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EXISTENCE AND MULTIPLICITY OF FAST HOMOCLINIC SOLUTIONS FOR A CLASS OF DAMPED VIBRATION PROBLEMS

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Abstract. In this paper, we investigate the existence and multiplicity of homoclinic solutions for the following damped vibration problems

$$(\mathscr{DV}) \qquad \ddot{u}(t)+q(t)\dot{u}(t)-L(t)u(t)+\nabla W(t,u(t))=0,$$

where $q:\mathscr{R}\longrightarrow\mathscr{R}$ is a continuous function, $L(t)\in C(\mathscr{R},\mathscr{R}^{n^2})$ is a symmetric matrix and $W(t,x)\in C^1(\mathscr{R}\times\mathscr{R}^n,\mathscr{R})$ are neither autonomous nor periodic in t. The novelty of this paper is that, supposing that $\lim_{|t|\longrightarrow\infty}Q(t)=+\infty$ ($Q(t)=\int_0^t q(s)ds$) and $L(t)$ is coercive unnecessary uniformly positively definite for all $t\in\mathscr{R}$, we establish one new compact embedding theorem. Subsequently, assuming $W(t,x)$ satisfies the super-quadratic condition $\frac{W(t,x)}{|x|^2}\longrightarrow+\infty$ as $|x|\longrightarrow\infty$ uniformly in t and need not satisfy the global Ambrosetti-Rabinowitz condition, we obtain some new criterion to guarantee the existence and multiplicity of nontrivial homoclinic solutions for $(\mathscr{DV})$ using the Minimax Methods in critical point theory.

Keywords. Damped vibration problems; Fast homoclinic solutions; Compact embedding; Minimax methods.

1. Introduction

Consider the following damped vibration problems

$$(\mathscr{DV}) \qquad \ddot{u}(t)+q(t)\dot{u}(t)-L(t)u(t)+\nabla W(t,u(t))=0,$$

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where $L(t) \in C(\mathscr{R},\mathscr{R}^{n^2})$ is a symmetric matrix, $W(t,x) \in C^1(\mathscr{R}\times\mathscr{R}^n,\mathscr{R})$ and $q:\mathscr{R}\longrightarrow\mathscr{R}$ is a continuous function satisfying

$$\lim_{|t|\longrightarrow\infty} Q(t) = +\infty, \tag{1.1}$$

where $Q(t) = \int_0^t q(s)ds$. When $q(t) = 0$, $(\mathscr{DV})$ is just the following second-order Hamiltonian system

$$\ddot{u}(t) - L(t)u(t) + \nabla W(t,u(t)) = 0. \tag{$\mathscr{HS}$}$$

It is well known that homoclinic orbits for dynamical systems are important in applications for a number of reasons. They may "organizing centers" for the dynamics in their neighborhood. From their existence one may, under certain conditions, infer the existence of chaos nearly or the bifurcation behavior of periodic orbits. In the past two decades many authors study homoclinic orbits for Hamiltonian systems via critical point theory and variational methods. Assuming that $L(t)$ and $W(t,x)$ are independent of t or $T-$periodic in t, many authors have studied the existence of homoclinic solutions for the Hamiltonian system $(\mathscr{HS})$, see e.g. [1-7] and the references therein. In this case, the existence of homoclinic solutions can be obtained by going to the limit of periodic solutions of approximating problems. If $L(t)$ and $W(t,x)$ are neither autonomous nor periodic in t, the existence of homoclinic solutions of $(\mathscr{HS})$ is quite different from the ones just described because of the lack of compactness of the Sobolev embedding, see e.g. [4-11] and the references therein. In the works mentioned above, W satisfies the global Ambrosetti-Rabinowitz condition, that is, there exists $\mu > 2$ such that

$$0 < \mu W(t,x) \leq \nabla W(t,x).x \tag{AR}$$

for all $t \in \mathscr{R}$ and $x \in \mathscr{R}^n - \{0\}$. However there are lots of potentials which are super-quadratic as $|x| \longrightarrow \infty$ but do not satisfy the (AR) condition. Therefore, many authors have been focusing their attention on obtaining the existence of homoclinic solutions under conditions different than the (AR) condition see e.g. [12-17] and the references therein.

As far as the case $q(t) \neq 0$ is concerned, to our best knowledge, there are few researchs about the existence of fast homoclinic orbits of $(\mathscr{DV})$. Recently the existence of fast homoclinic orbits for problem $(\mathscr{DV})$ have been studied in [9,18,19] with W being sub-quadratic at infinity and in [20] with W is of super-quadratic growth at infinity satisfying the global (AR) condition. In the present paper, we use some versions of the Mountain Pass Theorem to establish some new criterion to guarantee the existence and multiplicity of homoclinic solutions for $(\mathscr{DV})$ under the condition that $W(t,x)$ satisfies the following super-quadratic condition

$$\frac{W(t,x)}{|x|^2} \longrightarrow +\infty \; as \; |x| \longrightarrow \infty \; uniformly \; in \; t \in \mathscr{R}$$

and need not satisfy the (AR) condition.

2. Preliminaries

In the following, in order to introduce the concept of the fast homoclinic solutions of $(\mathscr{DV})$ conveniently, we firstly describe some properties of the weighted Sobolev space E.
Assume that Q satisfies (1.1) and we make the following assumption:
(L) there exists $\theta \in C(\mathscr{R},]0,\infty[)$ such that $\theta(t) \longrightarrow +\infty$ as $|t| \longrightarrow \infty$ and

$$L(t)x.x \geq \theta(t)\,|x|^2,\; \forall (t,x) \in \mathscr{R} \times \mathscr{R}^n. \tag{2.1}$$

Here $x.y$ denotes the standard inner product of $x, y \in \mathscr{R}^n$ and $|.|$ is the associated norm. Denote by A the self-adjoint extension of the operator $-\frac{d^2}{dt^2} + L(t)$ with the domain $\mathscr{D}(A) \subset L^2_Q(\mathscr{R},\mathscr{R}^n)$. Let $E = \mathscr{D}(|A|^{\frac{1}{2}})$, and define on E the inner product and norm

$$< u,v > = < |A|^{\frac{1}{2}} u, |A|^{\frac{1}{2}} v >_{L^2_Q(\mathscr{R})} + < u,v >_{L^2_Q(\mathscr{R})},$$

$$\|u\| = < u,u >^{\frac{1}{2}},$$

where $L^2_Q(\mathscr{R})$ denotes the Hilbert space of functions on $\mathscr{R}$ with values in $\mathscr{R}^n$ satisfying

$$\int_{\mathscr{R}} e^{Q(t)}\,|u(t)|^2\,dt < \infty,$$

under the inner product and norm

$$< u,v >_{L^2_Q(\mathscr{R})} = \int_{\mathscr{R}} e^{Q(t)} u(t).v(t)dt,$$

$$\|u\|_{L^2_Q(\mathscr{R})} = (\int_{\mathscr{R}} e^{Q(t)}\,|u(t)|^2\,dt)^{\frac{1}{2}}.$$

Similarly, $L^p_Q(\mathscr{R})$ $(1 \leq p < \infty)$ denotes the Banach space of functions on $\mathscr{R}$ with values in $\mathscr{R}^n$ under the norm

$$\|u\|_{L^p_Q(\mathscr{R})} = (\int_{\mathscr{R}} e^{Q(t)} |u(t)|^p \, dt)^{\frac{1}{p}}$$

and $L^\infty_Q(\mathscr{R})$ denotes the Banach space of functions on $\mathscr{R}$ with values in $\mathscr{R}^n$ under the norm

$$\|u\|_{L^\infty_Q(\mathscr{R})} = ess\ sup_{t\in\mathscr{R}} e^{\frac{Q(t)}{2}} |u(t)|.$$

Definition 2.1. If (1.1) holds, a solution u of $(\mathscr{D}\mathscr{V})$ is called a fast homoclinic solution if $u \in E$.

Lemma 2.1. [13] *Assume (L) holds. Then for $u \in H^1(\mathscr{R}, \mathscr{R}^n)$ such that $\int_{\mathscr{R}} [|\dot{u}(t)|^2 + L(t)u(t).u(t)]dt < \infty$, we have*

$$\|u\|_{L^\infty(\mathscr{R})} \leq \frac{1}{\sqrt{2\sqrt{\theta_0}}} \int_{\mathscr{R}} [|\dot{u}(t)|^2 + L(t)u(t).u(t)]dt,$$

$$|u(t)| \leq [\int_t^\infty \frac{1}{\sqrt{\theta(s)}} (|\dot{u}(s)|^2 + L(s)u(s).u(s))ds]^{\frac{1}{2}},\ t \in \mathscr{R},$$

and

$$|u(t)| \leq [\int_{-\infty}^t \frac{1}{\sqrt{\theta(s)}} (|\dot{u}(s)|^2 + L(s)u(s).u(s))ds]^{\frac{1}{2}},\ t \in \mathscr{R},$$

where $\theta_0 = \inf_{t\in\mathscr{R}} \theta(t)$.

Now, consider the following assumption:

$$(Q_\gamma) \qquad \begin{cases} q \text{ is bounded in } \mathscr{R},\ \lim_{|t|\longrightarrow\infty} Q(t) = +\infty, \\ Q \text{ is nondecreasing in } \mathscr{R}^+ \text{ and nonincreasing in } \mathscr{R}^-, \\ \text{there exists a constant } \gamma < 0 \text{ such that} \\ \int_{|t|\geq 1} e^{Q(t)} |t|^{-1+\gamma} dt < \infty. \end{cases}$$

Example 2.1. *Let $0 < s < -\gamma$ and set*

$$q(t) = \begin{cases} \frac{s|t|^{s-2}t}{1+|t|^s},\ if\ |t| \geq 1, \\ \frac{st}{2},\ if\ |t| \leq 1, \end{cases}$$

and $Q(t) = \int_0^t q(u)du$. It is clear that q is continuous and we have

$$Q(t) = \begin{cases} \frac{s}{4} + ln(1+|t|^s) - ln2,\ if\ |t| \geq 1, \\ \frac{st^2}{4},\ if\ |t| \leq 1. \end{cases}$$

It is easy to verify that Q is nondecreasing in $\mathscr{R}^+$, nonincreasing in $\mathscr{R}^-$ and $\lim_{|t|\longrightarrow\infty} Q(t) = +\infty$. *Moreover, since $1-\gamma > 1$ and $1-\gamma-s > 1$, we get*

$$\int_{|t|\geq 1} e^{Q(t)} |t|^{-1+\gamma} dt = \frac{1}{2} e^{\frac{s}{4}} \int_{|t|\geq 1} (1+|t|^s) |t|^{-1+\gamma} dt < +\infty.$$

It is easy to verify

Corollary 2.1. *Assume (Q_γ) and (L) hold. Then, for every $u \in E$, we have*

$$\|u\|_{L^\infty_Q(\mathscr{R})} \leq \frac{1}{\sqrt{2\sqrt{\theta_0}}} \int_{\mathscr{R}} e^{Q(t)} [|\dot{u}(t)|^2 + L(t)u(t).u(t)] dt,$$

$$e^{\frac{Q(t)}{2}} |u(t)| \leq [\int_t^\infty \frac{e^{Q(s)}}{\sqrt{\theta(s)}} (|\dot{u}(s)|^2 + L(s)u(s).u(s)) ds]^{\frac{1}{2}},\ t \in \mathscr{R}^+,$$

and

$$e^{\frac{Q(t)}{2}} |u(t)| \leq [\int_{-\infty}^t \frac{e^{Q(s)}}{\sqrt{\theta(s)}} (|\dot{u}(s)|^2 + L(s)u(s).u(s)) ds]^{\frac{1}{2}},\ t \in \mathscr{R}^-.$$

Definition 2.1. If (1.1) holds, a solution u of $(\mathscr{DV})$ is called a fast homoclinic solution if $u \in E$.

Compact embedding. Part of the difficulty in dealing with the existence and multiplicity of fast homoclinic solutions of $(\mathscr{DV})$ is of the lack of compactness of the Sobolev embedding theorem.

Lemma 2.2 *Suppose (Q_γ) holds and L satisfies*

(L_γ) For the smallest eigenvalue of $L(t)$, i.e., $l(t) = \inf_{|\xi|=1} L(t)\xi.\xi$, we have

$$l(t) |t|^{\gamma-1} \longrightarrow +\infty \text{ as } |t| \longrightarrow \infty,$$

where γ is defined in (Q_γ). Then E is compactly embedded in $L^p_Q(\mathscr{R})$ for $1 \leq p \leq \infty$, which implies that for all $p \in [1,\infty]$, there exists a constant $\lambda_p > 0$ such that

$$\|u\|_{L^p_Q(\mathscr{R})} \leq \lambda_p \|u\|,\ \forall u \in E. \tag{2.2}$$

Proof a) First, we suppose that the smallest eigenvalue $l(t) = \inf_{|\xi|=1} L(t)\xi.\xi \geq 1$ for all $t \in \mathscr{R}$. Let $(u_k) \subset E$ be a sequence such that $u_k \rightharpoonup u$ weakly in E. The Banach-Steinhauss theorem implies that

$$M = \sup_{k\in\mathcal{N}} \|u_k - u\| < \infty.$$

Let $T > 0$. Since $l(t) \geq 1$ in $I =]-T,T[$, the operator defined by $P : E \longrightarrow H^1_Q(I)$, $u \longmapsto u_{|I}$ is a continuous linear map. Here, $H^1_Q(I)$ denotes the weighted Sobolev space

$$H^1_Q(I) = \left\{ u \in H^1(I,\mathscr{R}^n) : \int_I e^{Q(t)}(|\dot{u}(t)|^2 + |u(t)|^2)dt < \infty \right\}.$$

Sobolev's theorem (see e.g. [20]) implies that $u_k \longrightarrow u$ uniformly on $\bar{I}$.

Step 1. We claim that E is compactly embedded in $L^2_Q(\mathscr{R})$.

Let $\varepsilon > 0$, there is $T > 0$ such that $\frac{1}{l(t)} \leq \varepsilon^2$ for all t such that $|t| \geq T$. Hence

$$\int_{|t|\geq T} e^{Q(t)}\,|u_k(t) - u(t)|^2\,dt \leq \varepsilon^2 \int_{|t|\geq T} e^{Q(t)} l(t)\,|u_k(t) - u(t)|^2\,dt \leq \varepsilon^2 M^2,\ \forall\, k \in \mathcal{N}. \tag{2.4}$$

On the other hand, since $u_k \longrightarrow u$ uniformly on $\bar{I}$, there is $k_0 \in \mathcal{N}$ such that

$$\int_I e^{Q(t)}\,|u_k(t) - u(t)|^2\,dt \leq \varepsilon^2 M^2,\ \forall\, k \geq k_0. \tag{2.5}$$

Combining (2.4) and (2.5), we get $u_k \longrightarrow u$ in $L^2_Q(\mathscr{R})$.

Step 2. E is compactly embedded in $L^\infty_Q(\mathscr{R})$.

In fact, let ε, T be as in step 1, then by Proposition 2.1, we have

$$e^{\frac{Q(t)}{2}}\,|u_k(t) - u(t)| \leq \Big(\int_t^\infty \frac{e^{Q(s)}}{\sqrt{l(s)}}[|\dot{u}_k(s) - \dot{u}(s)|^2 + L(s)(u_k(s) - u(s)).(u_k(s)) - u(s))ds\Big)^{\frac{1}{2}}$$

$$\leq \sqrt{\varepsilon}\Big(\int_t^\infty e^{Q(s)}[|\dot{u}_k(s) - \dot{u}(s)|^2 + L(s)(u_k(s) - u(s)).(u_k(s)) - u(s))]ds\Big)^{\frac{1}{2}}$$

$$\leq \sqrt{\varepsilon}M,\ \ \forall t \geq T. \tag{2.6}$$

Similarly, we have

$$e^{\frac{Q(t)}{2}}\,|u_k(t) - u(t)| \leq \sqrt{\varepsilon}M,\ \forall t \leq -T. \tag{2.7}$$

Since $u_k \longrightarrow u$ uniformly on $\bar{I}$, then by combining (2.5), (2.6) and (2.7), we get $u_k \longrightarrow u$ in $L^\infty_Q(\mathscr{R})$.

Step 3. E is compactly embedded in $L^p_Q(\mathscr{R})$ for $p \in]2,\infty[$.

Let t_0 be such that $Q(t_0)=\min_{t\in\mathscr{R}} Q(t)$. For any $p\in]2,\infty[$ and $v\in E$

$$\int_{\mathscr{R}} e^{Q(t)}\,|v(t)|^p\,dt = e^{Q(t_0)}\int_{\mathscr{R}} e^{Q(t)-Q(t_0)}\,|v(t)|^p\,dt$$

$$= e^{Q(t_0)}\int_{\mathscr{R}}\left|e^{\frac{Q(t)-Q(t_0)}{p}}v(t)\right|^p dt$$

$$\leq e^{Q(t_0)}\left\|e^{\frac{Q(t)-Q(t_0)}{p}}v\right\|^{p-2}_{L^\infty(\mathscr{R})}\int_{\mathscr{R}}\left|e^{\frac{Q(t)-Q(t_0)}{p}}v(t)\right|^2 dt$$

$$\leq e^{Q(t_0)}\left\|e^{\frac{Q(t)-Q(t_0)}{p}}v\right\|^{p-2}_{L^\infty(\mathscr{R})}\int_{\mathscr{R}} e^{Q(t)-Q(t_0)}\,|v(t)|^2\,dt$$

$$\leq e^{-\frac{p-2}{2}Q(t_0)}\left\|e^{\frac{Q(t)}{p}}v\right\|^{p-2}_{L^\infty(\mathscr{R})}\|v\|^2_{L^2_Q(\mathscr{R})}$$

$$\leq e^{-\frac{p-2}{2}Q(t_0)}\,\|v\|^{p-2}_{L^\infty_Q(\mathscr{R})}\,\|v\|^2_{L^2_Q(\mathscr{R})}\,. \tag{2.8}$$

By steps 1,2 and (2.8), we get $u_k\longrightarrow u$ in $L^p_Q(\mathscr{R})$.

Step 4. E is compactly embedded in $L^p_Q(\mathscr{R})$ for $p\in[1,2[$.

For $T\geq 1$, let $\beta(T)=\inf_{|t|\geq T} l(t)\,|t|^{\gamma-1}$. Then by (L_γ), $\beta(T)\longrightarrow\infty$ as $T\longrightarrow\infty$. Let $p\in[1,2[$ and set $r=\frac{1-\gamma}{2-p}$. Then $p>\frac{2}{2-\gamma}$ and $rp>1$. Note that for any $T\geq 1$ and $v\in E$

$$\int_{|t|\geq T} e^{Q(t)}\,|v(t)|^p\,dt = \int_{|t|\geq T,|t|^r|v(t)|\leq 1} e^{Q(t)}\,|v(t)|^p\,dt + \int_{|t|\geq T,|t|^r|v(t)|>1} e^{Q(t)}\,|v(t)|^p\,dt$$

$$\leq \int_{|t|\geq T} e^{Q(t)}\,|t|^{-rp}\,dt + \int_{|t|\geq T,|t|^r|v(t)|>1} e^{Q(t)}(|v(t)|\,|t|^r)^p\,|t|^{-rp}\,dt$$

$$\leq \int_{|t|\geq T} e^{Q(t)}\,|t|^{-rp}\,dt + \int_{|t|\geq T} e^{Q(t)}\,|v(t)|^2\,|t|^{(2-p)r}\,dt$$

$$\leq \int_{|t|\geq T} e^{Q(t)}\,|t|^{-1+\gamma}\,dt + \int_{|t|\geq T} e^{Q(t)} l(t)\,|v(t)|^2\,\frac{1}{l(t)\,|t|^{\gamma-1}}dt$$

$$\leq \int_{|t|\geq T} e^{Q(t)}\,|t|^{-1+\gamma}\,dt + \frac{1}{\beta(T)}\int_{|t|\geq T} e^{Q(t)} l(t)\,|v(t)|^2\,dt$$

$$\leq \int_{|t|\geq T} e^{Q(t)}\,|t|^{-1+\gamma}\,dt + \frac{1}{\beta(T)}\,\|v\|^2. \tag{2.9}$$

Set $\varepsilon>0$. There is $T\geq 1$ such that $\int_{|t|\geq T} e^{Q(t)}\,|t|^{-1+\gamma}\,dt+\frac{M^2}{\beta(T)}<\frac{\varepsilon}{2}$. Hence, by (2.9), we get

$$\int_{|t|\geq T} e^{Q(t)}\,|u_k(t)-u(t)|^p\,dt < \frac{\varepsilon}{2},\ \forall k\in\mathscr{N}. \tag{2.10}$$

Since $u_k \longrightarrow u$ uniformly in $[-T,T]$, there is $k_0 \in \mathcal{N}$ such that

$$\int_{[-T,T]} e^{Q(t)} |u_k(t)-u(t)|^p \, dt < \frac{\varepsilon}{2}, \ \forall k \geq k_0. \tag{2.11}$$

Combining (2.10) and (2.11), we get $u_k \longrightarrow u$ in $L^p_Q(\mathscr{R})$.

b) Next, in general, by (L_γ), $l(t)$ is bounded from below and so there is a constant $a>0$ such that $l(t)+a \geq 1$ for all $t \in \mathscr{R}$. Since, as a set, $E = \mathscr{D}((A+aI)^{\frac{1}{2}})$, one may introduce a norm on E

$$\|u\|_a = \left\|(A+aI)^{\frac{1}{2}} u\right\|_{L^2_Q(\mathscr{R})} + \|u\|_{L^2_Q(\mathscr{R})}$$

By the previous arguments, $(E, \|.\|_a)$ is compactly embedded in $L^p_Q(\mathscr{R})$ for $p \in [1,\infty]$. Hence the proof will be completed by showing that $\|.\|_0$ is equivalent to $\|.\|_a$. In fact, for $u \in \mathscr{D}(A) = \mathscr{D}(A+aI)$

$$\begin{aligned}
\left\|(A+aI)^{\frac{1}{2}} u\right\|^2_{L^2_Q(\mathscr{R})} &= < Au, u >_{L^2_Q(\mathscr{R})} + a\|u\|^2_{L^2_Q(\mathscr{R})} \\
&= < U|A|^{\frac{1}{2}} u, |A|^{\frac{1}{2}} u > + a\|u\|^2_{L^2_Q(\mathscr{R})} \\
&\leq \left\||A|^{\frac{1}{2}} u\right\|^2_{L^2_Q(\mathscr{R})} + a\|u\|^2_{L^2_Q(\mathscr{R})}
\end{aligned}$$

$$\leq max(1,a)\|u\|^2_0, \tag{2.12}$$

and on the other hand

$$\begin{aligned}
\left\||A|^{\frac{1}{2}} u\right\|^2_{L^2_Q(\mathscr{R})} &= < |A|u, u >_{L^2_Q(\mathscr{R})} \\
&= < (A+aI)Uu, u >_{L^2_Q(\mathscr{R})} - a < Uu, u >_{L^2_Q(\mathscr{R})} \\
&\leq \left\|(A+aI)^{\frac{1}{2}} u\right\|^2_{L^2_Q(\mathscr{R})} + a\|u\|^2_{L^2_Q(\mathscr{R})}
\end{aligned}$$

$$\leq max(1,a)\|u\|^2_a. \tag{2.13}$$

Since $\mathscr{D}(A)$ is dense in E, (2.12) and (2.13) imply that $\|.\|_0$ and $\|.\|_a$ are equivalent proving Lemma 2.2.

By Lemma 2.2, we see that, since the selfadjoint operator A in $L^2_Q(\mathscr{R})$ is bounded from below, it possesses a compact resolvent. Therefore the spectrum $\sigma(A)$ consists of eigenvalues numbered by $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_k \leq ... \nearrow \infty$ (counted in their multiplicities), and a corresponding system of eigenfunctions (e_k), $Ae_k = \lambda_k e_k$ which forms an orthogonal basis of

$L^2_Q(\mathscr{R})$. Assume $\lambda_1 \leq ... \leq \lambda_{n^-} < 0 = \lambda_{n^-+1} = ... = \lambda_{n^-+n^0} < \lambda_{n^-+n^0+1} \leq ... < \infty$, and let $E^- = span(\{e_1,...,e_{n^-}\})$, $E^0 = span(\{e_{n^-+1},...,e_{n^-+n^0}\})$ and $E^+ = (E^- \oplus E^0)^\perp$. Then $E = E^- \oplus E^0 \oplus E^+$. For later use, let

$$a(u,v) = <Au,v>_{L^2_Q(\mathscr{R})},\ \forall u,v \in E$$

be the bilinear form associated with A. For any $u \in \mathscr{D}(A)$ and $v \in E$, we have

$$a(u,v) = \int_{\mathscr{R}} e^{Q(t)}(\dot{u}(t)\dot{v}(t) + L(t)u(t).v(t))dt,\ u \in \mathscr{D}(A),$$

and so, since $\mathscr{D}(A)$ is dense in E, (2.15) holds for all $u,v \in E$. Moreover, by definition

$$a(u,u) = <(P^+ - P^-)u,u> = \left\|u^+\right\|^2 - \left\|u^-\right\|^2$$

for all $u = u^- + u^0 + u^+ \in E$ where $P^\pm : E \longrightarrow E^\pm$ are the projectors.

For the existence and multiplicity of homoclinic solutions of $(\mathscr{DV})$, we appeal to the following abstract critical lemmas. Let E be a Banach space and $f \in C^1(E,\mathscr{R})$. As usual we say f satisfies the Palais-Smale condition ((PS) for short) if any sequence $(u_k) \subset E$ for which $(f(u_k))$ is bounded and $f'(u_k) \longrightarrow 0$ as $k \longrightarrow \infty$ possesses a convergent subsequence.

Lemma 2.3. (A special version of the Generalized Mountain Pass Theorem) [21]. *Let E an infinite dimensional Banach space such that $E = V \oplus X$, where V is finite dimensional. If $f \in C^1(E,\mathscr{R})$ satisfies*

(f_1) *the (PS) condition;*

(f_2) *there are constants $\rho, \delta > 0$ such that*

$$f_{|\partial B_\rho \cap X} \geq \delta,$$

where $\partial B_\rho = \{u \in E : \|u\| = \rho\}$;

(f_3) *there are constants $r > \rho$, $M > 0$ and $e \in X$ with $\|e\| = 1$ such that*

$$f_{|\partial\Lambda} \leq 0 \text{ and } f_{|\Lambda} \leq M,$$

where

$$\Lambda = (B_r \cap V) \oplus \{se : 0 \leq s \leq r\};$$

then f has a critical point u with $f(u) \geq \delta$.

Lemma 2.4. [21] *Let E be an infinite dimensional Banach space such that $E = V \oplus X$, where V is finite dimensional. If $f \in C^1(E, \mathscr{R})$ is even and satisfies $f(0) = 0$, (f_1), (f_2) and $(f_3^{'})$ for each finite dimensional subspace $\tilde{E} \subset E$, there is an $R = R(\tilde{E}) > 0$ such that $f \leq 0$ on $\tilde{E} - B_R$; then f possesses an unbounded sequence of critical values.*

3. Main results

Let $W : \mathscr{R} \times \mathscr{R}^n \longrightarrow \mathscr{R}$, $(t,x) \longrightarrow W(t,x)$ be a continuous function, differentiable with respect to the second variable with continuous derivative $\nabla W(t,x) = \frac{\partial W}{\partial x}(t,x)$. We make the following assumptions:

(W_1)
$$\frac{W(t,x)}{|x|^2} \longrightarrow +\infty \; as \; |x| \longrightarrow \infty, \; uniformly \; in \; t \in \mathscr{R};$$

(W_2)
$$\nabla W(t,x).x \geq 2W(t,x) \geq 0, \; \forall (t,x) \in \mathscr{R} \times \mathscr{R}^n;$$

(W_3)
$$\frac{|\nabla W(t,x)|}{|x|} \longrightarrow 0 \; as \; |x| \longrightarrow \infty, \; uniformly \; in \; t \in \mathscr{R};$$

(W_4) there exist constants $\alpha > 0$ and $a > 0$ such that

$$|\nabla W(t,x)| \leq a(|x|^{\alpha} + 1), \; \forall (t,x) \in \mathscr{R} \times \mathscr{R}^n;$$

(W_5) there exist constants $\beta \geq \alpha$, $\beta > 1$, $b > 0$ and $r > 0$ such that

$$\nabla W(t,x).x - 2W(t,x) \geq b\,|x|^{\beta}, \; \forall t \in \mathscr{R}, \; \forall |x| \geq r.$$

Our first main result is the following one:

Theorem 3.1. *Assume that (Q_γ), (L_γ) and $(W_1) - (W_5)$ hold. Then the damped viration system*

$(\mathscr{DV})$
$$\ddot{u}(t) + q(t)\dot{u}(t) - L(t)u(t) + \nabla W(t,u(t)) = 0,$$

possesses at least one nontrivial fast homoclinic solution.
Moreover, if $W(t,x)$ is even in $x \in \mathscr{R}^n$, then $(\mathscr{DV})$ has infinitely many distinct fast homoclinic solutions.

Example 3.1. *Let*

$$L(t) = (t^2 - 1)I_n, \; W(t,x) = |x|^2\, ln(1 + |x|^2),$$

where I_n is the unit matrix of order n. A straightforward computation shows that L and W satisfy our Theorem 3.1, but they don't satisfy the corresponding results in [1] because that W don't satisfy the (AR) condition and $L(t)$ is not positively definite for $|t| \leq 1$.

Proof of Theorem 3.1.

Now we are going to establish the corresponding variational framework to obtain fast homoclinic solutions for $(\mathscr{DV})$. To this end, define the functional $f : E \longrightarrow \mathscr{R}$ by

$$f(u) = \frac{1}{2}\int_{\mathscr{R}} e^{Q(t)}(|\dot{u}(t)|^2 + L(t)u(t).u(t))dt - \int_{\mathscr{R}} e^{Q(t)}W(t,u(t))dt$$
$$= \frac{1}{2}\left\|u^+\right\|^2 - \frac{1}{2}\left\|u^-\right\|^2 - g(u), \text{ where } g(u) = \int_{\mathscr{R}} e^{Q(t)}W(t,u(t))dt.$$

Lemma 3.1. *Suppose* $\lim_{|t|\longrightarrow\infty} Q(t) = +\infty$ *and* (L_γ), (W_2), (W_3) *hold. Then g is continuously differentiable in E and for all $u, v \in E$, we have*

$$Dg(u)v = \int_{\mathscr{R}} e^{Q(t)}\nabla W(t,u(t)).v(t)dt.$$

Moreover, Dg is compact.

Proof. By (W_3), for all $\varepsilon > 0$, there exists $\eta > 0$ such that

$$|\nabla W(t,x)| \leq \varepsilon |x|, \ \forall t \in \mathscr{R}, \ \forall |x| \leq \eta. \tag{3.1}$$

By (W_2), (3.1) and the Mean Value Theorem, we get

$$W(t,x) = \int_0^1 \nabla W(t,sx).xds \leq \frac{\varepsilon}{2}|x|^2, \ \forall t \in \mathscr{R}, \ \forall |x| \leq \eta. \tag{3.2}$$

Letting $u_0 \in E$, then $u_0 \in C^0(\mathscr{R},\mathscr{R}^n)$, the space of continuous functions u on $\mathscr{R}$ such that $u(t) \longrightarrow 0$ as $|t| \longrightarrow \infty$. Therefore, there is $R > 0$ such that $|t| \geq R$ implies $|u(t)| \leq \frac{\eta}{2}$. Hence by (3.2), we obtain

$$0 \leq \int_{\mathscr{R}} e^{Q(t)}W(t,u_0(t))dt \leq \int_{[-R,R]} e^{Q(t)}W(t,u_0(t))dt + \frac{\varepsilon}{2}\int_{|t|\geq R} e^{Q(t)}|u_0(t)|^2 dt$$

so g is well defined. Next we prove that $g \in C^1(E,\mathscr{R})$. For $v \in E$ such that $\|v\| < inf(1, \frac{\eta\sqrt{c_0}}{2\lambda_\infty})$, we have

$$\|v\|_{L^\infty(\mathscr{R})} \leq \frac{1}{\sqrt{c_0}}\|v\|_{L^\infty_Q(\mathscr{R})} < \frac{\eta}{2}. \tag{3.3}$$

The Mean Value Theorem, (3.1) and (3.3) show that for $|t| \geq R$

$$\begin{aligned}|W(t,u_0(t)+v(t))-W(t,u_0(t))| &= \left|\int_0^1 \nabla W(t,u_0(t)+sv(t)).v(t)ds\right| \\ &\leq \varepsilon(|u_0(t)|+|v(t)|)\,|v(t)|.\end{aligned}$$

Hence by Lemma 2.2 and Hölder's inequality, we deduce that

$$\begin{aligned}\int_{|t|\geq R} e^{Q(t)}\,|W(t,u_0(t)+v(t))-W(t,u_0(t))|\,dt &\leq \varepsilon\int_{|t|\geq R} e^{Q(t)}(|u_0(t)|+|v(t)|)\,|v(t)|\,dt \\ &\leq \varepsilon(\|u_0\|_{L^2_Q(\mathscr{R})}+\|v\|_{L^2_Q(\mathscr{R})})\,\|v\|_{L^2_Q(\mathscr{R})} \\ &\leq \varepsilon\lambda_2^2(\|u_0\|+1)\,\|v\|. \qquad (3.4)\end{aligned}$$

Likewise, by (3.1), Hölder's inequality and Lemma 2.2, we get

$$\begin{aligned}\int_{|t|\geq R} e^{Q(t)}\,|\nabla W(t,u_0(t)).v(t)|\,dt &\leq \varepsilon\int_{|t|\geq R} e^{Q(t)}\,|u_0(t)|\,|v(t)|\,dt \\ &\leq \varepsilon\lambda_2^2\,\|u_0\|\,\|v\|. \qquad (3.5)\end{aligned}$$

On the other hand, it is well known that $u \longrightarrow \int_{[-R,R]} e^{Q(t)}W(t,u(t))dt$ is continuously differentiable on E, hence there exists a constant $\alpha > 0$ such that for $v\in E, \|v\| < \alpha$,

$$\left|\int_{[-R,R]} e^{Q(t)}[W(t,u_0(t)+v(t))-W(t,u_0(t))-\nabla W(t,u_0(t)).v(t)]dt\right| \leq \varepsilon\,\|v\|. \qquad (3.6)$$

Combining (3.4), (3.5) and (3.6) yields for $\|v\| < inf(1,\alpha,\frac{\eta\sqrt{c_0}}{2\lambda_\infty})$

$$\left|\int_{\mathscr{R}} e^{Q(t)}[W(t,u_0(t)+v(t))-W(t,u_0(t))-\nabla W(t,u_0(t)).v(t)]dt\right| \leq \varepsilon[\lambda_2^2(2\,\|u_0\|+1)+1]\,\|v\|,$$

which proves that g is differentiable and

$$Dg(u_0)v = \int_{\mathscr{R}} e^{Q(t)}\nabla W(t,u_0(t)).v(t)dt.$$

Next we prove that Dg is continuous. Let $u_j \longrightarrow u$ in E. Then $u_j \longrightarrow u$ in $L^2_Q(\mathscr{R})$. By Hölder's inequality, we have

$$\begin{aligned}\left\|Dg(u_j)-Dg(u)\right\|_{E'} &= \sup_{\|v\|=1}\left\|Dg(u_j)v-Dg(u)v\right\| \\ &= \sup_{\|v\|=1}\left|\int_{\mathscr{R}} e^{Q(t)}(\nabla W(t,u_j(t))-\nabla W(t,u(t))).v(t)dt\right|\end{aligned}$$

$$\leq \sup_{\|v\|=1} \left\|\nabla W(t,u_j(t)) - \nabla W(t,u(t))\right\|_{L^2_Q(\mathscr{R})} \|v\|_{L^2_Q(\mathscr{R})}$$

$$\leq \lambda_2 \left\|\nabla W(t,u_j(t)) - \nabla W(t,u(t))\right\|_{L^2_Q(\mathscr{R})}. \tag{3.7}$$

Let $\varepsilon > 0$ and let η be defined as in (3.1), there exists $j_1 \in \mathscr{N}$ such that $\left\|u_j - u\right\| \leq \frac{\eta\sqrt{c_0}}{2\lambda_\infty}$ for all $j \geq j_1$, so

$$\left\|u_j - u\right\|_{L^\infty(\mathscr{R})} \leq \frac{1}{\sqrt{c_0}} \left\|u_j - u\right\|_{L^\infty_Q(\mathscr{R})} \leq \frac{\eta}{2}. \tag{3.8}$$

Since $u \in C^0(\mathscr{R},\mathscr{R}^n)$, there exists $R > 0$ such that

$$|u(t)| \leq \frac{\eta}{2},\ \forall |t| \geq R, \tag{3.9}$$

which with (3.8) imply that

$$\left|u_j(t)\right| \leq \eta,\ \forall j \geq j_1,\ \forall |t| \geq R. \tag{3.10}$$

Therefore by (3.1), (3.9) and (3.10), we get

$$\left(\int_{|t|\geq R} e^{Q(t)} \left|\nabla W(t,u_j(t)) - \nabla W(t,u(t))\right|^2 dt\right)^{\frac{1}{2}} \leq \varepsilon\left(\int_{|t|\geq R} e^{Q(t)} (\left|u_j(t)\right| + |u(t)|)^2 dt\right)^{\frac{1}{2}}$$

$$\leq \varepsilon\lambda_2(\left\|u_j\right\| + \|u\|).$$

Since (u_j) is bounded in E, there exists a constant $M > 0$ such that

$$\left(\int_{|t|\geq R} e^{Q(t)} \left|\nabla W(t,u_j(t)) - \nabla W(t,u(t))\right|^2 dt\right)^{\frac{1}{2}} \leq \varepsilon\lambda_2 M,\ \forall j \geq j_1. \tag{3.11}$$

Now, since $u_j \longrightarrow u$ in $L^\infty_Q(\mathscr{R})$, it is well known that

$$\int_{[-R,R]} e^{Q(t)} \left|\nabla W(t,u_j(t)) - \nabla W(t,u(t))\right|^2 dt \longrightarrow 0 \ as\ j \longrightarrow \infty.$$

So there exists $j_2 \in \mathscr{N}$ such that

$$\left(\int_{[-R,R]} e^{Q(t)} \left|\nabla W(t,u_j(t)) - \nabla W(t,u(t))\right|^2 dt\right)^{\frac{1}{2}} \leq \varepsilon,\ \forall j \geq j_2. \tag{3.12}$$

Combining (3.7), (3.11) and (3.12) and taking $j_0 = max(j_1, j_2)$, we have

$$\left\|Dg(u_j) - Dg(u)\right\|_{E'} \leq \varepsilon(1 + \lambda_2 M),\ \forall j \geq j_0.$$

This shows that $Dg(u_j) \longrightarrow Dg(u)$ in E' and $g \in C^1(E,\mathscr{R})$.

It remains to prove that Dg is compact. Let $u_j \rightharpoonup u$ weakly in E. By Lemma 2.2, there exists a subsequence (u_{j_k}) such that $u_{j_k} \longrightarrow u$ in $L^\infty_Q(\mathscr{R})$. Let $\varepsilon > 0$ and η be defined as in (3.1), there exists $k_1 \in \mathscr{N}$ such that $\|u_{j_k} - u\|_{L^\infty_Q(\mathscr{R})} \leq \frac{\eta\sqrt{c_0}}{2}$ for all $k \geq k_1$, so

$$\|u_{j_k} - u\|_{L^\infty(\mathscr{R})} \leq \frac{\eta}{2},\ \forall k \geq k_1. \tag{3.13}$$

Combining (3.9) and (3.13) yields

$$|u_{j_k}(t)| \leq \eta,\ \forall k \geq k_1,\ \forall |t| \geq R. \tag{3.14}$$

Hence by (3.1), (3.9) and (3.14), we get as above

$$\Big(\int_{|t|\geq R} e^{Q(t)} \left|\nabla W(t,u_{j_k}(t)) - \nabla W(t,u(t))\right|^2 dt\Big)^{\frac{1}{2}} \leq \varepsilon\delta_2 M,\ \forall k \geq k_1 \tag{3.14}$$

for a positive constant M, which proves that

$$\int_{|t|\geq R} e^{Q(t)} \left|\nabla W(t,u_{j_k}(t)) - \nabla W(t,u(t))\right|^2 dt \longrightarrow 0\ as\ k \longrightarrow \infty. \tag{3.15}$$

As above, we have

$$\int_{[-R,R]} e^{Q(t)} \left|\nabla W(t,u_{j_k}(t)) - \nabla W(t,u(t))\right|^2 dt \longrightarrow 0\ as\ k \longrightarrow \infty. \tag{3.16}$$

Combining (3.15) and (3.16) yields $Dg(u_{j_k}) \longrightarrow Dg(u)$ in E'. So Dg is compact. The proof of Lemma 3.1 is complete.

Now, it is easy to see that $h : u \longrightarrow \frac{1}{2}\int_{\mathscr{R}} e^{Q(t)}[|\dot{u}(t)|^2 + L(t)u(t).u(t)]dt$ is continuously differentiable and

$$h'(u)v = \int_{\mathscr{R}} e^{Q(t)}[\dot{u}(t).\dot{v}(t) + L(t)u(t).v(t)]dt,\ \forall u,v \in E.$$

Therefore $f = h - g \in C^1(E,\mathscr{R})$ and for all $u,v \in E$

$$f'(u)v = \int_{\mathscr{R}} e^{Q(t)}[\dot{u}(t).\dot{v}(t) + L(t)u(t).v(t)]dt - \int_{\mathscr{R}} e^{Q(t)}\nabla W(t,u(t)).v(t)dt$$

If u is a critical point of f, for any $v \in E \subset C^0(\mathscr{R},\mathscr{R}^n)$, we have

$$\int_{\mathscr{R}} e^{Q(t)}\dot{u}(t).\dot{v}(t)dt = -\int_{\mathscr{R}} e^{Q(t)}[L(t)u(t) - \nabla W(t,u(t))].v(t)dt,$$

which implies that $e^{Q(t)}[L(t)u - \nabla W(t,u)]$ is the weak derivative of $e^{Q(t)}\dot{u}$. Since $L \in C(\mathscr{R},\mathscr{R}^{n^2})$, $W \in C^1(\mathscr{R}\times\mathscr{R}^n,\mathscr{R})$ and $E \subset C^0(\mathscr{R},\mathscr{R}^n)$, we see that $\frac{d}{dt}(e^{Q(t)}\dot{u})$ is continuous, which yields that $\ddot{u}$ is continuous and $u \in C^2(\mathscr{R},\mathscr{R}^n)$, i.e., u is a classical solution of $(\mathscr{DV})$.

Lemma 3.2. *If* (Q_γ), (L_γ), (W_2), (W_4) *and* (W_5) *hold, then* f *satisfies the* (PS) *condition.*

Proof. Let $(u_k) \subset E$ be a (PS) sequence, i.e., there exists a constant $M > 0$ such that

$$|f(u_k)| \leq M,\ \forall k \in \mathcal{N} \ and\ f'(u_k) \longrightarrow 0,\ as\ k \longrightarrow \infty. \tag{3.17}$$

We claim that (u_k) is bounded. If not, passing to a subsequence if necessary, we may assume that $\|u_k\| \longrightarrow \infty$ as $k \longrightarrow \infty$. By (W_2) and (W_5), we have

$$2f(u_k) - f'(u_k).u_k = \int_{\mathscr{R}} e^{Q(t)}[\nabla W(t,u_k).u_k - 2W(t,u_k)]dt$$

$$\geq b\int_{\{t\in\mathscr{R}:|u_k(t)|\geq r\}} e^{Q(t)}\,|u_k(t)|^\beta\, dt \tag{3.18}$$

for all positive integer k, which implies that

$$\frac{1}{\|u_k\|}\int_{\{t\in\mathscr{R}:|u_k(t)|\geq r\}} e^{Q(t)}\,|u_k(t)|^\beta\, dt \longrightarrow 0 \tag{3.19}$$

as $k \longrightarrow \infty$. Let

$$v_k(t) = \begin{cases} u_k(t),\ if\ |u_k(t)| \leq r, \\ 0,\ if\ |u_k(t)| > r, \end{cases} \tag{3.20}$$

and

$$w_k(t) = u_k(t) - v_k(t) \tag{3.21}$$

for all positive integer k and all $t \in \mathscr{R}$. By (3.18) and (3.21), we have

$$c_1(1+\|u_k\|) \geq b\,\|w_k\|^\beta_{L^\beta_Q(\mathscr{R})} \tag{3.22}$$

for all positive integer k and some constant $c_1 > 0$. It follows from Hölder's inequality, (3.20), (3.21) and the equivalence of the norms on the finite dimensional subspace $E^- \oplus E^0$ that

$$\left\|u_k^- + u_k^0\right\|^2_{L^2_Q(\mathscr{R})} = < u_k^- + u_k^0, u_k >_{L^2_Q(\mathscr{R})}$$

$$= < u_k^- + u_k^0, v_k >_{L^2_Q(\mathscr{R})} + < u_k^- + u_k^0, w_k >_{L^2_Q(\mathscr{R})}$$

$$\leq \frac{1}{\sqrt{c_0}}\left\|u_k^- + u_k^0\right\|_{L^1_Q(\mathscr{R})}\|v_k\|_{L^\infty_Q(\mathscr{R})} + \left\|u_k^- + u_k^0\right\|^2_{L^{\beta'}_Q(\mathscr{R})}\|w_k\|_{L^\beta_Q(\mathscr{R})}$$

$$\leq c_2\left\|u_k^- + u_k^0\right\|_{L^2_Q(\mathscr{R})}(1+\|w_k\|_{L^\beta_Q(\mathscr{R})}) \tag{3.23}$$

for all positive integer k and some constant $c_2 > 0$, where $\beta' = \frac{\beta}{\beta-1}$ $(\beta > 1)$ is the Hölder's conjugate of β.

From the equivalence of the norms on the finite dimensional subspace $E^- \oplus E^0$, (3.22) and (3.23) we obtain

$$\|u_k^- + u_k^0\| \leq c_3 \|u_k^- + u_k^0\|_{L_Q^2(\mathscr{R})} \leq c_4(1 + \|w_k\|_{L_Q^\beta(\mathscr{R})})$$
$$\leq c_5(1 + \|u_k\|^{\frac{1}{\beta}})$$

for all positive integer k and some positive constants c_3, c_4, c_5, which implies that

$$\frac{\|u_k^- + u_k^0\|}{\|u_k\|} \longrightarrow 0 \tag{3.24}$$

as $k \longrightarrow \infty$. By (W_4) and Lemma 2.2, one sees that

$$f'(u_k).u_k^+ = \|u_k^+\|^2 - \int_{\mathscr{R}} e^{Q(t)} \nabla W(t, u_k).u_k^+ dt$$
$$\geq \|u_k^+\|^2 - \int_{\mathscr{R}} e^{Q(t)} |\nabla W(t, u_k)| |u_k^+| dt$$
$$\geq \|u_k^+\|^2 - a \int_{\mathscr{R}} e^{Q(t)} |u_k|^\alpha |u_k^+| dt - a \int_{\mathscr{R}} e^{Q(t)} |u_k^+| dt$$
$$\geq \|u_k^+\|^2 - \frac{a}{\sqrt{c_0}} \|u_k^+\|_{L_Q^\infty(\mathscr{R})} \int_{\{t\in\mathscr{R}:|u_k(t)|\geq r\}} e^{Q(t)} |u_k|^\alpha dt$$
$$-ar^\alpha \int_{\{t\in\mathscr{R}:|u_k(t)|<r\}} e^{Q(t)} |u_k^+| dt - a \int_{\mathscr{R}} e^{Q(t)} |u_k^+| dt$$
$$\geq \|u_k^+\|^2 - \frac{a}{\sqrt{c_0}} \|u_k^+\|_{L_Q^\infty(\mathscr{R})} r^{\alpha-\beta} \int_{\{t\in\mathscr{R}:|u_k(t)|\geq r\}} e^{Q(t)} |u_k|^\beta dt$$
$$-ar^\alpha \|u_k^+\|_{L_Q^1(\mathscr{R})} - a \|u_k^+\|_{L_Q^1(\mathscr{R})}$$
$$\geq \|u_k^+\|^2 - \frac{a\lambda_\infty}{\sqrt{c_0}} \|u_k^+\| r^{\alpha-\beta} \int_{\{t\in\mathscr{R}:|u_k(t)|\geq r\}} e^{Q(t)} |u_k|^\beta dt$$
$$-ar^\alpha \lambda_1 \|u_k^+\| - a\lambda_1 \|u_k^+\|,$$

which, by (3.19), implies

$$\frac{\|u_k^+\|}{\|u_k\|} \longrightarrow 0 \tag{3.25}$$

as $k \longrightarrow \infty$. Hence by (3.24) and (3.25), we obtain

$$1 = \frac{\|u_k\|}{\|u_k\|} \leq \frac{\|u_k^- + u_k^0\| + \|u_k^+\|}{\|u_k\|} \longrightarrow 0$$

as $k \longrightarrow \infty$, which is a contradiction. Hence (u_k) must be bounded. Moreover we have

$$\left\|u_k^+ - u^+\right\|^2 = (f'(u_k) - f'(u)).(u_k^+ - u^+) + (g'(u_k) - g'(u)).(u_k^+ - u^+).$$

Going to a subsequence if necessary, we may assume that $u_k \rightharpoonup u$ weakly in E and

$$u_k \longrightarrow u \text{ in } L_Q^2(\mathscr{R}) \text{ as } k \longrightarrow \infty. \tag{3.26}$$

From Lemma 3.1, we deduce that $u_k^+ \longrightarrow u^+$ in E. From (3.26) and the equivalence of the norms on the finite dimensional subspace $E^- \oplus E^0$ we obtain that $u_k^0 \longrightarrow u^0$ and $u_k^- \longrightarrow u^-$ in E as $k \longrightarrow \infty$. Hence (u_k) has a convergent subsequence, which shows that the (PS) condition holds.

Lemma 3.3. *There are constants $\rho > 0$ and $\delta > 0$ such that*

$$f_{|S} \geq \delta,$$

where

$$S = \left\{u \in E^+ : \|u\| = \rho\right\}.$$

Proof. Chose in (3.2) $\varepsilon = (2\lambda_2^2)^{-1}$, $\rho = \frac{\eta\sqrt{c_0}}{\lambda_\infty}$ and $\delta = \frac{\rho^2}{4}$, then by Lemma 2.2, we get

$$f(u) = \frac{1}{2}\|u\|^2 - \int_{\mathscr{R}} e^{Q(t)} W(t,u(t))dt \geq \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\int_{\mathscr{R}} e^{Q(t)}|u(t)|^2 dt$$

$$\geq \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\lambda_2^2\|u\|^2 = \frac{1}{4}\|u\|^2 = \frac{\rho^2}{4} = \delta$$

for all $u \in S$. The proof is complete.

Lemma 3.4. *Let $e \in E^+$ with $\|e\| = 1$. There exist $r_1, r_2 > 0$ such that*

$$f(u) \leq 0, \ \forall u \in \partial\Lambda,$$

where

$$\Lambda = \{se : 0 \leq s \leq r_1\} \oplus \left\{u \in E^- \oplus E^0 : \|u\| \leq r_2\right\}.$$

Proof. Let $e \in E^+$ with $\|e\| = 1$ and $F = span\{e\} \oplus E^- \oplus E^0$. We claim that there exists $\varepsilon_1 > 0$ such that

$$meas(\{t \in \mathscr{R} : |u(t)| \geq \varepsilon_1 \|u\|\}) \geq \varepsilon_1, \ \forall u \in F - \{0\}. \tag{3.27}$$

Otherwise, for any positive integer k, there exists $u_k \in F - \{0\}$ such that

$$meas(\left\{t \in \mathscr{R} : |u_k(t)| \geq \frac{1}{k}\|u_k\|\right\}) < \frac{1}{k}. \tag{3.28}$$

If necessary, replacing u_k by $\frac{u_k}{\|u_k\|}$, we may assume that $\|u_k\| = 1$ and

$$meas(\left\{t \in \mathscr{R} : |u_k(t)| \geq \frac{1}{k}\right\}) < \frac{1}{k}$$

for a positive integer k. Since $dim(F) < \infty$, it follows from the compactness of the unit sphere of F that there exists a subsequence, say (u_k), such that (u_k) converges to some u_0 in F. Hence one has $\|u_0\| = 1$. By the equivalence of the norms on the finite dimensional subspace F, we have $u_k \longrightarrow u_0$ in $L^2_Q(\mathscr{R})$

$$\int_{\mathscr{R}} e^{Q(t)} |u_k - u_0|^2 dt \longrightarrow 0 \; as \; k \longrightarrow \infty. \tag{3.29}$$

Thus there exist constants $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$meas(\{t \in \mathscr{R} / |u_0(t)| \geq \delta_1\} \geq \delta_2. \tag{3.30}$$

In fact, if not, we have

$$meas(\left\{t \in \mathscr{R} / |u_0(t)| \geq \frac{1}{k}\right\}) = 0,$$

for all positive integer k, which by Lemma 2.2 implies that

$$0 \leq \int_{\mathscr{R}} e^{Q(t)} |u_0|^4 dt \leq \|u_0\|^2_{L^\infty(\mathscr{R})} \|u_0\|^2_{L^2_Q(\mathscr{R})} \leq \frac{\lambda_2^2}{k^2} \longrightarrow 0$$

as $k \longrightarrow \infty$. Hence $u_0 = 0$ which contradicts that $\|u_0\| = 1$. Therefore (3.30) holds. Now let

$$I_0 = \{t \in \mathscr{R} / |u_0(t)| \geq \delta_1\}, \; I_k = \left\{t \in \mathscr{R} / |u_k(t)| < \frac{1}{k}\right\},$$

and $I_k^c = \mathscr{R} - I_k$. By (3.28) and (3.30), we have for all positive integer k

$$meas(I_k \cap I_0) = meas(I_0 - (I_k^c \cap I_0)) \geq meas(I_0) - meas(I_k^c \cap I_0) \geq \delta_2 - \frac{1}{k}.$$

Let k be large enough such that

$$\delta_1 - \frac{1}{k} \geq \frac{1}{2}\delta_1 \; and \; \delta_2 - \frac{1}{k} \geq \frac{1}{2}\delta_2.$$

Therefore one has

$$|u_k(t) - u_0(t)|^2 \geq ||u_k(t)| - |u_0(t)||^2 \geq (\delta_1 - \frac{1}{k})^2 \geq \frac{1}{4}\delta_1^2, \; \forall t \in I_k \cap I_0,$$

which implies that

$$\int_{\mathscr{R}} e^{Q(t)} |u_k(t)-u_0(t)|^2 dt \ge \int_{I_k\cap I_0} c_0 |u_k(t)-u_0(t)|^2 dt \ge \frac{1}{4}\delta_1^2 meas(I_k \cap I_0)$$

$$\ge \frac{1}{4}\delta_1^2(\delta_2 - \frac{1}{k}) \ge \frac{1}{8}\delta_1^2\delta_2 > 0$$

for all large integer k. This is a contradiction with (3.29). Therefore (3.27) holds. For $u = u^- + u^0 + u^+ \in F$, let

$$\Omega_u = \{t \in \mathscr{R} / |u(t)| \ge \varepsilon_1 \|u\|\}.$$

By (W_1), for $d = \frac{1}{2\varepsilon_1^3 c_0} > 0$, there exists $R_1 > 0$ such that

$$W(t,x) \ge d|x|^2, \ \forall |x| \ge R_1, \ \forall t \in \mathscr{R}.$$

Hence one has

$$e^{Q(t)}W(t,u(t)) \ge de^{Q(t)}|u(t)|^2 \ge dc_0\varepsilon_1^2 \|u\|^2 \tag{3.31}$$

for all $u \in F$ with $\|u\| \ge \frac{R_1}{\varepsilon_1}$ and $t \in \Omega_u$. It follows from (W_5), (3.27) and (3.31) that

$$f(u) = \frac{1}{2}\left\|u^+\right\|^2 - \frac{1}{2}\left\|u^-\right\|^2 - \int_{\mathscr{R}} e^{Q(t)}W(t,u(t))dt$$

$$\le \frac{1}{2}\left\|u^+\right\|^2 - \int_{\Omega_u} e^{Q(t)}W(t,u(t))dt$$

$$\le \frac{1}{2}\left\|u^+\right\|^2 - dc_0\varepsilon_1^2\|u\|^2 meas(\Omega_u)$$

$$\le \frac{1}{2}\left\|u^+\right\|^2 - dc_0\varepsilon_1^3\|u\|^2 = \frac{1}{2}\left\|u^+\right\|^2 - \frac{1}{2}\|u\|^2 \le 0, \tag{3.32}$$

for all $u \in F$ with $\|u\| \ge \frac{R_1}{\varepsilon_1}$. Let $r_1 > 0$ and denote

$$\Lambda = \{se/0 \le s \le r_1\} \oplus \left\{u \in E^- \oplus E^0 : \|u\| \le r_1\right\}.$$

Then we have

$$\partial\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3,$$

where

$$\Lambda_1 = \left\{u \in E^- \oplus E^0 : \|u\| \le r_1\right\},$$

$$\Lambda_2 = r_1 e + \left\{u \in E^- \oplus E^0 / \|u\| \le r_1\right\},$$

$$\Lambda_3 = \{se/0 \le s \le r_1\} \oplus \left\{u \in E^- \oplus E^0 / \|u\| = r_1\right\}.$$

By (3.32), one has

$$f(u) \leq 0,\ \forall u \in \Lambda_2 \cup \Lambda_3$$

for all $r_1 \geq \frac{R}{\varepsilon_1}$. From (W_5), we have

$$f(u) \leq 0,\ \forall u \in E^- \oplus E^0,$$

which implies that $f(u) \leq 0, \forall u \in \Lambda_1$. Hence we have $f(u) \leq 0, \forall u \in \partial\Lambda$, for all $r_1 > max\left\{\rho, \frac{R_1}{\varepsilon_1}\right\}$, where ρ is defined in Lemma 3.3, which completes the proof of Lemma 3.4.

By Lemma 2.3, f has a critical point u satisfying $f(u) \geq \delta > 0$ where δ is given by Lemma 3.3. Since $f(0) = 0$, then u is nontrivial and $(\mathscr{DV})$ possesses a nontrivial homoclinic solution. It remains to study the multiplicity of homoclinic solutions for $(\mathscr{DV})$ when $W(t,.)$ is even. We have $f(0) = 0$ and since $W(t,x)$ is even with respect to the second variable, then f is even. The assumptions (f_1), (f_2) are proved above. Let us prove $(f_3^{'})$. Let $\tilde{E} \subset E$ be a finite dimensional subspace of E, there exists $m \geq 1$ such that $\tilde{E} \subset E^- \oplus E^0 \oplus span\{w_1,...,w_m\} = X^m$, where $w_k = e_{n^- + n^0 + k}$, $k \geq 1$. Replacing the subspace $F = span\{e\} \oplus E^- \oplus E^0$ by the subspace X^m in the proof of Lemma 3.4 and following the same steps, we obtain $R_m > 0$ such that

$$f(u) \leq \frac{1}{2}\left\|u^+\right\|^2 - \frac{1}{2}\left\|u^-\right\|^2 \leq 0,\ \forall u \in X^m,\ \|u\| \geq R_m.$$

Therefore $(f_3^{'})$ is verified. Therefore, by Lemma 2.4, f possesses a unbounded sequence of critical points. Hence $(\mathscr{DV})$ possesses infinitely may homoclinic solutions.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H, where the criterion for the approximate computation of $y_{n,i}$ in (2.1) is $\|y_{n,i} - P_C(x_n - \lambda_i A_i x_n)\| \leq e_{n,i}$, where $\lim_{n\to\infty}\|e_{n,i}\| = 0$ for each $1 \leq i \leq r$. Assume that the above control sequences satisfies the following conditions. Then sequence $\{x_n\}$ converges in norm to $p = P_{\mathscr{F}}u$.*

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