim_{n\to\infty}\|x_n-I_nx_n\|=0. \tag{18}$$

For each $k\in J$, we have

$$\begin{aligned}\|x_n-I_{n+k}x_n\| &\leq \|x_n-x_{n+k}\|+\|x_{n+k}-I_{n+k}x_{n+k}\|+\|I_{n+k}x_{n+k}-I_{n+k}x_n\| \\ &\leq \|x_n-x_{n+k}\|+\|x_{n+k}-I_{n+k}x_{n+k}\|+\Gamma\|x_n-x_{n+k}\|.\end{aligned}$$

Thus, from (17) and (18), we have

$$\lim_{n\to\infty}\|x_n-I_{n+k}x_n\|=0. \tag{19}$$

Consequently, for each $k \in J$, we obtain

$$\lim_{n\to\infty} \|x_n - I_k x_n\| = 0. \tag{20}$$

On the other hand, we have

$$\begin{aligned}
\|x_n - T_n x_n\| &\leq \left\|x_n - T_n (PT_n)^{n-1} x_n\right\| + \left\|T_n (PT_n)^{n-1} x_n - T_n x_n\right\| \\
&\leq \left\|x_n - T_n (PT_n)^{n-1} x_n\right\| + L\left\|I_n (PI_n)^{n-2} x_n - I_n x_n\right\| \\
&\leq \left\|x_n - T_n (PT_n)^{n-1} x_n\right\| + L\Big\{\left\|I_{n-N} (PI_{n-N})^{n-2} x_{n-N} - x_{n-N}\right\| \\
&\quad + \|x_{n-N} - x_n\| + \left\|I_n (PI_n)^{n-2} x_n - I_{n-N} (PI_{n-N})^{n-2} x_{n-N}\right\|\Big\} \\
&\leq \left\|x_n - T_n (PT_n)^{n-1} x_n\right\| + L\Big\{\left\|I_{n-N} (PI_{n-N})^{n-2} x_{n-N} - x_{n-N}\right\| \\
&\quad + \|x_{n-N} - x_n\| + \Gamma \|x_n - x_{n-N}\|\Big\}.
\end{aligned}$$

Thus it follows from $(13), (15)$ and (17), we obtain that $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$. For each $k \in J$, similar to the proof of (19), we have $\lim_{n\to\infty} \|x_n - T_{n+k} x_n\| = 0$. Consequently, for each $k \in J$, we obtain

$$\lim_{n\to\infty} \|x_n - T_k x_n\| = 0. \tag{21}$$

Finally, by using the same methods above the mentioned, we can show that

$$\lim_{n\to\infty} \|x_n - S_k x_n\| = 0. \tag{22}$$

This completes the proof.

Theorem 3.3. *Let E be a uniformly convex Banach space, $K, T_i, S_i, I_i, \{x_n\}$ be same as in Theorem . If T_i, S_i and I_i satisfy the Condition (B), then $\{x_n\}$ converges strongly to a common fixed point of T_i, S_i and I_i.*

Proof. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\| = d \geq 0$ exists for all $p \in F$. If $d = 0$, there is nothing to prove.

Suppose that $d > 0$, by Theorem 3.2, we have

$$\lim_{n\to\infty} \|x_n - T_k x_n\| = \lim_{n\to\infty} \|x_n - S_k x_n\| = \lim_{n\to\infty} \|x_n - I_k x_n\| = 0.$$

From Lemma 3.1, we know that $\lim_{n\to\infty} d(x_n, F)$ exists. Because $\{T_i : i \in J\}, \{S_i : i \in J\}$ and $\{I_i : i \in J\}$ satisfy the Condition (B), therefore

$$f(d(x_n, F)) \leq \max_{k\in J} \left\{ \frac{1}{3} (\|x_n - T_k x_n\| + \|x_n - S_k x_n\| + \|x_n - I_k x_n\|) \right\}.$$

It follows from $(20), (21)$ and (22) that $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is a nondecreasing function and $f(0) = 0$, so $\lim_{n\to\infty} d(x_n, F) = 0$. In the proof of Lemma 3.1, we showed that $\{x_n\}$ is a Cauchy sequence in K. So it must converge to a point q in K. It follows from $\lim_{n\to\infty} d(x_n, F) = 0$ that $d(q, F) = 0$. By the routine method we can easily show that F is closed, therefore $q \in F$. This completes the proof.

Note that the Condition (B) is weaker than both the compactness of K and the semi-compactness of the mappings $\{T_i : i \in J\}$, $\{S_i : i \in J\}$ and $\{I_i : i \in J\}$, therefore we already have the following theorem proved.

Theorem 3.4. *Let E be a uniformly convex Banach space, $K, T_i, S_i, I_i, \{x_n\}$ be same as in Theorem 3.2. Assume that either K is compact or one of the mappings in $\{T_i : i \in J\}$, $\{S_i : i \in J\}$ and $\{I_i : i \in J\}$ is semi-compact. Then $\{x_n\}$ converges strongly to a common fixed point of T_i, S_i and I_i.*

Proof. For any $k \in J$, we suppose that T_k, S_k and I_k are semi-compact. Then from $(20), (21)$ and (22), we have

$$\lim_{n\to\infty} \|x_n - T_k x_n\| = 0, \lim_{n\to\infty} \|x_n - S_k x_n\| = 0 \text{ and } \lim_{n\to\infty} \|x_n - I_k x_n\| = 0.$$

From the semi-compactness T_k, S_k and I_k, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to a $q \in K$. Again, using $(20), (21)$ and (22), we obtain

$$\lim_{j\to\infty} \left\|x_{n_j} - T_k x_{n_j}\right\| = \|q - T_k q\| = 0, \lim_{j\to\infty} \left\|x_{n_j} - S_k x_{n_j}\right\| = \|q - S_k q\| = 0$$

and $\lim_{j\to\infty} \left\|x_{n_j} - I_k x_{n_j}\right\| = \|q - I_k q\| = 0$ for all $k \in J$. It follows from Lemma 2.6, $q \in F = \cap_{i=1}^{N} (F(T_i) \cap F(S_i) \cap F(I_i))$. Since $\lim_{n\to\infty} \left\|x_{n_j} - q\right\| = 0$ and $\lim_{n\to\infty} \|x_n - q\|$ exists for all $q \in F$ by Lemma 3.1, therefore$\lim_{n\to\infty} \|x_n - q\| = 0$. This completes the proof.

Theorem 3.5. *Let E be a uniformly convex Banach space, $K, T_i, S_i, I_i, \{x_n\}$ be same as in Theorem 3.2. Assume that one of the mappings in $\{T_i : i \in J\}$, $\{S_i : i \in J\}$ and $\{I_i : i \in J\}$ is completely continuous mapping, then $\{x_n\}$ converges strongly to a common fixed point of T_i, S_i and I_i.*

Proof. By Lemma 3.1, $\{x_n\}$ is bounded. If one of T_i, S_i and I_i's is completely continuous, say T_k, then there exists a subsequence $\{T_k x_{n_j}\}$ of $\{T_k x_n\}$ such that $T_k x_{n_j} \to q$ as $j \to \infty$. By Theorem 3.2, $\lim_{n\to\infty} \|x_{n_j} - T_k x_{n_j}\| = 0$, and using the continuity of T_k, we have $\lim_{n\to\infty} \|x_{n_j} - q\| = 0$. As above, by Lemma , $q \in \cap_{i=1}^{N} F(T_i)$. Since $\lim_{n\to\infty} \|x_{n_j} - q\| = 0$ and $\lim_{n\to\infty} \|x_n - q\|$ exists for all $q \in F$ by Lemma 3.1, therefore $\lim_{n\to\infty} \|x_n - q\| = 0$. This completes the proof.

Remark 3.6. Under suitable conditions, sequence $\{x_n\}$ defined by (2) can also be generalized to the iterative sequences with errors. Thus all the results proved in this paper can also be proved for the iterative process with errors. In this case, our main iterative process (2) is as, for any given $x_1 \in K$ and $n \geq 1$,

(23)
$$x_{n+1} = P\Big((1-\alpha_n-\beta_n-\gamma_n) I_n (PI_n)^{n-1} x_n + \alpha_n S_n (PS_n)^{n-1} x_n + \beta_n T_n (PT_n)^{n-1} x_n + \gamma_n u_n\Big),$$

where $\{T_i, S_i, I_i : i \in J\}$ are non-self mappings from K to E and P is a nonexpansive retraction from E onto K. Also, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three real sequences in $(0,1)$ and $\{u_n\}$ is a bounded sequence.

Remark 3.7. (1) If we take $u_n = 0$, iterative process (23) reduces to iterative process (2).

(2) Our results improve and extend the corresponding results of Rhoades and Temir [10], Temir and Gul [15], Temir [14], Yao and Wang [18], Zhang and Wang [21] to two finite families of non-self I-asymptotically quasi-nonexpansive mappings.

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References

[1] C. Byrne, A unified treatment of some iterative algorithms in signal processing and imagine reconstruction, Inverse Prob. 20 (2004), 103-120.

[2] Y.J. Chen, G. Wang, An iteration process for common fixed points of a finite family of nonself I-asymptotically quasi-nonexpansive mappins in Banach spaces, Int. Math. Forum 5 (2010), 429-440.

[3] C.E. Chidume, E.U. Ofoedu, H. Zegeye, Strong and weak convergence theorems for asymptotically nonexpansive mappings, J. Math. Anal. Appl. 280 (2003), 364-374.

[4] K. Goebel, W.A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 35 (1972), 171-174.

[5] B. Gunduz, S. Akbulut, On weak and strong convergence theorems for a finite family of nonself I-asymptotically nonexpansive mappings, Math. Moravica 19 (2015), 49-64.

[6] S.H. Khan, N. Hussain, Convergence theorems for nonself asymptotically nonexpansive mappings, Comp. Math. Appl. 55 (2008), 2544-2553.

[7] H. Kiziltunc, M. Ozdemir, S. Akbulut, Common fixed points for two nonexpansive nonself-mappings with errors in Banach spaces, Arabian J. Sci. Eng. 35 (2010), 215–224.

[8] M. Maiti, M.K. Ghosh, Approximating fixed points by Ishikawa iterates, Bull. Austral. Math. Soc. 40 (1989), 113-117.

[9] H.K. Pathak, Y.J. Cho, S.M. Kang, Strong and weak convergence theorems for nonself asymptotically peturbed nonexpansive mappings, Nonlinear Anal. 70 (2009), 1929-1938.

[10] B.E. Rhoades, S. Temir, Convergence theorems for I-nonexpansive mappings, Int. J. Math. Math. Sci. 2006 (2006), Article ID 63435.

[11] J. Schu, Iterative construction of fixed points of asymptotically nonexpansive mappings, J. Math. Anal. Appl. 158 (1991) 407-413.

[12] H.F. Senter, W.G. Dotson, Approximating fixed points of nonexpansive mappings, Proc. Amer. Math. Soc. 44 (1974), 375-380.

[13] K.K. Tan, H.K. Xu, Approximating fixed points of nonexpansive mappings by Ishikawa iteration process, J. Math. Anal. Appl. 178 (1993), 301-308.

[14] S. Temir, On the convergence theorems of implicit iteration process for a finite family of I-asymptotically nonexpansive mappings, J. Comp. Appl. Math. 225 (209) 398-405.

[15] S. Temir, O. Gul, Convergence theorem for I-asymptotically nonexpansive mapping in Hilbert space, J. Math. Anal. Appl. 329 (2007) 759-765.

[16] L. Wang, Strong and weak convergence theorems for common fixed points of nonself asymptotically nonexpansive mappings, J. Math. Anal. Appl. 323 (2006), 550-557.

[17] L.P. Yang, Modified multistep iterative process for some common fixed points of a finite family of nonself asymptotically nonexpansive mappings, Math. Comp. Modell. 45 (2007), 1157-1169.

[18] S. Yao, L. Wang, Strong convergence theorems for nonself I-asymptotically quasi-nonexpansive mappings, Thai J. Math. 5 (2007), 199-207.

[19] I. Yildirim, On the convergence theorems of an implicit iteration process for asymptotically quasi I-nonexpansive mappings, Hacettepe J. Math. Statistics 42 (2013), 617-626.

[20] I. Yildirim, F. Gu, A new iterative process for approximating common fixed points of nonself I-asymptotically quasi-nonexpansive, Appl. Math. J. Chinese Univ. 27 (2012), 489-502.

[21] Y. Zhang, L. Wang, Strong convergence theorems of I-nonexpansive mapings, Math. Sci. Res. J. 11 (2007), 384-388.