



TWO-DIMENSIONAL POMPEIU'S MEAN VALUE THEOREMS AND RELATED RESULTS

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Abstract. In this work, a mean value theorem of Pompeiu's type for functions of two variables is presented. Other related results are given as well.

Keywords. Mean-Value Theorem, Pompeiu's MVT, Cauchy MVT.

1. Introduction

The mean-value theorem (MVT) or Lagrange MVT is considered as one of the most useful and fundamental result in analysis, named after the French mathematician Joseph Lagrange, where he presented his mean value theorem in his book *Theorie des fonctions analytiques* in 1797; he states that: "If f is continuous on $[a, b]$, differentiable on (a, b) then there is a point c , $a < c < b$, such that

$$(1.1) \quad f'(c) = \frac{f(b) - f(a)}{b - a}."$$

We call such a point c a mean-value point of f . Further note that the mean-value theorem of differentiation can be used to prove that: if $f' > 0$ and every sub-interval contains a point at which $f' > 0$, in particular if $f' > 0$ with $f'(x) = 0$ at only a finite number of points, then f is strictly increasing.

In recent years several authors modified and generalized various types of mean value theorems in different ways and interesting approach. For recent works the reader may refer for example to [1], [6], [7], for general reading the interested researcher may find a hundreds of works cited in the book [5].

In 1946, Pompeiu established another MVT for real functions defined on a real interval that not containing '0'; nowadays known as Pompeiu's mean value theorem, which states [4](see also, p., 83; [5]):

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Theorem 1. *For every real valued function f differentiable on an interval $[a, b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a, b]$ there exists a point $\xi \in (x_1, x_2)$ such that*

$$(1.2) \quad \frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi).$$

The geometrical interpretation of this theorem as given in [5]: the tangent at the point $(\xi, f(\xi))$ intersects on the y -axis at the same point as the secant line connecting the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

In 1947, Boggio [2] (see also, p.,92; [5]) established the following generalization of Pompeiu's mean value theorem 1:

Theorem 2. *For every real valued functions f and g differentiable on an interval $[a, b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a, b]$ there exists a point $\xi \in (x_1, x_2)$ such that*

$$(1.3) \quad \frac{g(x_1) f(x_2) - g(x_2) f(x_1)}{g(x_2) - g(x_1)} = f(\xi) - \frac{g(\xi)}{g'(\xi)} f'(\xi).$$

In their famous book [5], Sahoo and Riedel have discussed various type of mean value theorems for functions of one or more variables. Among others, they stated in the end of Chapter four (p., 145; [5]) that: "We have not been able to generalize Pompeiu's mean value theorem for functions in two variables..".

The aim of this work is to answer Sahoo and Riedel problem about the characterization of Pompeiu's MVT for functions of two variables. Namely, for functions of two variables; we prove three mean value theorems; the Cauchy-, Pompeiu's-, and Cauchy-Pompeiu's-mean value theorems.

Two useful results concerning Rectangular MVT for functions of two variables have been recently obtained in (p., 93; [3]):

Theorem 3. *(Rectangular Rolle's Theorem) Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, and let $f : \Delta := [a, b] \times [c, d]$ satisfy the following:*

- (1) *For each fixed $y_0 \in [c, d]$, the function given by $x \mapsto f(x, y_0)$ is continuous on $[a, b]$ and differentiable on (a, b) .*
- (2) *For each fixed $x_0 \in [a, b]$, the function given by $y \mapsto f(x_0, y)$ is continuous on $[c, d]$ and differentiable on (c, d) .*
- (3) $f(a, c) + f(b, d) = f(a, d) + f(b, c)$.

Then, there exists $(x_0, y_0) \in (a, b) \times (c, d)$ such that $f_{xy}(x_0, y_0) = 0$.

Theorem 4. *(Rectangular Mean Value Theorem) Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, and let $f : \Delta := [a, b] \times [c, d]$ satisfy the following:*

- (1) For each fixed $y_0 \in [c, d]$, the function given by $x \mapsto f(x, y_0)$ is continuous on $[a, b]$ and differentiable on (a, b) .
- (2) For each fixed $x_0 \in [a, b]$, the function given by $y \mapsto f(x_0, y)$ is continuous on $[c, d]$ and differentiable on (c, d) .

Then, there exists $(x_0, y_0) \in (a, b) \times (c, d)$ such that

$$f(b, d) - f(b, c) - f(a, d) + f(a, c) = (b - a)(d - c)f_{xy}(x_0, y_0).$$

2. The results

We begin with the following generalization of Theorem 3 and Theorem 4:

Theorem 5. (Rectangular Cauchy Mean Value Theorem) Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, and let $f, g : \Delta := [a, b] \times [c, d]$ satisfy the following:

- (1) For each fixed $y_0 \in [c, d]$, the functions given by $x \mapsto f(x, y_0)$ and $x \mapsto g(x, y_0)$ are continuous on $[a, b]$ and differentiable on (a, b) .
- (2) For each fixed $x_0 \in [a, b]$, the functions given by $y \mapsto f(x_0, y)$ and $y \mapsto g(x_0, y)$ are continuous on $[c, d]$ and differentiable on (c, d) .

Then, there exists $(x_0, y_0) \in (a, b) \times (c, d)$ such that

$$(2.1) \quad \frac{f_{xy}(x_0, y_0)}{g_{xy}(x_0, y_0)} = \frac{f(b, d) - f(b, c) - f(a, d) + f(a, c)}{g(b, d) - g(b, c) - g(a, d) + g(a, c)}.$$

Proof. Define the function $H : \Delta \rightarrow \mathbb{R}$, given by

$$\begin{aligned} H(x, y) &= [f(b, d) - f(b, c) - f(a, d) + f(a, c)]g(x, y) \\ &\quad - [g(b, d) - g(b, c) - g(a, d) + g(a, c)]f(x, y). \end{aligned}$$

It is easy to see that H is continuous and differentiable on D , and

$$\begin{aligned} H(a, c) &= [f(b, d) - f(b, c) - f(a, d) + f(a, c)]g(a, c) \\ &\quad - [g(b, d) - g(b, c) - g(a, d) + g(a, c)]f(a, c), \end{aligned}$$

$$\begin{aligned} H(a, d) &= [f(b, d) - f(b, c) - f(a, d) + f(a, c)]g(a, d) \\ &\quad - [g(b, d) - g(b, c) - g(a, d) + g(a, c)]f(a, d), \end{aligned}$$

$$\begin{aligned} H(b, c) &= [f(b, d) - f(b, c) - f(a, d) + f(a, c)]g(b, c) \\ &\quad - [g(b, d) - g(b, c) - g(a, d) + g(a, c)]f(b, c), \end{aligned}$$

and

$$H(b, d) = [f(b, d) - f(b, c) - f(a, d) + f(a, c)]g(b, d) \\ - [g(b, d) - g(b, c) - g(a, d) + g(a, c)]f(b, d),$$

then we have

$$H(a, c) - H(a, d) - H(b, c) + H(b, d) \\ = [f(b, d) - f(b, c) - f(a, d) + f(a, c)][g(a, c) - g(a, d) - g(b, c) + g(b, d)] \\ - [g(a, c) - g(a, d) - g(b, c) + g(b, d)][f(b, d) - f(b, c) - f(a, d) + f(a, c)] \\ = 0,$$

which gives that $H(a, d) + H(b, c) = H(a, c) + H(b, d)$. So by the Rectangular Rolle's Theorem 3, there is $(x_0, y_0) \in (a, b) \times (c, d)$ such that $H_{xy}(x_0, y_0) = 0$, therefore

$$H_{xy}(x_0, y_0) = [f(b, d) - f(b, c) - f(a, d) + f(a, c)]g_{xy}(x_0, y_0) \\ - [g(b, d) - g(b, c) - g(a, d) + g(a, c)]f_{xy}(x_0, y_0) = 0,$$

which gives that

$$\frac{f_{xy}(x_0, y_0)}{g_{xy}(x_0, y_0)} = \frac{f(b, d) - f(b, c) - f(a, d) + f(a, c)}{g(b, d) - g(b, c) - g(a, d) + g(a, c)}.$$

This yields the desired result. □

Remark 1. In Theorem 5, if one chooses, $g(t, s) = ts$, then we recapture the rectangular Mean-Value Theorem 4.

Our first main result concerning Pompeiu's Mean Value Theorem for functions of two variables may be considered as follows:

Theorem 6. (*Pompeiu's Mean Value Theorem*) Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, and let $f : \Delta := [a, b] \times [c, d] \rightarrow \mathbb{R}$ satisfy the following:

- (1) Δ not containing the points $(0, \cdot), (\cdot, 0), (0, 0)$.
- (2) For each fixed $y_0 \in [c, d]$, the function given by $x \mapsto f(x, y_0)$ is continuous on $[a, b]$ and differentiable on (a, b) .
- (3) For each fixed $x_0 \in [a, b]$, the function given by $y \mapsto f(x_0, y)$ is continuous on $[c, d]$ and differentiable on (c, d) .
- (4) For all pair $x_1, x_2 \in (a, b)$ with $x_1 \neq x_2$ and $y_1, y_2 \in (c, d)$ with $y_1 \neq y_2$.

Then, there exists $(\xi_1, \xi_2) \in (x_1, y_1) \times (x_2, y_2)$ such that

$$(2.2) \quad \xi_1 \xi_2 \frac{\partial^2 f}{\partial t \partial s}(\xi_1, \xi_2) - \xi_1 \frac{\partial f}{\partial t}(\xi_1, \xi_2) - \xi_2 \frac{\partial f}{\partial s}(\xi_1, \xi_2) + f(\xi_1, \xi_2) \\ = \frac{x_2 y_2 f(x_1, y_1) - x_2 y_1 f(x_1, y_2) - x_1 y_2 f(x_2, y_1) + x_1 y_1 f(x_2, y_2)}{(x_2 - x_1)(y_2 - y_1)}.$$

Proof. Define a real valued function $F : [\frac{1}{b}, \frac{1}{a}] \times [\frac{1}{d}, \frac{1}{c}] \rightarrow \mathbb{R}$, given by

$$(2.3) \quad F(t, s) = tsf\left(\frac{1}{t}, \frac{1}{s}\right).$$

By the assumptions (1)-(3), its easy to see that

- (1) F is defined on $[\frac{1}{b}, \frac{1}{a}] \times [\frac{1}{d}, \frac{1}{c}]$, since Δ does not containing the points $(0, \cdot), (\cdot, 0), (0, 0)$.
- (2) For each fixed $y_0 \in [\frac{1}{d}, \frac{1}{c}]$, the function given by $x \mapsto F(x, y_0)$ is continuous on $[\frac{1}{b}, \frac{1}{a}]$ and differentiable on $(\frac{1}{b}, \frac{1}{a})$.
- (3) For each fixed $x_0 \in [\frac{1}{b}, \frac{1}{a}]$, the function given by $y \mapsto F_x(x_0, y)$ is continuous on $[\frac{1}{d}, \frac{1}{c}]$ and differentiable on $(\frac{1}{d}, \frac{1}{c})$.

Therefore, simple calculations yield that

$$F_t(t, s) = sf\left(\frac{1}{t}, \frac{1}{s}\right) - \frac{s}{t} f_t\left(\frac{1}{t}, \frac{1}{s}\right), \\ F_s(t, s) = tf\left(\frac{1}{t}, \frac{1}{s}\right) - \frac{t}{s} f_s\left(\frac{1}{t}, \frac{1}{s}\right),$$

and

$$F_{ts}(t, s) = \frac{1}{ts} f_{st}\left(\frac{1}{t}, \frac{1}{s}\right) - \frac{1}{t} f_t\left(\frac{1}{t}, \frac{1}{s}\right) - \frac{1}{s} f_s\left(\frac{1}{t}, \frac{1}{s}\right) + f\left(\frac{1}{t}, \frac{1}{s}\right) \\ (2.4) \quad = F_{st}(t, s).$$

Applying the Rectangular Mean Value Theorem 4 to F on the interval $[u, v] \times [z, w] \subset [\frac{1}{b}, \frac{1}{a}] \times [\frac{1}{d}, \frac{1}{c}]$, we get

$$(2.5) \quad (v - u)(w - z) F_{ts}(\eta_1, \eta_2) = F(v, w) - F(v, z) - F(u, w) + F(u, z)$$

for some $(\eta_1, \eta_2) \in (u, v) \times (z, w)$. Let $x_1 = \frac{1}{v}$, $x_2 = \frac{1}{u}$, $y_1 = \frac{1}{w}$, $y_2 = \frac{1}{z}$, $\xi_1 = \frac{1}{\eta_1}$, and $\xi_2 = \frac{1}{\eta_2}$. Then, since $(\eta_1, \eta_2) \in (u, v) \times (z, w)$, we have

$$x_1 \leq \xi_1 \leq x_2, \quad \text{and} \quad y_1 \leq \xi_2 \leq y_2.$$

Now, using (2.3) and (2.4) on (2.5), we have

$$\frac{1}{\eta_1 \eta_2} f_{st}\left(\frac{1}{\eta_1}, \frac{1}{\eta_2}\right) - \frac{1}{\eta_1} f_t\left(\frac{1}{\eta_1}, \frac{1}{\eta_2}\right) - \frac{1}{\eta_2} f_s\left(\frac{1}{\eta_1}, \frac{1}{\eta_2}\right) + f\left(\frac{1}{\eta_1}, \frac{1}{\eta_2}\right) \\ = \frac{1}{(v - u)(w - z)} \left[vw f\left(\frac{1}{v}, \frac{1}{w}\right) - vz f\left(\frac{1}{v}, \frac{1}{z}\right) - uw f\left(\frac{1}{u}, \frac{1}{w}\right) + uz f\left(\frac{1}{u}, \frac{1}{z}\right) \right],$$

that is,

$$\begin{aligned} \xi_1 \xi_2 \frac{\partial^2 f}{\partial t \partial s}(\xi_1, \xi_2) - \xi_1 \frac{\partial f}{\partial t}(\xi_1, \xi_2) - \xi_2 \frac{\partial f}{\partial s}(\xi_1, \xi_2) + f(\xi_1, \xi_2) \\ = \frac{x_2 y_2 f(x_1, y_1) - x_2 y_1 f(x_1, y_2) - x_1 y_2 f(x_2, y_1) + x_1 y_1 f(x_2, y_2)}{(x_2 - x_1)(y_2 - y_1)}. \end{aligned}$$

This completes the proof of the theorem. \square

Next, a characterization of Boggio MVT which is of Cauchy's type for functions of two variables is stated as follows:

Theorem 7. (*Boggio Mean Value Theorem*) *Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, and let $f, g : \Delta := [a, b] \times [c, d] \rightarrow \mathbb{R}$ satisfy the following:*

- (1) Δ not containing the points $(0, \cdot), (\cdot, 0), (0, 0)$.
- (2) For each fixed $y_0 \in [c, d]$, the functions given by $x \mapsto f(x, y_0)$ and $x \mapsto g(x, y_0)$ are continuous on $[a, b]$ and differentiable on (a, b) .
- (3) For each fixed $x_0 \in [a, b]$, the functions given by $y \mapsto f(x_0, y)$ and $y \mapsto g(x_0, y)$ are continuous on $[c, d]$ and differentiable on (c, d) .
- (4) For all pair $x_1, x_2 \in (a, b)$ with $x_1 \neq x_2$ and $y_1, y_2 \in (c, d)$ with $y_1 \neq y_2$.

Then, there exists $(\xi_1, \xi_2) \in (x_1, y_1) \times (x_2, y_2)$ such that

$$\begin{aligned} (2.6) \quad & \frac{\xi_1 \xi_2 \frac{\partial^2 g}{\partial t \partial s}(\xi_1, \xi_2) - \xi_1 \frac{\partial g}{\partial t}(\xi_1, \xi_2) - \xi_2 \frac{\partial g}{\partial s}(\xi_1, \xi_2) + g(\xi_1, \xi_2)}{g(x_2, y_2) - g(x_2, y_1) - g(x_1, y_2) + g(x_1, y_1)} \\ & - \frac{\xi_1 \xi_2 \frac{\partial^2 f}{\partial t \partial s}(\xi_1, \xi_2) - \xi_1 \frac{\partial f}{\partial t}(\xi_1, \xi_2) - \xi_2 \frac{\partial f}{\partial s}(\xi_1, \xi_2) + f(\xi_1, \xi_2)}{f(x_2, y_2) - f(x_2, y_1) - f(x_1, y_2) + f(x_1, y_1)} \\ & = \frac{x_2 y_2 g(x_1, y_1) - x_1 y_2 g(x_2, y_1) - x_2 y_1 g(x_1, y_2) + x_1 y_1 g(x_2, y_2)}{(x_2 - x_1)(y_2 - y_1)[g(x_2, y_2) - g(x_2, y_1) - g(x_1, y_2) + g(x_1, y_1)]} \\ & - \frac{x_2 y_2 f(x_1, y_1) - x_2 y_1 f(x_1, y_2) - x_1 y_2 f(x_2, y_1) + x_1 y_1 f(x_2, y_2)}{(x_2 - x_1)(y_2 - y_1)[f(x_2, y_2) - f(x_2, y_1) - f(x_1, y_2) + f(x_1, y_1)]} \end{aligned}$$

Proof. As in the proof of Theorem 6, let $[u, v] \times [z, w] \subset [\frac{1}{b}, \frac{1}{a}] \times [\frac{1}{d}, \frac{1}{c}]$, and setting $x_1 = \frac{1}{v}$, $x_2 = \frac{1}{u}$, $y_1 = \frac{1}{w}$, $y_2 = \frac{1}{z}$. Define the following two real valued functions $H : \Delta \rightarrow \mathbb{R}$, given by

$$(2.7) \quad \begin{aligned} H(t, s) = [f(x_1, y_1) - f(x_1, y_2) - f(x_2, y_1) + f(x_2, y_2)]g(t, s) \\ - [g(x_1, y_1) - g(x_1, y_2) - g(x_2, y_1) + g(x_2, y_2)]f(t, s), \end{aligned}$$

and $F : [\frac{1}{b}, \frac{1}{a}] \times [\frac{1}{d}, \frac{1}{c}] \rightarrow \mathbb{R}$, given by

$$(2.8) \quad F(t, s) = tsH\left(\frac{1}{t}, \frac{1}{s}\right)$$

By the assumptions (1)-(3), it is easy to see that

- (1) F is defined on $[\frac{1}{b}, \frac{1}{a}] \times [\frac{1}{d}, \frac{1}{c}]$, since Δ does not containing the points $(0, \cdot), (\cdot, 0), (0, 0)$.

- (2) For each fixed $y_0 \in [\frac{1}{d}, \frac{1}{c}]$, the function given by $x \mapsto F(x, y_0)$ is continuous on $[\frac{1}{b}, \frac{1}{a}]$ and differentiable on $(\frac{1}{b}, \frac{1}{a})$.
- (3) For each fixed $x_0 \in [\frac{1}{b}, \frac{1}{a}]$, the function given by $y \mapsto F_x(x_0, y)$ is continuous on $[\frac{1}{d}, \frac{1}{c}]$ and differentiable on $(\frac{1}{d}, \frac{1}{c})$.

Therefore, simple calculations yield that

$$(2.9) \quad \begin{aligned} F_{ts}(t, s) &= \frac{1}{ts} H_{st} \left(\frac{1}{t}, \frac{1}{s} \right) - \frac{1}{t} H_t \left(\frac{1}{t}, \frac{1}{s} \right) - \frac{1}{s} H_s \left(\frac{1}{t}, \frac{1}{s} \right) + H \left(\frac{1}{t}, \frac{1}{s} \right) \\ &= F_{st}(t, s). \end{aligned}$$

Applying the Rectangular Mean Value Theorem 4 to F on the interval $[u, v] \times [z, w] \subset [\frac{1}{b}, \frac{1}{a}] \times [\frac{1}{d}, \frac{1}{c}]$, we get

$$(2.10) \quad (v - u)(w - z) F_{ts}(\eta_1, \eta_2) = F(v, w) - F(v, z) - F(u, w) + F(u, z).$$

for some $(\eta_1, \eta_2) \in (u, v) \times (z, w)$.

Using the assumption that $x_1 = \frac{1}{v}$, $x_2 = \frac{1}{u}$, $y_1 = \frac{1}{w}$, $y_2 = \frac{1}{z}$, $\xi_1 = \frac{1}{\eta_1}$, and $\xi_2 = \frac{1}{\eta_2}$. Then, since $(\eta_1, \eta_2) \in (u, v) \times (z, w)$, we have

$$x_1 \leq \xi_1 \leq x_2, \quad \text{and} \quad y_1 \leq \xi_2 \leq y_2.$$

Now, using (2.7)–(2.9) on (2.10), we have

$$\begin{aligned} & \frac{\xi_1 \xi_2 \frac{\partial^2 g}{\partial t \partial s}(\xi_1, \xi_2) - \xi_1 \frac{\partial g}{\partial t}(\xi_1, \xi_2) - \xi_2 \frac{\partial g}{\partial s}(\xi_1, \xi_2) + g(\xi_1, \xi_2)}{g(x_2, y_2) - g(x_2, y_1) - g(x_1, y_2) + g(x_1, y_1)} \\ & \quad - \frac{\xi_1 \xi_2 \frac{\partial^2 f}{\partial t \partial s}(\xi_1, \xi_2) - \xi_1 \frac{\partial f}{\partial t}(\xi_1, \xi_2) - \xi_2 \frac{\partial f}{\partial s}(\xi_1, \xi_2) + f(\xi_1, \xi_2)}{f(x_2, y_2) - f(x_2, y_1) - f(x_1, y_2) + f(x_1, y_1)} \\ & = \frac{x_2 y_2 g(x_1, y_1) - x_1 y_2 g(x_2, y_1) - x_2 y_1 g(x_1, y_2) + x_1 y_1 g(x_2, y_2)}{(x_2 - x_1)(y_2 - y_1)[g(x_2, y_2) - g(x_2, y_1) - g(x_1, y_2) + g(x_1, y_1)]} \\ & \quad - \frac{x_2 y_2 f(x_1, y_1) - x_2 y_1 f(x_1, y_2) - x_1 y_2 f(x_2, y_1) + x_1 y_1 f(x_2, y_2)}{(x_2 - x_1)(y_2 - y_1)[f(x_2, y_2) - f(x_2, y_1) - f(x_1, y_2) + f(x_1, y_1)]}. \end{aligned}$$

This completes the proof of the theorem. \square

Remark 2. In (2.6), if one chooses $g(t, s) = ts$, then we recapture the Pompeiu's Mean Value Theorem 6.

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