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SMALLEST EIGENVALUES FOR FRACTIONAL BOUNDARY VALUE PROBLEMS WITH A FRACTIONAL BOUNDARY CONDITION

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Abstract. Let $n \in \mathbb{N}$, $n \geq 2$. For $n-1 < \alpha \leq n$, we use the theory of u_0-positive operators to show the existence of and then compare smallest eigenvalues of the fractional boundary value problems $D_{0+}^{\alpha} u + \lambda_1 p(t) u = 0$, $D_{0+}^{\alpha} u + \lambda_2 q(t) u = 0$, $0 < t < 1$, satisfying boundary conditions $u^{(i)}(0) = 0$, $i = 0, 1, \ldots, n-2$, $D_{0+}^{\beta} u(1) = 0$, $0 \leq \beta \leq n-1$, where p and q are nonnegative continuous functions on $[0,1]$ which do not vanish identically on any nondegenerate compact subinterval of $[0,1]$. The cases where $\beta = 0$ and $\beta > 0$ are treated separately and then compared.

Keywords. Fractional boundary value problem; u_0-positive operator; Smallest eigenvalue.

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1. Introduction

Let $n \in \mathbb{N}$, $n \geq 2$, and $n-1 < \alpha \leq n$. In this paper, we will consider boundary value problems consisting of fractional differential equations

$$D_{0+}^{\alpha} u + \lambda_1 p(t) u = 0, \quad 0 < t < 1, \tag{1}$$

$$D_{0+}^{\alpha} u + \lambda_2 q(t) u = 0, \quad 0 < t < 1, \tag{2}$$

which satisfy the boundary conditions

$$u^{(i)}(0) = 0, \; i = 0, 1, \ldots, n-2, \quad D_{0+}^{\beta} u(1) = 0, \tag{3_β}$$

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$0 \le \beta \le n-1$, where $D_{0^+}^{\alpha}$ and $D_{0^+}^{\beta}$ are the standard Riemann-Liouville derivatives. Here p and q are continuous nonnegative functions on $[0,1]$ that do not vanish identically on any nondegenerate compact subinterval of $[0,1]$. The real numbers λ_1 and λ_2 such that these boundary value problems yield a nontrivial solution are called eigenvalues.

The purpose of this paper is to show the existence of smallest eigenvalues by using the theory of u_0-positive operators with respect to a cone in a Banach space. Then, a comparison of those eigenvalues can be made. The technique for showing the existence and then comparing these smallest eigenvalues involve the application of sign properties of the Green's function for the specified boundary value problem, followed by the application of u_0-positive operators with respect to a cone in a Banach space. These applications are presented in books by Krasnosel'skii [14] and by Krein and Rutman [15].

These cone theoretic techniques have been used by many authors to study the existence of smallest eigenvalues of ordinary boundary value problems, difference equations, and dynamic equations on time scales. See [1, 2, 3, 4, 5, 6, 7, 16, 17] and the references therein for some examples. Recently, Eloe and Neugebauer [9] showed the existence of and compared smallest eigenvalues for fractional boundary value problems with Dirichlet boundary conditions. These results have been used and extended in [10, 11, 12, 18]. Here, we look to extend the results to a fractional boundary value problem with fractional boundary conditions. The cases when $0 < \beta \le n-1$ and when $\beta = 0$ are treated separately.

We point out that the Banach space used in this paper differs from the space used when working with ordinary boundary value problems, even if α and β are integers. This method has is benefits. Many times, when working with higher order problems, the problem needs to be reduced to a lower order problem and then an appropriate Banach space, cone and interior of the cone are chosen. See, for example, [16]. However, with this method, there is no need to reduce the higher order problem to a lower order problem. The boundary conditions here cover many problems that have not been dealt with before.

2. Preliminaries

For completeness, we first introduce the definition of the Riemann-Liouville fractional derivative. For $n \in \mathbb{N}$ and $n-1 < \alpha \leq n$, the α-th Riemann-Liouville fractional derivative of the function $u : [0,1] \to \mathbb{R}$, denoted $D^{\alpha}_{0+}u$, is defined as

$$D^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s) ds,$$

provided the right-hand side exists.

Let $\mathscr{B}$ be a Banach space over $\mathbb{R}$. A closed nonempty subset $\mathscr{P}$ of $\mathscr{B}$ is said to be a cone provided the following:

(i) $\alpha u + \beta v \in \mathscr{P}$, for all $u, v \in \mathscr{P}$ and all $\alpha, \beta \geq 0$, and

(ii) $u \in \mathscr{P}$ and $-u \in \mathscr{P}$ implies $u = 0$.

A cone $\mathscr{P}$ is solid if the interior, $\mathscr{P}^{\circ}$, of $\mathscr{P}$, is nonempty. A cone $\mathscr{P}$ is reproducing if $\mathscr{B} = \mathscr{P} - \mathscr{P}$; i.e., given $w \in \mathscr{B}$, there exist $u, v \in \mathscr{P}$ such that $w = u - v$. Krasnosel'skii [14] proved that every solid cone is reproducing.

Cones generate a natural partial ordering on a Banach space. Let $\mathscr{P}$ be a cone in a real Banach space $\mathscr{B}$. If $u, v \in \mathscr{B}$, $u \leq v$ with respect to $\mathscr{P}$ if $v - u \in \mathscr{P}$. If both $M, N : \mathscr{B} \to \mathscr{B}$ are bounded linear operators, $M \leq N$ with respect to $\mathscr{P}$ if $Mu \leq Nu$ for all $u \in \mathscr{P}$.

A bounded linear operator $M : \mathscr{B} \to \mathscr{B}$ is u_0-positive with respect to $\mathscr{P}$ if there exists $u_0 \in \mathscr{P} \backslash \{0\}$ such that for each $u \in \mathscr{P} \backslash \{0\}$, there exist $k_1(u) > 0$ and $k_2(u) > 0$ such that $k_1 u_0 \leq Mu \leq k_2 u_0$ with respect to $\mathscr{P}$.

The following two results are fundamental to our existence and comparison results and are attributed to Krasnosel'skii [14]. The proof of Theorem 2.1 can be found in [14], and the proof of Theorem 2.2 is provided by Keener and Travis [13] as an extension of Krasonel'skii's results.

Theorem 2.1. *Let $\mathscr{B}$ be a real Banach space and let $\mathscr{P} \subset \mathscr{B}$ be a reproducing cone. Let $L : \mathscr{B} \to \mathscr{B}$ be a compact, u_0-positive, linear operator. Then L has an essentially unique eigenvector in $\mathscr{P}$, and the corresponding eigenvalue is simple, positive and larger than the absolute value of any other eigenvalue.*

Theorem 2.2. *Let $\mathscr{B}$ be a real Banach space and $\mathscr{P} \subset \mathscr{B}$ be a cone. Let both $M, N : \mathscr{B} \to \mathscr{B}$ be bounded, linear operators and assume that at least one of the operators is u_0-positive. If*

$M \leq N$, $Mu_1 \geq \lambda_1 u_1$ for some $u_1 \in \mathscr{P}$ and some $\lambda_1 > 0$, and $Nu_2 \leq \lambda_2 u_2$ for some $u_2 \in \mathscr{P}$ and some $\lambda_2 > 0$, then $\lambda_1 \leq \lambda_2$. Furthermore, $\lambda_1 = \lambda_2$ implies u_1 is a scalar multiple of u_2.

3. The case when $0 < \beta \leq n-1$

First, we consider the case where $\beta > 0$. We derive existence and comparison results by applying the two previous theorems. To do this, we will define integral operators whose kernels are the Green's function for $-D^{\alpha}_{0+}u = 0$, (3_β), which is given by (see [8])

$$G(\beta;t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}, & 0 \leq t \leq s < 1. \end{cases}$$

So u solves (1), (3_β) if and only if

$$u(t) = \lambda_1 \int_0^1 G(\beta;t,s)p(s)u(s)ds.$$

Similarly, u solves (2), (3_β) if and only if

$$u(t) = \lambda_2 \int_0^1 G(\beta;t,s)q(s)u(s)ds.$$

Notice that $G(\beta;t,s) \geq 0$ for $(t,s) \in [0,1] \times [0,1)$ and $G(\beta;t,s) > 0$ for $(t,s) \in (0,1] \times [0,1)$.

Now, define the Banach Space

$$\mathscr{B} = \{u : u = t^{\alpha-1}v,\ v \in C[0,1]\},$$

with the norm

$$\|u\| = |v|_0,$$

where $|v|_0 = \sup_{t\in[0,1]} |v(t)|$ denotes the usual supremum norm. Notice that for $u \in \mathscr{B}$,

$$|u|_0 = |t^{\alpha-1}v|_0 \leq t^{\alpha-1}\|u\|,$$

implying

$$|u|_0 \leq \|u\|.$$

Define the linear operators

$$Mu(t) = \int_0^1 G(\beta;t,s)p(s)u(s)ds, \tag{3}$$

and

$$Nu(t) = \int_0^1 G(\beta;t,s)q(s)u(s)ds. \tag{4}$$

Now,

$$\begin{aligned} Mu(t) &= \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} p(s)u(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds \\ &= t^{\alpha-1}\left(\int_0^1 \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} p(s)u(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds\right). \end{aligned}$$

Notice that since $n-1<\alpha\le n$ and $0<\beta<n-1$,

$$\begin{aligned} \left|\int_0^1 \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} p(s)u(s)ds\right| &\le \frac{|p|_0|v|_0}{\Gamma(\alpha)}\left|\int_0^1 s^{\alpha-1}(1-s)^{\alpha-1-\beta}ds\right| \\ &= \frac{|p|_0|v|_0}{\Gamma(\alpha)}\left|B(\alpha,\alpha-\beta)\right| \\ &= \frac{|p|_0|v|_0}{\Gamma(\alpha)}\left|\frac{\Gamma(\alpha)\Gamma(\alpha-\beta)}{\Gamma(\alpha+\beta)}\right| \\ &= \frac{|p|_0|v|_0\Gamma(\alpha-\beta)}{\Gamma(\alpha+\beta)} \\ &< \infty. \end{aligned}$$

Therefore, the first term inside the parentheses is well-defined.

Set

$$g(t) = \begin{cases} 0, & t=0, \\ t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds, & 0<t\le 1. \end{cases}$$

Then, for $|p_0| = P$, $\|u\| = L$,

$$\begin{aligned}
|g(t)| &= \left| t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds \right| \\
&= \left| t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)s^{\alpha-1}v(s)ds \right| \\
&\leq PLt^{1-\alpha} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} ds \\
&\leq PLt^{1-\alpha} t^{\alpha-1} \int_0^t (t-s)^{\alpha-1} ds \\
&= \frac{PLt^{\alpha}}{\alpha},
\end{aligned}$$

where $\dfrac{PL}{\alpha} \geq 0$. So, $\lim_{t\to 0^+} g(t) = g(0) = 0$. Thus, $g \in C[0,1]$. Therefore, $M : \mathscr{B} \to \mathscr{B}$.

A similar argument to the argument made in [9] shows that M is compact. This can also be applied to N to show that $N : \mathscr{B} \to \mathscr{B}$ is compact, which gives the following theorem.

Theorem 3.1. *The operators $M, N : \mathscr{B} \to \mathscr{B}$ are compact.*

Next, we define the cone

$$\mathscr{P} = \{u \in \mathscr{B} : u(t) \geq 0 \text{ for } t \in [0,1]\}.$$

Lemma 3.2. *The cone $\mathscr{P}$ is solid in $\mathscr{B}$ and hence reproducing.*

Proof. Define

$$\Omega := \{u = t^{\alpha-1}v \in \mathscr{B} : u(t) > 0 \text{ for } t \in (0,1],\ v(0) > 0\} \tag{5}$$

We will show that $\Omega \subset \mathscr{P}^{\circ}$. Since $v(0) > 0$, there exists an $\varepsilon_1 > 0$ such that $v(0) - \varepsilon_1 > 0$. Since $v \in C[0,1]$, there exists an $a \in (0,1)$ such that $v(t) > \varepsilon_1$ for all $t \in (0,a)$. So $u(t) = t^{\alpha-1}v(t) > \varepsilon_1 t^{\alpha-1}$ for all $t \in (0,a)$. Now, on the interval $[a,1]$, $u(t) > 0$. Thus there exists an $\varepsilon_2 > 0$ with $u(t) - \varepsilon_2 > 0$ for all $t \in [a,1]$.

Let $\varepsilon = \min\{\frac{\varepsilon_1}{2}, \frac{\varepsilon_2}{2}\}$. Define $B_\varepsilon(u) = \{\hat{u} \in \mathscr{B} : \|u - \hat{u}\| < \varepsilon\}$. Let $\hat{u} \in B_\varepsilon(u)$. So $\hat{u} = t^{\alpha-1}\hat{v}$, where $\hat{v} \in C[0,1]$. Now $|\hat{u}(t) - u(t)| \leq t^{\alpha-1}\|\hat{u} - u\| < \varepsilon t^{\alpha-1}$. So for $t \in (0,a)$, $\hat{u}(t) > u(t) - t^{\alpha-1}\varepsilon > t^{\alpha-1}\varepsilon_1 - t^{\alpha-1}\varepsilon_1/2 = t^{\alpha-1}\varepsilon_1/2$. So $\hat{u}(t) > 0$ for $t \in (0,a)$. Also, $|\hat{u}(t) - u(t)| \leq \|\hat{u} - u\| < \varepsilon$. So for $t \in [a,1]$, $\hat{u}(t) > u(t) - \varepsilon > \varepsilon_2 - \varepsilon_2/2 > 0$. So $\hat{u}(t) > 0$ for all $t \in [a,1]$. So $\hat{u} \in \mathscr{P}$ and thus $B_\varepsilon(u) \subset \mathscr{P}$. So $\Omega \subset \mathscr{P}^{\circ}$ and the proof is complete.

Lemma 3.3. *The bounded linear operators M and N are u_0-positive with respect to $\mathscr{P}$.*

Proof. First, we show $M : \mathscr{P}\backslash\{0\} \to \Omega \subset \mathscr{P}^\circ$. Let $u \in \mathscr{P}$. So $u(t) \geq 0$ on $[0,1]$. Then since $G(\beta;t,s) \geq 0$ on $[0,1]\times[0,1)$, and $p(t) \geq 0$ on $[0,1]$,

$$Mu(t) = \int_0^1 G(\beta;t,s)p(s)u(s)ds \geq 0,$$

for $0 \leq t \leq 1$. So $M : \mathscr{P} \to \mathscr{P}$.

Now, let $u \in \mathscr{P}\backslash\{0\}$. So there exists a compact interval $[a,b] \subset [0,1]$ such that $u(t) > 0$ and $p(t) > 0$ for all $t \in [a,b]$. Then, since $G(\beta;t,s) > 0$ on $(0,1]\times(0,1)$,

$$\begin{aligned} Mu(t) &= \int_0^1 G(\beta;t,s)p(s)u(s)ds \\ &\geq \int_a^b G(\beta;t,s)p(s)u(s)ds \\ &> 0, \end{aligned}$$

for $0 < t \leq 1$.

Now,

$$Mu(t) = t^{\alpha-1}\left(\int_0^1 \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}p(s)u(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds\right).$$

Let

$$v(t) = \int_0^1 \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}p(s)u(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds.$$

Thus, $v(0) = \displaystyle\int_0^1 \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}p(s)u(s)ds > 0$. So $M : \mathscr{P}\backslash\{0\} \to \Omega \subset \mathscr{P}^\circ$.

Now, choose $u_0 \in \mathscr{P}\backslash\{0\}$, and let $u \in \mathscr{P}\backslash\{0\}$. So $Mu \in \Omega \subset \mathscr{P}^\circ$. Choose $k_1 > 0$ sufficiently small and k_2 sufficiently large so that $Mu - k_1u_0 \in \mathscr{P}^\circ$ and $u_0 - \frac{1}{k_2}Mu \in \mathscr{P}^\circ$. So $k_1u_0 \leq Mu$ with respect to $\mathscr{P}$ and $Mu \leq k_2u_0$ with respect to $\mathscr{P}$. Thus $k_1u_0 \leq Mu \leq k_2u_0$ with respect to $\mathscr{P}$ and so M is u_0-positive with respect to P. A similar argument shows N is u_0-positive. This completes the proof.

Theorem 3.4. *Let $\mathscr{B}$, $\mathscr{P}$, M, and N be defined as earlier. Then M (and N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathscr{P}^\circ$.*

Proof. Since M is a compact linear operator that is u_0-positive with respect to $\mathscr{P}$, by Theorem 2.1, M has an essentially unique eigenvector, say $u \in \mathscr{P}$, and eigenvalue Λ with the above properties. Since $u \neq 0$, $Mu \in \Omega \subset \mathscr{P}^\circ$ and $u = M\left(\frac{1}{\Lambda}u\right) \in \mathscr{P}^\circ$.

Theorem 3.5. *Let $\mathscr{B}$, $\mathscr{P}$, M, and N be defined as earlier. Let $p(t) \leq q(t)$ on $[0,1]$. Let Λ_1 and Λ_2 be the eigenvalues defined in Theorem 3.4 associated with M and N, respectively, with the essentially unique eigenvectors u_1 and $u_2 \in \mathscr{P}^\circ$. Then $\Lambda_1 \leq \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $p(t) = q(t)$ on $[0,1]$.*

Proof. Let $p(t) \leq q(t)$ on $[0,1]$. So for any $u \in \mathscr{P}$ and $t \in [0,1]$,

$$(Nu - Mu)(t) = \int_0^1 G(\beta;t,s)(q(s)-p(s))u(s)ds \geq 0.$$

So $Nu - Mu \in \mathscr{P}$ for all $u \in \mathscr{P}$, or $M \leq N$ with respect to $\mathscr{P}$. Then, by Theorem 2.2, $\Lambda_1 \leq \Lambda_2$.

If $p(t) = q(t)$, then $\Lambda_1 = \Lambda_2$. Now suppose $p(t) \neq q(t)$. So $p(t) < q(t)$ on some subinterval $[a,b] \subset [0,1]$, which implies $(N-M)u_1(t) > 0$ for $t \in (0,1]$. Let $(N-M)u_1(t) = t^{\alpha-1}v(t)$. So

$$v(t) = \int_0^1 \frac{(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}(q(s)-p(s))u_1(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(q(s)-p(s))u_1(s)ds.$$

Since $p(t) < q(t)$ on $[a,b] \subset [0,1]$, then $v(0) > 0$. So, $(N-M)u_1 \in \Omega \subset \mathscr{P}^\circ$. So there exists $\varepsilon > 0$ such that $(N-M)u_1 - \varepsilon u_1 \in \mathscr{P}$. So $\Lambda_1 u_1 + \varepsilon u_1 = Mu_1 + \varepsilon u_1 \leq Nu_1$, implying $Nu_1 \geq (\Lambda_1 + \varepsilon)u_1$. Since $N \leq N$ and $Nu_2 = \Lambda_2 u_2$, by Theorem 2.2, $\Lambda_1 + \varepsilon \leq \Lambda_2$, or $\Lambda_1 < \Lambda_2$.

Lemma 3.6. *The eigenvalues of* (1), (3_β) *are reciprocals of eigenvalues of M, and conversely. Similarly, eigenvalues of* (2), (3_β) *are reciprocals of eigenvalues of N, and conversely.*

Proof. Let Λ be an eigenvalue of M with corresponding eigenvector $u(t)$. Notice that

$$\Lambda u(t) = Mu(t) = \int_0^1 G(\beta;t,s)p(s)u(s)ds,$$

if and only if

$$u(t) = \frac{1}{\Lambda}\int_0^1 G(\beta;t,s)p(s)u(s)ds,$$

if and only if

$$D_{0+}^{\alpha}u(t) + \frac{1}{\Lambda}p(t)u(t) = 0, \quad 0 < t < 1,$$

with

$$u^{(i)}(0) = 0,\ i = 0,1,\ldots,n-2, \quad D_{0+}^{\beta}u(1) = 0.$$

So $\dfrac{1}{\Lambda}$ is an eigenvalue of (1), (3_β), if and only if Λ is an eigenvalue of M. A similar argument can be made that the reciprocals of eigenvalues of N are eigenvalues of (2), (3_β) and vice versa, which completes the proof.

Since the eigenvalues of (1), (3_β) are reciprocals of eigenvalues of M and conversely, and the eigenvalues of (2), (3_β) are reciprocals of eigenvalues of N and conversely, the following theorem is an immediate consequence of Theorems 3.4 and 3.5.

Theorem 3.7. *Assume the hypotheses of Theorem 3.5. Then there exists smallest positive eigenvalues λ_1 and λ_2 of* (1), (3_β) *and* (2), (3_β), *respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to λ_1 and λ_2 may be chosen to belong to $\mathscr{P}^\circ$. Finally, $\lambda_1 \geq \lambda_2$, and $\lambda_1 = \lambda_2$ if and only if $p(t) = q(t)$ for all $t \in [0,1]$.*

4. The case when $\beta = 0$

Next, we consider the case where $\beta = 0$. Again, we will define integral operators whose kernels are the Green's function for $-D^\alpha_{0+}u = 0$, (3_0), which is given by

$$G(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s < 1. \end{cases} \tag{6}$$

Therefore, $u(t) = \lambda_1 \int_0^1 G(t,s)p(s)u(s)ds$ if and only if u solves (1), (3_0). Similarly, $u(t) = \lambda_2 \int_0^1 G(t,s)q(s)u(s)ds$ if u solves (2), (3_0). Notice that $G(t,s) \geq 0$ on $[0,1]\times[0,1)$ and $G(t,s) > 0$ on $(0,1)\times(0,1)$. We point out that $G(1,s) = 0$, so the Banach space $\mathbb{B}$ and the interior of the cone Ω used in the previous section are not appropriate for this problem.

Define the Banach Space

$$\mathscr{B} = \{u : u = t^{\alpha-1}v,\ v \in C^{(1)}[0,1],\ v(1) = 0\},$$

with the norm

$$\|u\| = |v'|_0.$$

Notice that,

$$|v(t)| = |v(t) - v(1)| = \left|\int_1^t v'(s)ds\right| \leq (1-t)|v'|_0 \leq \|u\|.$$

Therefore, $|v|_0 \le \|u\| = |v'|_0$ and

$$|u|_0 = |t^{\alpha-1}v|_0 \le t^{\alpha-1}\|u\|,$$

implying

$$|u|_0 \le \|u\|.$$

Define the linear operators

$$Mu(t) = \int_0^1 G(t,s)p(s)u(s)ds, \tag{7}$$

and

$$Nu(t) = \int_0^1 G(t,s)q(s)u(s)ds. \tag{8}$$

Theorem 4.1. *The operators $M, N : \mathscr{B} \to \mathscr{B}$ are compact.*

Proof. First, we show $M : \mathscr{B} \to \mathscr{B}$. Let $u \in \mathscr{B}$. So there is a $v \in C^{(1)}[0,1]$ such that $u = t^{\alpha-1}v$. Since $v \in C^{(1)}[0,1]$ and $p \in C[0,1]$, let $L = |v|_0$ and $P = |p|_0$. Now,

$$\begin{aligned} Mu(t) &= \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds \\ &= t^{\alpha-1}\left(\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds\right). \end{aligned}$$

Define

$$g(t) = \begin{cases} 0, & t = 0, \\ t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds, & 0 < t \le 1. \end{cases}$$

Notice $g \in C^{(1)}(0,1]$. Now,

$$\begin{aligned} |g(t)| &= \left| t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)u(s)ds \right| \\ &= \left| t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} p(s)s^{\alpha-1}v(s)ds \right| \\ &\le PLt^{1-\alpha}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}ds \\ &\le PLt^{1-\alpha}t^{\alpha-1}\int_0^t (t-s)^{\alpha-1}ds \\ &= \frac{PLt^{\alpha}}{\alpha}, \end{aligned}$$

where $\frac{PL}{\alpha} \geq 0$. Thus, $\lim_{t\to 0^+} g(t) = g(0) = 0$ and $g \in C[0,1]$. Also, for $t > 0$,

$$\begin{aligned}
|g'(t)| &= \left|(1-\alpha)t^{-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds + (\alpha-1)t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)}p(s)u(s)ds\right| \\
&\leq \left|(1-\alpha)t^{-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)s^{\alpha-1}v(s)ds\right| \\
&\quad + \left|(\alpha-1)t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)}p(s)s^{\alpha-1}v(s)(s)ds\right| \\
&\leq (\alpha-1)PLt^{-\alpha}t^{\alpha-1}\int_0^t (t-s)^{\alpha-1}ds + (\alpha-1)PLt^{1-\alpha}t^{\alpha-1}\int_0^t (t-s)^{\alpha-2}ds \\
&= (\alpha-1)PL\left(t^{-1}\int_0^t (t-s)^{\alpha-1}ds + \int_0^t (t-s)^{\alpha-2}ds\right) \\
&= (\alpha-1)PL\left(\frac{t^{\alpha-1}}{\alpha} + \frac{t^{\alpha-1}}{\alpha-1}\right) \\
&= \left(\frac{\alpha-1}{\alpha}+1\right)PLt^{\alpha-1}.
\end{aligned}$$

So, $\lim_{t\to 0^+} g'(t) = 0$. Moreover, using the definition of derivative and L'Hôpital's rule,

$$g'(0) = \lim_{t\to 0^+}\frac{g(t)-g(0)}{t} = \lim_{t\to 0^+}\frac{g(t)}{t} = \lim_{t} \to 0^+ g'(t) = 0.$$

So $g' \in C[0,1]$.

Now let

$$\hat{v}(t) = \int_0^t \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds.$$

Then, $\hat{v}(1) = 0$. Thus, $Mu \in \mathscr{B}$. So, $M : \mathscr{B} \to \mathscr{B}$. A similar argument can be made that $N : \mathscr{B} \to \mathscr{B}$. An argument similar to the one made in [9] shows M and N are compact.

Next, we define the cone

$$\mathscr{P} = \{u \in \mathscr{B} : u(t) \geq 0 \text{ for } t \in [0,1]\}.$$

Lemma 4.2. *The cone $\mathscr{P}$ is solid in $\mathscr{B}$ and hence reproducing.*

Proof. Define

$$\Omega := \{u = t^{\alpha-1}v \in \mathscr{B} : u(t) > 0, \text{ for } t \in (0,1),\ v(0) > 0,\ v'(1) < 0\} \tag{9}$$

We will show $\Omega \subset \mathscr{P}^{\circ}$. Let $u \in \Omega$. Since $v(0) > 0$, there exists an $\varepsilon_1 > 0$ such that $v(0) - \varepsilon_1 > 0$. Since $v \in C[0,1]$, there exists an $a \in (0,1)$ such that $v(t) > \varepsilon_1$ for all $t \in (0,a)$. Thus, $u(t) = t^{\alpha-1}v(t) > \varepsilon_1 t^{\alpha-1}$ for all $t \in (0,a)$. Now, since $v'(1) < 0$, there exists an $\varepsilon_2 > 0$ such that $v'(1) + \varepsilon_2 < 0$, implying that $-v'(1) > \varepsilon_2$. Then, by the definition of derivative, $\lim_{t\to 1^-} \dfrac{-v(t)+v(1)}{t-1} > \varepsilon_2$. Since $v(1) = 0$, $\lim_{t\to 1^-} \dfrac{v(t)}{1-t} > \varepsilon_2$. Thus, there exists a $b \in (a,1)$ such that for $t \in (b,1)$, $\dfrac{v(t)}{1-t} > \varepsilon_2$. This implies $v(t) > (1-t)\varepsilon_2$. Therefore, $u(t) > b^{\alpha-1}(1-t)\varepsilon_2$ for all $t \in (b,1)$. Also, since $u(t) > 0$ on $[a,b]$, there exists an $\varepsilon_3 > 0$ such that $u(t) - \varepsilon_3 > 0$ for all $t \in [a,b]$.

Let $\varepsilon = \min\left\{\dfrac{\varepsilon_1}{2}, \dfrac{b^{\alpha-1}\varepsilon_2}{2}, \dfrac{\varepsilon_3}{2}\right\}$. Define $B_\varepsilon(u) = \{\hat{u} \in \mathscr{B} : \|u - \hat{u}\| < \varepsilon\}$. Let $\hat{u} \in B_\varepsilon(u)$. Thus, $\hat{u} = t^{\alpha-1}\hat{v}$, where $\hat{v} \in C^{(1)}[0,1]$ with $\hat{v}(1) = 0$. Now

$$|\hat{u}(t) - u(t)| \leq t^{\alpha-1}\|\hat{u} - u\| < \varepsilon t^{\alpha-1}.$$

So, for $t \in (0,a)$, $\hat{u}(t) > u(t) - t^{\alpha-1}\varepsilon > t^{\alpha-1}\varepsilon_1 - t^{\alpha-1}\varepsilon_1/2 = t^{\alpha-1}\varepsilon_1/2$. So, $\hat{u}(t) > 0$ for $t \in (0,a)$. By the Mean Value Theorem, there exists $c \in (t,1)$ such that

$$\frac{\hat{v}(1) - v(1) - \hat{v}(t) + v(t)}{1-t} = \hat{v}'(c) - v'(c).$$

Since $\hat{v}(1) = 0$ and $v(1) = 0$, then

$$\left|\frac{v(t) - \hat{v}(t)}{1-t}\right| = \left|\hat{v}'(c) - v'(c)\right| \leq \left|\hat{v}' - v'\right|_0.$$

However,

$$\left|\frac{u(t) - \hat{u}(t)}{1-t}\right| \leq \left|\frac{v(t) - \hat{v}(t)}{1-t}\right|.$$

So, $|u(t) - \hat{u}(t)| \leq (1-t)\|\hat{u} - u\| < (1-t)\varepsilon$, for $t \in (b,1)$. Thus, for $t \in (b,1)$

$$\hat{u}(t) > u(t) - (1-t)\varepsilon > b^{\alpha-1}(1-t)\varepsilon_2 - (1-t)b^{\alpha-1}\varepsilon - 2/2 = (1-t)b^{\alpha-1} > 0.$$

Therefore, for $t \in (b,1)$, $\hat{u}(t) > 0$. Also, $|\hat{u}(t) - u(t)| \leq ||\hat{u} - u|| < \varepsilon$. So for $t \in [a,b], \hat{u}(t) > u(t) - \varepsilon > \varepsilon_3 - \varepsilon_3/2 > 0$. So, $\hat{u}(t) > 0$ for all $t \in [a,b]$. So, $\hat{u} \in \mathscr{P}$ and therefore $B_\varepsilon(u) \subset \mathscr{P}$. Thus, $\Omega \subset \mathscr{P}^{\circ}$, completing the proof.

Lemma 4.3. *The bounded linear operators M and N are u_0-positive with respect to $\mathscr{P}$.*

Proof. First, we show $M : \mathscr{P}\backslash\{0\} \to \Omega \subset \mathscr{P}^{\circ}$. Let $u \in \mathscr{P}$. So $u(t) \geq 0$ on $[0,1]$. Then since $G(t,s) \geq 0$ on $[0,1]\times[0,1)$ and $p(t) \geq 0$ on $[0,1]$,

$$Mu(t) = \int_0^1 G(t,s)p(s)u(s)ds \geq 0,$$

for $0 \leq t \leq 1$. So $M : \mathscr{P} \to \mathscr{P}$.

Now, let $u \in \mathscr{P}\backslash\{0\}$. So there exists a compact interval $[a,b] \subset [0,1]$ such that $u(t) > 0$ and $p(t) > 0$ for all $t \in [a,b]$. Then, since $G(t,s) > 0$ on $(0,1)\times(0,1)$,

$$\begin{aligned} Mu(t) &= \int_0^1 G(t,s)p(s)u(s)ds \\ &\geq \int_a^b G(t,s)p(s)u(s)ds \\ &> 0, \end{aligned}$$

for $0 < t < 1$. Now,

$$Mu(t) = t^{\alpha-1}\left(\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds\right).$$

Let

$$v(t) = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds.$$

Thus, $v(0) = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds > 0$ and

$$\begin{aligned} v'(1) &= (1-\alpha)\left(\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)}p(s)u(s)ds - \int_0^t \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)u(s)ds\right) \\ &= (1-\alpha)\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)}p(s)u(s)\left(1-(1-s)\right)ds \\ &< 0. \end{aligned}$$

So $M : \mathscr{P}\backslash\{0\} \to \Omega \subset \mathscr{P}^{\circ}$.

Now, choose $u_0 \in \mathscr{P}\backslash\{0\}$, and let $u \in \mathscr{P}\backslash\{0\}$. So $Mu \in \Omega \subset \mathscr{P}^{\circ}$. Choose $k_1 > 0$ sufficiently small and k_2 sufficiently large so that $Mu - k_1u_0 \in \mathscr{P}^{\circ}$ and $u_0 - \frac{1}{k_2}Mu \in \mathscr{P}^{\circ}$. So $k_1u_0 \leq Mu$ with respect to $\mathscr{P}$ and $Mu \leq k_2u_0$ with respect to $\mathscr{P}$. Thus $k_1u_0 \leq Mu \leq k_2u_0$ with respect to $\mathscr{P}$ and so M is u_0-positive with respect to P. A similar argument shows N is u_0-positive.

Theorem 4.4. *Let $\mathscr{B}$, $\mathscr{P}$, M, and N be defined as earlier. Then M (and N) has an eigenvalue that is simple, positive, and larger than the absolute value of any other eigenvalue, with an essentially unique eigenvector that can be chosen to be in $\mathscr{P}^\circ$.*

Proof. Since M is a compact linear operator that is u_0-positive with respect to $\mathscr{P}$, by Theorem 2.1, M has an essentially unique eigenvector, say $u \in \mathscr{P}$, and eigenvalue Λ with the above properties. Since $u \neq 0$, $Mu \in \Omega \subset \mathscr{P}^\circ$ and $u = M\left(\frac{1}{\Lambda}u\right) \in \mathscr{P}^\circ$.

Theorem 4.5. *Let $\mathscr{B}$, $\mathscr{P}$, M, and N be defined as earlier. Let $p(t) \leq q(t)$ on $[0,1]$. Let Λ_1 and Λ_2 be the eigenvalues defined in Theorem 3.4 associated with M and N, respectively, with the essentially unique eigenvectors u_1 and $u_2 \in \mathscr{P}^\circ$. Then $\Lambda_1 \leq \Lambda_2$, and $\Lambda_1 = \Lambda_2$ if and only if $p(t) = q(t)$ on $[0,1]$.*

Proof. Let $p(t) \leq q(t)$ on $[0,1]$. So for any $u \in \mathscr{P}$ and $t \in [0,1]$,

$$(Nu - Mu)(t) = \int_0^1 G(t,s)(q(s) - p(s))u(s)ds \geq 0.$$

So $Nu - Mu \in \mathscr{P}$ for all $u \in \mathscr{P}$, or $M \leq N$ with respect to $\mathscr{P}$. Then, by Theorem 2.2, $\Lambda_1 \leq \Lambda_2$.

If $p(t) = q(t)$, then $\Lambda_1 = \Lambda_2$. Now suppose $p(t) \neq q(t)$. So $p(t) < q(t)$ on some subinterval of $[0,1]$. So $(N-M)u_1(t) > 0$ for $t \in (0,1)$. Let $(N-M)u_1(t) = t^{\alpha-1}v(t)$. Thus

$$v(t) = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}(q(s) - p(s))u_1(s)ds - t^{1-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(q(s) - p(s))u_1(s)ds.$$

Then, $v(0) > 0$ and

$$v'(1) = (1-\alpha)\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)}(q(s) - p(s))\,u_1(s)ds - \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} < 0.$$

So, $(N-M)u_1 \in \Omega \subseteq \mathscr{P}^\circ$. Then there exists $\varepsilon > 0$ such that $(N-M)u_1 - \varepsilon u_1 \in \mathscr{P}$. So $\Lambda_1 u_1 + \varepsilon u_1 = Mu_1 + \varepsilon u_1 \leq Nu_1$, implying $Nu_1 \geq (\Lambda_1 + \varepsilon)u_1$. Since $N \leq N$ and $Nu_2 = \Lambda_2 u_2$, by Theorem 2.2, $\Lambda_1 + \varepsilon \leq \Lambda_2$, or $\Lambda_1 < \Lambda_2$, which completes the proof.

The proof of the following lemma is similar to that of Lemma 3.6, and is therefore omitted.

Lemma 4.6. *The eigenvalues of (1), (3_0) are reciprocals of eigenvalues of M, and conversely. Similarly, eigenvalues of (2), (3_0) are reciprocals of eigenvalues of N, and conversely.*

Again, since the eigenvalues of (1), (3_0) are reciprocals of eigenvalues of M and conversely, and the eigenvalues of (2), (3_0) are reciprocals of eigenvalues of N and conversely, the following theorem is an immediate consequence of Theorems 4.4 and 4.5.

Theorem 4.7 *Assume the hypotheses of Theorem 3.5. Then there exists smallest positive eigenvalues λ_1 and λ_2 of* (1), (3_0) *and* (2), (3_0), *respectively, each of which is simple, positive, and less than the absolute value of any other eigenvalue of the corresponding problems. Also, eigenfunctions corresponding to λ_1 and λ_2 may be chosen to belong to $\mathscr{P}^\circ$. Finally, $\lambda_1 \geq \lambda_2$, and $\lambda_1 = \lambda_2$ if and only if $p(t) = q(t)$ for all $t \in [0,1]$.*

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