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COMMON FIXED POINT THEOREMS OF TWO CONTRACTION MAPPINGS IN CONE b-METRIC SPACES WITH BANACH ALGEBRAS

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Abstract. In the framework of complete cone b-metric spaces with Banach algebras, we prove the existence and uniqueness of two contractive mappings of coincidence points and common fixed points. The conclusions obtained in this paper extend and improve the corresponding results announced recently.

Keywords. Banach algebra; Cone b-metric spaces; Coincidence point; Common fixed point.

1. Introduction

In 2007, Huang and Zhang [1] introduced the concept of a cone metric space, as a generalization of the metric space. Recently, Hussain and Shah [5] introduced the cone b-metric space, as a generalization of the b-metric space and the cone metric space. Following Hussain and Shah [5], Huang and Xu [4] further obtained some interesting fixed point results for contractive mappings in the framework of cone b-metric spaces. In 2013, Shi and Xu [10] proved the common fixed point theorems for two weakly compatible self-mappings in cone b-metric spaces.

In 2013, Liu and Xu [2] introduced the concept of cone metric spaces over Banach algebras, replacing Banach spaces E by a Banach algebra A. It is significant to introduce the concept of cone metric spaces with Banach algebras since one proved that cone metric spaces with Banach algebras are not equivalent to metric spaces. This dose bring about prosperity in the study of cone metric spaces.

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In this paper, we prove common fixed point theorems of two weakly compatible contractive mappings in cone b-metric spaces with Banach algebras. As consequences, our results generalize, extend and unify several well-known comparable results.

2. Preliminaries

Let A be a Banach algebra, where θ denotes the null and e denotes the unit, suppose $x \in A$. If the spectra radius $r(x)$ of x is less than 1, i.e., $r(x) < 1$, then $e - x$ is invertible. Actually, $(e-x)^{-1} = \sum_{i=0}^{\infty} x^i$. A subset P of A is called a cone of A, if

(i) P is non-empty closed and $\{\theta, e\} \subset P$;

(ii) $\forall \alpha, \beta \in R$ and $\alpha, \beta \geq 0, \alpha P + \beta P \subset P$;

(iii) $P^2 = PP \subset P$;

(iv) $P \bigcap (-P) = \{\theta\}$,

for a given cone $P \subset A$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ stands for $x \preceq y$ and $x \neq y$, while $x \ll y$ stands for $y - x \in intP$, where $intP$ denotes the interior of P. If $intP \neq \emptyset$, then P is called a solid cone.

Definition 2.1. [4] Let X be a non-empty set and A be a Banach algebra. Suppose the mapping $d : X \times X \to A$, such that $\forall x, y, z \in X$ and $s \geq 1$ the following three conditions hold:

(1) $\theta \preceq d(x,y)$ and $d(x,y) = \theta \Leftrightarrow x = y$;

(2) $d(x,y) = d(y,x)$;

(3) $d(x,y) \preceq s(d(x,z) + d(z,y))$,

then (X,d) is called a cone b-metric space over a Banach algebra A.

Definition 2.2. [4] Let (X,d) be a cone b-metric space over a Banach algebra A, $x \in X$, $\{x_n\}$ be a sequence in X, then

(1) $\{x_n\}$ converges to x whenever for each $c \in A$ with $\theta \ll c$ there is natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x (n \to \infty)$;

(2) $\{x_n\}$ is a Cauchy sequence whenever for each $c \in A$ with $\theta \ll c$ there is natural number N such that $d(x_m, x_n) \ll c$, for all $m, n \geq N$;

(3) (X,d) is a complete cone metric space if every Cauchy sequence is convergent.

Definition 2.3. [8] Let P be a solid cone in a Banach space A. A sequence $\{x_n\} \subseteq P$ is a c-sequence if for each $\theta \ll c$ there exists $n_0 \in N$ such that $x_n \ll c$ for $n \geq n_0$.

Definition 2.4. [3] Let f and g be self maps of a set X. If $w = fx = gx$, for some $x \in X$, then x is called a coincidence point of f and g, w is called a point of coincidence of f and g.

Definition 2.5. [3] The mappings $f, g : X \to X$ be called weakly compatible if $x \in X$ and $fx = gx$, then$fgx = gfx$.

Proposition 2.6. [9] *Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is a unique common fixed point of f and g.*

Lemma 2.7. [6] *Let $x, y, z \in A$. If $x \leq y$ and $y \ll x$, then $x \ll z$.*

Lemma 2.8. [6] *Let P be a cone. If $\forall c \in intP$, which implies that $\theta \leq u \ll c$, then $u = \theta$.*

Lemma 2.9. [7] *Let P be a cone. If $\|x_n\| \to 0 (n \to \infty)$then for any $c \in A$ and $\theta \ll c$, there exists N, for any $n \geq N$, we have $x_n \ll c$.*

Lemma 2.10. [3] *Let (Xd) be a complete cone b-metric space with Banach algebra A, P be a solid cone, $\{x_n\}, \{y_n\} \subseteq P$, if $\{x_n\}, \{y_n\}$ are c-sequences and $\alpha, \beta > 0$, $k \in P$, then $\{\alpha x_n + \beta y_n\}, \{kx_n\}$ are c-sequences, if $\{x_n\}$ converges to $x \in X$, then $\{d(x_n, x\}$ is also a c-sequence.*

Lemma 2.11. [3] *Let A be a Banach algebra and let x, y, k be vectors in A. If x, y commute, then the following hold:*

(1) $r(sx) = sr(x)$;

(2) $r(xy) \leq r(x)r(y)$;

(3) $r((e-k)^{-1}) \leq (1-r(k))^{-1}$, where $0 \leq r(k) < 1$.

3. Main results

Theorem 3.1. *Let (X, d) be a complete cone b-metric space with Banach algebra A, where $s \geq 1$ and let P be a solid cone with $k \in P$, where $r(k) < \frac{1}{s}$. Suppose that the mappings $f, g : X \to X$ satisfy: $d(fx, fy) \preceq kd(gx, gy), \forall x, y \in X$, where $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X. Then f and g have a unique point of coincidence in X. If f and g are weakly compatible, then f and g have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point in X. Choose a point x_1 in X such that $f(x_0) = g(x_1)$. This can be done, since the $f(X) \subset g(X)$. Continuing this process, having chosen x_n in X, we obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$. It follows that

$$\begin{aligned} d(gx_{n+1}, gx_n) &= d(fx_n, fx_{n-1}) \\ &\preceq kd(gx_n, gx_{n-1}) \\ &\preceq k^2 d(gx_{n-1}, gx_{n-2}) \\ &\preceq \cdots \\ &\preceq k^n d(gx_1, gx_o). \end{aligned}$$

Then, for $m \geq 1, p \geq 1$, we have

$$\begin{aligned} d(gx_m, gx_{m+p}) &\preceq s(d(gx_m, gx_{m+1}) + d(gx_{m+1}, gx_{m+p})) \\ &= sd(gx_m, gx_{m+1}) + sd(gx_{m+1}, gx_{m+p}) \\ &\preceq sd(gx_m, gx_{m+1}) + s^2 d(gx_{m+1}, gx_{m+2}) + s^2 d(gx_{m+2}, gx_{m+p}) \\ &\vdots \\ &\preceq sd(gx_m, gx_{m+1}) + s^2 d(gx_{m+1}, gx_{m+2}) + \cdots + s^{p-1} d(gx_{m+p-1}, gx_{m+p}) \\ &\preceq sk^m d(gx_0, gx_1) + s^2 k^{m+1} d(gx_0, gx_1) + \cdots + s^{p-1} k^{m+p-1} d(gx_0, gx_1) \\ &= sk^m (e + sk + (sk)^2 + \cdots + (sk)^{p-1}) d(gx_0, gx_1) \\ &\preceq \sum_{i=0}^{\infty} (sk)^i sk^m d(gx_0, gx_1) \\ &= (e - sk)^{-1} sk^m d(gx_0, gx_1). \end{aligned}$$

Since $r(k) < \frac{1}{s} < 1$, we have $\|k^m\| \to 0 (m \to \infty)$. So

$$\|(e - sk)^{-1} sk^m d(gx_0, gx_1)\| \to 0, m \to \infty.$$

From Lemma 2.7 and Lemma 2.9, we get

$$d(gx_m, gx_{m+p}) \preceq (e - sk)^{-1} sk^m d(gx_0, gx_1) \ll c.$$

Hence gx_n is a Cauchy sequence.

Since $g(X)$ is complete, there $\exists q \in g(X)$ such that $gx_n \to q(n \to \infty)$. Consequently, we find $p \in X$ such that $g(p) = q$. Further

$$d(gx_n, fp) = d(fx_{n-1}, fp) \preceq kd(gx_{n-1}, gp).$$

From Definition 2.2 and Lemma 2.10, which implies that $kd(gx_{n-1}, gp) \ll c$. From Lemma 2.7, which gives $d(gx_n, fp) \ll c$, so $\lim_{n\to\infty} gx_n = fp$, thus $fp = gp$.

Now, we show that f and g have a unique point of coincidence. Assume that there exists another point $z \in X, fz = gz$, now

$$d(gz, gp) = d(fz, fp) \preceq kd(gz, gp),$$

which gives $(e-k)d(gz, gp) \preceq \theta$, which implies that $d(gz, gp) \preceq \theta$, so, $d(gz, gp) = \theta$, then $gz = gpf$ and g have a unique point of coincidence. From Proposition 2.6, f and g have a unique common fixed point.

Theorem 3.2. *Let (Xd) be a complete cone b-metric space with a Banach algebra A, where $s \geq 1$ and let P be a solid cone with $k \in P$, where $r(k) < \frac{1}{2s}$. Suppose that the mappings $f, g : X \to X$ satisfy: $d(fx, fy) \preceq k(d(fx, gx) + d(fy, gy)), \forall x, y \in X$, where $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. If f and g are weakly compatible, then f and g have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point in X. Choose a point x_1 in X such that $f(x_0) = g(x_1)$. This can be done, since the $f(X) \subset g(X)$. Continuing this process, having chosen x_n in X, we obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$. Then

$$\begin{aligned} d(gx_{n+1}, gx_n) &= d(fx_n, fx_{n-1}) \\ &\preceq k(d(fx_n, gx_n) + d(fx_{n-1}, gx_{n-1}) \\ &= kd(gx_{n+1}, gx_n) + kd(gx_n, gx_{n-1}). \end{aligned}$$

Thus, $(e-k)d(gx_{n+1}, gx_n) \preceq kd(gx_n, gx_{n-1})$, which gives

$$d(gx_{n+1}, gx_n) \preceq k(e-k)^{-1} d(gx_n, gx_{n-1}).$$

Now, we turn to a proof that $r(k(e-k)^{-1}) < 1$. In fact,

$$r(k(e-k)^{-1}) \leq r(k)r((e-k)^{-1}) \leq \frac{r(k)}{1-r(k)} < 1.$$

Using Theorem 3.1, we get $\{gx_n\}$ is a Cauchy sequence. By the same argument to the proof in Theorem 3.1, we have f and g have a unique common fixed point. This completes the proof.

Theorem 3.3. *Let (Xd) be a complete cone b-metric space with Banach algebra A, where $s \geq 1$ and let P be a solid cone with $k \in P$, where $r(k) < \frac{1}{2s^2}$. Suppose the mappings $f, g : X \to X$ satisfy: $d(fx, fy) \leq k(d(fx, gy) + d(fy, gx)), \forall x, y \in X$, where $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. If f and g are weakly compatible, then f and g have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point in X. Choose a point x_1 in X such that $f(x_0) = g(x_1)$. This can be done, since the $f(X) \subset g(X)$. Continuing this process, having chosen x_n in X, we obtain x_{n+1} in X such that $f(x_n) = g(x_{n+1})$. Then

$$\begin{aligned}
d(gx_{n+1}, gx_n) &= d(fx_n, fx_{n-1}) \\
&\preceq k(d(fx_n, gx_{n-1}) + d(fx_{n-1}, gx_n) \\
&= kd(gx_{n+1}, gx_{n-1}) + kd(gx_n, gx_n) = kd(gx_{n+1}, gx_{n-1}) \\
&\preceq k(sd(gx_{n+1}, gx_n) + sd(gx_{n+1}, gx_n)).
\end{aligned}$$

Thus, $(e-ks)d(gx_{n+1}, gx_n) \preceq ksd(gx_n, gx_{n-1})$, which in turn gives

$$d(gx_{n+1}, gx_n) \preceq ks(e-ks)^{-1}d(gx_n, gx_{n-1}).$$

Now, we turn to a proof that $r(ks(e-ks)^{-1}) < 1$. In fact,

$$r(ks(e-ks)^{-1}) \leq r(ks)r((e-ks)^{-1}) \leq \frac{r(ks)}{1-r(ks)} < 1,$$

from Theorem 3.1, we get that $\{gx_n\}$ is a Cauchy sequence. By the same argument to the proof in Theorem 3.1, we have f and g have a unique common fixed point. This competes the proof.

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