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SUFFICIENT CONDITIONS FOR EXISTENCE OF POSITIVE SOLUTIONS FOR THIRD-ORDER NONLOCAL BOUNDARY VALUE PROBLEMS WITH SIGN CHANGING NONLINEARITY

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Abstract. In this paper, we obtain the sufficient conditions for the existence and multiplicity of positive solutions of a kind of nonlocal boundary value problem of third-order nonlinear differential equations. The interesting point is that the nonlinear term may change sign.

Keywords. Third-order multi-point boundary value problem; Positive solution; Cone; Fixed point.

1. Introduction

Let $f : [0,1] \times [0,+\infty) \to (-\infty,+\infty))$ be a continuous function and $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1,\ \beta > 0,\ \alpha_i \geq 0,\ 0 < \sum_{i=1}^{m-2} \alpha_i < 1$ be given. In this paper, we study the existence of positive solutions for the third-order differential equation

$$x'''(t) + f(t,\ x(t)) = 0,\ t \in [0,1] \tag{1.1}$$

subject to the m-point boundary value conditions

$$x''(0) = 0,\ x(0) = \beta x'(0),\ x(1) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i). \tag{1.2}$$

Throughout this paper, we always suppose that f satisfies the condition

(H) $f(t,0) \geq 0,\ t \in [0,\ 1]$ *and there exist* $t_0 \in [0,1]$ *such that* $f(t_0,0) \neq 0$.

Third order differential equations arise in a variety of different areas of applied mathematics and physics, as the deflection of a curved beam having a constant or varying cross section, three layer beam, electromagnetic waves, gravity driven flows, and so on [1]. In recent years, existence of positive solutions for third-order boundary

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value problems has been extensively investigated. For instance, Anderson [2] established the existence of at least three positive solution to the problem

$$\begin{cases} -x'''(t) + f(x(t)) = 0, \ t \in (0,1), \\ x(0) = x'(t_2) = x''(1) = 0, \quad t_2 \in (0,1), \end{cases}$$

where $f : R \to [0, +\infty)$ is continuous and $\frac{1}{2} \le t_2 < 1$. Palamides and Smyrlis [3] proved that there exist at least one positive solution for the third-order three-point Bvp

$$\begin{cases} x'''(t) = a(t)f(t, x(t)), \ t \in (0,1), \\ x''(\eta) = 0, \ x(0) = x(1) = 0, \ \eta \in (0,1). \end{cases}$$

The results are based on the well-known Guo-Krasnoselski$\breve{i}$ fixed point theorem [4]. Guo, Sun and Zhao [5] studied the positive solutions for the third-order three-point problem

$$\begin{cases} x'''(t) = a(t)f(x(t)), \ t \in (0,1), \\ x(0) = x'(0) = 0, \ x'(1) = x'(\eta), \ \eta \in (0,1), \end{cases}$$

By using Guo-Krasnoselski$\breve{i}$ fixed point theorem, they obtained the existence of positive solutions for problem above. For more existence results for third-order bvps, one can see [6-12] and references along this line.

In order to apply the concavity of solutions in the proofs, all the above works were done under the assumption that the nonlinear term is nonnegative. Few paper has appeared in the literature which concerns the multi-point boundary value problem for third-order differential equation when the nonlinearity in the differential equation may change sign. This paper attempts to fill this gap in the literature. Motivated by Ji, Feng and Ge [13] and Lan [14], in this paper,we will use a fixed point theorem for operators in a cone to derive several new results of positive solutions for problem (1.1-1.2). An example is given to illustrate the results.

The rest of this paper is organized as follows. Section 2 contains some preliminary lemmas, Section 3 contains the main results of this paper.

2. Background

In this section, we present the necessary definitions from cone theory in Banach spaces and a fixed point theorem in cones.

Definition 2.1. Let E be a real Banach space over R. A nonempty convex closed set $P \subset E$ is said to be a cone provided that:

(1) $au \in P$, for all $u \in P, \ a \ge 0$;

(2) $u, -u \in P$ implies $u = 0$.

Definition 2.2. An operator is called completely continuous if it is continuous and maps bounded sets into pre-compact sets.

Definition 2.3. A function x is said to be concave on $[0,1]$, if

$$x(\lambda t_1 + (1-\lambda)t_2) \geq \lambda x(t_1) + (1-\lambda)x(t_2), \lambda,\ t_1,\ t_2 \in [0,\ 1].$$

Lemma 2.4. *Let K be a cone in a Banach space X. Let D be an open bounded subset of X with with $D_K = D \cap K \neq \emptyset$. Assume that $A : \overline{D_K} \to K$ is completely continuous such that $x \neq Ax$ for $x \in \partial D_K$. Then the following results hold:*

(1) If $\|Ax\| \leq \|x\|,\ x \in \partial D_K$, then $i_K(A,\ D_K) = 1$.

(2) If there exists $e \in K \backslash \{0\}$ such that $x \neq Ax + \lambda e$ for all $x \in \partial D_K,\ \lambda > 0$, then $i_K(A,\ D_K) = 0$.

(3) Let U be open in X such that $\overline{U} \subset D_K$. If $i_K(A,\ D_K) = 1,\ i_K(A,\ U_K) = 0$, then A has a fixed point in $D_k \backslash \overline{U_k}$. The same result holds if $i_K(A,\ D_K) = 0,\ i_K(A,\ U_K) = 1$.

3. Main results

Firstly we consider the third-order problem

$$x'''(t) + y(t) = 0,\ t \in [0,1], \tag{3.1}$$

$$x''(0) = 0,\ x(0) = \beta x'(0),\ x(1) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \tag{3.2}$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1,\ 0 < \beta_i < 1,\ i = 1, 2, \cdots, m-2$, and $\sum_{i=1}^{m-2} \alpha_i < 1$.

Lemma 3.1. *Denote $\xi_0 = 0, \xi_{m-1} = 1, \rho = \beta(1 - \sum_{i=0}^{m-1} \alpha_i) + 1 - \sum_{i=0}^{m-1} \alpha_i \xi_i$,then boundary value problem (3.1-3.2) has the unique solution*

$$x(t) = \int_0^1 G(t,s) \int_0^s y(\tau) d\tau ds,$$

where

$$G(t,s) = \frac{1}{\rho} \begin{cases} [(1-s) + \sum_{k=i}^{m-1} \alpha_k (s - \xi_k)](\beta + t) & t \leq s,\ \xi_{i-1} \leq s \leq \xi_i \\ (s+\beta)(1-t) + \sum_{k=0}^{i-1} \alpha_k (t-s)(\beta + \xi_k) + \sum_{k=i}^{m-1} \alpha_k (t - \xi_k)(s + \beta) & t \geq s,\ \xi_{i-1} \leq s \leq \xi_i. \end{cases}$$

Proof. Integrating both sides of (3.1) and considering the boundary condition $x''(0) = 0$, we have

$$-x''(t) = \int_0^s y(s) ds. \tag{3.3}$$

Let $G(t,s)$ is the Green's function of problem

$$-x''(t) = 0, \tag{3.4}$$

$$x(0) = \beta x'(0),\ x(1) = \sum_{i=0}^{m-1} \beta_i x(\xi_i). \tag{3.5}$$

From (3.4), we can suppose

$$G(t,s)=\begin{cases} A+Bt & t\le s,\\ C+Dt & t\ge s.\end{cases}$$

By the definition and properties of Green's function together with (3.5), we have

$$\begin{cases} A+Bs=C+Ds,\\ B-D=1,\\ A=\beta B,\\ C+D=\sum_{k=0}^{i-1}\alpha_k(A+B\xi_k)+\sum_{k=i}^{m-1}\alpha_k(C+D\xi_k).\end{cases}$$

We get

$$A=\frac{\beta}{\rho}[(1-s)+\sum_{k=i}^{m-1}\alpha_k(s-\xi_k)],$$

$$B=\frac{1}{\rho}[(1-s)+\sum_{k=i}^{m-1}\alpha_k(s-\xi_k)],$$

$$C=\frac{1}{\rho}[-\sum_{k=0}^{i-1}\alpha_k\beta s+s(1-\sum_{k=0}^{m-1}\alpha_k\xi_k)+\beta(1-\sum_{k=i}^{m-1}\alpha_k\xi_k)],$$

$$D=\frac{1}{\rho}[(\beta+s)(\sum_{k=i}^{m-1}\alpha_k-1)+\sum_{k=0}^{i-1}\alpha_k(\beta+\xi_k)].$$

Thus

$$G(t,s)=\frac{1}{\rho}\begin{cases} [(1-s)+\sum_{k=i}^{m-1}\alpha_k(s-\xi_k)](\beta+t), & t\le s,\\ (s+\beta)(1-t)+\sum_{k=0}^{i-1}\alpha_k(t-s)(\beta+\xi_k)+\sum_{k=i}^{m-1}\alpha_k(t-\xi_k)(s+\beta), & t\ge s.\end{cases}$$

Considering (3.3) together, we obtain that problem $(3.1),(3.2)$ has the unique solution

$$x(t)=\int_0^1 G(t,s)\int_0^s y(\tau)d\tau ds.$$

Lemma 3.2. $G(t,s)\ge 0$ *for* $t,\ s\in[0,1]$.

Proof. For $\xi_{i-1}\le s\le \xi_i$, if $t\le s$,

$$(1-s)+\sum_{k=i}^{m-1}\alpha_k(s-\xi_k)\ge\sum_{k=i}^{m-1}\alpha_k(1-\xi_k)\ge 0.$$

If $t\ge s$,

$$\begin{aligned}&(s+\beta)(1-t)+\sum_{k=0}^{i-1}\alpha_k(t-s)(\beta+\xi_k)+\sum_{k=i}^{m-1}\alpha_k(t-\xi_k)(s+\beta)\\ &\ge\sum_{k=0}^{i-1}\alpha_k(t-s)(\beta+\xi_k)+\sum_{k=i}^{m-1}\alpha_k(1-\xi_k)(s+\beta)\ge 0.\end{aligned}$$

Then $G(t,s) \geq 0, t, s \in [0,1]$.

Lemma 3.3. *If $y(t) \geq 0, t \in [0,1]$, $x(t)$ is the solution of problem (3.1), (3.2), we claim that*

$$\min_{\xi_{j-1} \leq t \leq \xi_j} |x(t)| \geq \delta \max_{0 \leq t \leq 1} |x(t)|,$$

where $\xi_j \in \{\xi_1,\ \xi_2,\ \cdots,\ \xi_{m-1}\}$, $\delta = min\{\xi_{j-1}, 1-\xi_j\} < 1$ is a constant.

Proof. For $x'''(t) = -y(t) \leq 0,\ t \in [0,1]$, we see that $x''(t)$ is decreasing on $[0,1]$. Considering $x''(0) = 0$, we have $x''(t) \leq 0,\ t \in (0,1)$. Let $\max\limits_{0 \leq t \leq 1} x(t) = x(\eta),\ \eta \in [0,1]$. We distinguish some cases to show the conclusion.
(1) $\xi_j < \eta$. From the concavity of $x(t)$, $min_{\xi_{j-1} \leq t \leq \xi_j} x(t) = x(\xi_{j-1})$ and

$$\frac{x(\xi_{j-1}) - x(0)}{\xi_{j-1}} \geq \frac{x(\eta) - x(0)}{\eta}.$$

We have $x(\xi_{j-1}) \geq \xi_{j-1} x(\eta)$.
(2) $\xi_{j-1} > \eta$. Here $min_{\xi_{j-1} \leq t \leq \xi_j} x(t) = x(\xi_j)$.From

$$\frac{x(1) - x(\eta)}{1-\eta} \geq \frac{x(1) - x(\xi_j)}{1-\xi_j}.$$

We see $x(\xi_j) \geq (1-\xi_j) x(\eta)$.
Considering the case $\xi_{j-1} \leq \eta \leq \xi_j$ similarly, we can get the conclusion. □

Let Banach space $E = C[0,\ 1]$ be endowed with the norm

$$\|x\| = \max_{0 \leq t \leq 1} |x(t)|,\ x \in E.$$

We define the cone $P \subset E$ by $K = \{x \in E \mid x(t) \geq 0,\ x''(0) = 0,\ x(0) = \beta x'(0),\ x(1) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i),$ *$x(t)$ is concave on* $[0,1]\}$. We define

$$\phi(t) = \min\{t,\ 1-t\},\ t \in (0,1),$$

$$K_\rho = \{x \in K : \|x\| < \rho\},\ K_\rho^* = \{x \in K : \rho\phi(t) < x(t) <\ \rho\},$$

$$\Omega_\rho = \{x \in C[0,\ 1] : x(t) \geq 0, x(t) \textit{ is concave on } [0,1],\ \min_{0 \leq t \leq 1} x(t) < \delta\rho\}.$$

Lemma 3.4. [14] *Ω_ρ has the following properties:*
(1) $\Omega_\rho \subset K_\rho$ is open relative to K_ρ. (2) If $x \in \partial\Omega_\rho$, then $\delta\rho \leq x(t) \leq \rho,\ t \in [0,\ 1]$.
We denote

$$f_{\delta\rho}^{\rho} = \min\{\frac{f(t,\ x)}{\rho} :\ t \in [0,\ 1],\ x \in [\delta\rho,\ \rho]\},$$

$$f_{\phi(t)\rho}^{\rho} = \max\{\frac{f(t,\ x)}{\rho} :\ t \in [0,\ 1],\ x \in [\rho\phi(t),\ \rho]\},$$

$$f_0^{\rho} = \max\{\frac{f(t,\ x)}{\rho} :\ t \in [0,\ 1],\ x \in [0,\ \rho]\},$$

$$m = \frac{1}{\max\limits_{0 \leq t \leq 1} \int_0^1 sG(t,\ s)ds},\ M = \frac{1}{\delta \min\limits_{0 \leq t \leq 1} \int_0^1 sG(t,s)ds}.$$

Theorem 3.5. *Assume* (H) *holds. Furthermore, the following condition* $(H1)$ *holds:*

$(H1)$ *There exist positive constants* $\rho_1,\ \rho_2,\ \rho_3 \in (0,\ \infty)$ *with* $\rho_1 < \delta\rho_2 < \rho_2 < \rho_3$ *such that*

(1) $f(t,\ u) > 0,\ t \in [0,\ 1], u \in [\rho_1\phi(t),\ \infty)$,

(2) $f^{\rho_1}_{\rho_1\phi(t)} < m, f^{\rho_2}_{\delta\rho_2} > M,\ f^{\rho_3}_{\rho_3\phi(t)} \leq m$,

then problem (1.1-1.2) has three positive solutions in K*. Assume* (H) *holds and following condition* $(H2)$ *holds:*

$(H2)$ *There exist positive constants* $\rho_1,\ \rho_2,\ \rho_3 \in (0,\ \infty)$ *with* $\rho_1 < \rho_2 < \rho_3$ *such that*

(3) $f(t,\ u) > 0,\ t \in [0,\ 1], u \in [\min\{\rho_1,\ \rho_2\phi(t)\},\ \infty)$,

(4) $f^{\rho_1}_{\delta\rho_1} > M,\ f^{\rho_2}_{\rho_2\phi(t)} < m,\ f^{\rho_3}_{\delta\rho_3} \geq M$,

then problem (1.1-1.2) has two positive solutions in K*.*

Proof. We first assume that $(H1)$ holds. Define an auxiliary function $f^*(t,\ x) \in C([0,\ 1] \times [0,\infty),\ [0,\infty))$:

$$f^*(t,\ x) = \begin{cases} f(t,\ x), & x \geq \rho_1\phi(t), \\ f(t,\ \rho_1\phi(t), & 0 \leq x < \rho_1\phi(t)). \end{cases}$$

Now, we consider the following auxiliary boundary value problem

$$\begin{cases} x'''(t) + f^*(t,\ x(t)) = 0,\ t \in [0,1], \\ x''(0) = 0,\ x(0) = \beta x'(0),\ x(1) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i). \end{cases} \tag{3.6}$$

Define the operator

$$(Tx)(t) = \int_0^1 G(t,s) \int_0^s f^*(\tau,\ x(\tau)) d\tau ds.$$

It is easy to check that $T : K \to K$ is completely continuous. From the condition $(H1)$, we have

$$f^{*\rho_1}_{\rho_1\phi(t)} < m, f^{*\rho_2}_{\delta\rho_2} > M,\ f^{*\rho_3}_{\rho_3\phi(t)} \leq m.$$

Then, for $x \in \partial K^*_{\rho_1}$, we get

$$\begin{aligned} \|(Tx)(t)\| &= \max_{0\leq t\leq 1} |\int_0^1 G(t,s) \int_0^s f^*(\tau,\ x(\tau)) d\tau ds| \\ &\leq \int_0^1 G(t,s) \int_0^s |f^*(\tau,\ x(\tau))| d\tau ds \\ &< m\rho_1 \int_0^1 sG(t,s) ds \leq \rho_1, \end{aligned}$$

which gives that $\|Tx\| \leq \|x\|$ for $x \in \partial K^*_{\rho_1}$. This induces that $i_K(T,\ K^*_{\rho_1}) = 1$. Let $e(t) \equiv 1$ for $t \in [0,\ 1]$. Then $e \in \partial K_1$. We claim that

$$x \neq Tx + \lambda e,\ x \in \partial\Omega_{\rho_2},\ \lambda \geq 0.$$

Otherwise, there exist $x_0 \in \partial\Omega_{\rho_2}$ and $\lambda_0 \geq 0$ such that $x_0 = Tx_0 + \lambda_0 e$. But

$$\begin{aligned} x_0(t) &= (Tx_0)(t) + \lambda_0 e(t) \geq \delta\|Tx_0\| + \lambda_0 \\ &= \delta \max_{0\leq t\leq 1} \left| \int_0^1 G(t,s) \int_0^s f^*(\tau,\ x(\tau))d\tau ds \right| + \lambda_0 \\ &\geq \delta\rho_2 M \int_0^1 sG(t,\ s)ds + \lambda_0 \geq \rho_2 + \lambda_0. \end{aligned}$$

This gives that $\rho_2 \geq \rho_2 + \lambda_0$, which is a contradiction. Thus, an application of Lemma 2.4 ensures that $i_K(T,\ \Omega_{\rho_2}) = 0$. Similar with above, we can prove that $i_K(T,\ K^*_{\rho_2}) = 1$ and $i_K(T,\ K^*_{\rho_3}) = 1$. Thus, there exist positive solution $x_1,\ x_2,\ x_3$ such that $x_1 \in K^*_{\rho_1},\ x_2 \in \Omega_{\rho_2}\backslash\overline{K^*_{\rho_1}}, x_3 \in K^*_{\rho_3}$. Then auxiliary boundary value problem has at least three positive solutions $x_1,\ x_2,\ x_3 \in [\rho_1\phi(t),\ \infty)$, which ensure that $x_1,\ x_2,\ x_3$ are positive solutions of boundary value problem (1.1-1.2). The proof of (H2) is similar to that in (H1), we omit it here.

Theorem 3.6. *Assume condition* (H) *holds. Furthermore, one of the following conditions hold:*

$(H3)$ *There exist positive constants* $\rho_1,\ \rho_2 \in (0,\ \infty)$ *with* $\rho_1 < \delta\rho_2$ *such that*

(5) $f(t,\ x) > 0,\ t \in [0,\ 1], x \in [\rho_1\phi(t),\ \infty)$

(6) $f^{\rho_1}_{\rho_1\phi(t)} \leq m, f^{\rho_2}_{\delta\rho_2} \geq M,$

$(H4)$ *There exist positive constants* $\rho_1,\ \rho_2 \in (0,\ \infty)$ *with* $\rho_1 < \rho_2$ *such that*

(7) $f(t,\ x) > 0,\ t \in [0,\ 1], x \in [\min\{\rho_1,\ \rho_2\phi(t)\},\ \infty)$

(8) $f^{\rho_1}_{\delta\rho_1} \geq M, f^{\rho_2}_{\rho_2\phi(t)} \leq m,$

then problem (1.1-1.2) has at least one positive solutions in K*.*

Remark 3.7. The proof of Theorem 3.6 is similar with Theorem 3.5. So we omit it here.

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