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THE GEOMETRY OF BANACH SPACES AND FIXED POINTS OF NON-EXPANSIVE MAPPINGS

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Abstract. Let X be a Banach space, and $B(X)$ and $S(X)$ be the unit ball and unit sphere of X. In this paper, we study the further properties of some known geometric parameters that related to convexity of $B(X)$, inscribed parallelograms and curves on $S(X)$. The relationships of values of these parameters with normal structures, uniformly non-square are obtained. Some existing results on fixed points of non-expansive mappings are improved.

Keywords. Fixed point; Normal structure; Ultra-product; Uniformly non-square Banach space.

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1. Introduction

Let X be a normed linear space. Let $B(X) = \{x \in X : \|x\| \leq 1\}$ and $S(X) = \{x \in X : \|x\| = 1\}$ be the unit ball, and the unit sphere of X, respectively. Let X^* be the dual space of X.

Brodskiĭ and Mil'man [2] introduced the following geometric concepts in 1948.

Definition 1.1. A bounded and convex subset K of a Banach space X is said to have normal structure if every convex subset H of K that contains more than one point contains a point $x_0 \in H$, such that $\sup\{\|x_0 - y\| : y \in H\} < d(H)$, where $d(H) = \sup\{\|x - y\| : x, y \in H\}$ denotes the diameter of H. A Banach space X is said to have the normal structure if every bounded and convex subset of X has normal structure. A Banach space X is said to have the weak

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normal structure if for each weakly compact convex set K in X has normal structure. X is said to have uniform normal structure if there exists $0 < c < 1$ such that for any subset bounded closed convex subset K of X that contains more than one point, there exists $x_0 \in K$ such that $\sup\{\|x_0 - y\| : y \in K\} \leq c \cdot d(K)$.

For a reflexive Banach space, the normal structure and weak normal structure coincide.

Kirk [16] proved that if a Banach space X has weak normal structure then it has weak fixed point property, that is, every non-expansive mapping from a weakly compact and convex subset of X into itself has a fixed point.

In this paper, we study the further properties of some known geometric parameters that related to convexity of $B(X)$, inscribed parallelograms and curves on $S(X)$. The relationships of values of these parameters with normal structure, uniformly non-square are obtained. Finally we study uniformly normal structure in section 3. Some existing results about fixed points of non-expansive mapping are improved.

2. Preliminaries and main results

Definition 2.1. [11] Let X be a Banach space. A hexagon H in X is called a normal hexagon if the length of each side is 1 and each pair of two opposite sides are parallel.

Lemma 2.1. [13] *Let X be a Banach space without weak normal structure. Then for any $0 < \delta < 1$, there are x_1, x_2, and x_3 in $S(X)$ satisfying*

(i) $x_2 - x_3 = x_1$;

(ii) $\|\frac{x_1+x_2}{2}\| > 1 - \delta$; *and*

(iii) $\|\frac{x_3+(-x_1)}{2}\| > 1 - \delta$.

The geometric meaning of lemma 2.1 is that if a Banach space X fails to have weak normal structure then there is an inscribed normal hexagons with four sides arbitrarily closed to the unit sphere $S(X)$.

Lemma 2.2. [12] *If $x, y \in B(X)$ and $0 < \varepsilon < 1$ are such that $\frac{\|x+y\|}{2} > 1 - \varepsilon$, then for all $0 \leq c \leq 1$ and $z = cx + (1-c)y \in [x, y]$, the line segment connecting x and y, it follows that $\|z\| > 1 - 2\varepsilon$.*

Let $\delta(\varepsilon) = \inf\{1 - \frac{\|x+y\|}{2} : x, y \in S(X), \|x-y\| \geq \varepsilon\}$ where $0 \leq \varepsilon \leq 2$ be the modulus of convexity of X [6], and $C(\varepsilon) = \sup\{1 - \frac{\|x+y\|}{2} : x, y \in S(X), \|x-y\| \leq \varepsilon\}$ where $0 \leq \varepsilon \leq 2$ be the modulus of flatness of X ([10], [14]);

$J(X) = \sup\{\|x+y\| \bigwedge \|x-y\| : x, y \in S(X)\}$, and $g(X) = \inf\{\|x+y\| \bigvee \|x-y\| : x, y \in S(X)\}$ [7];

$E(X) = \sup\{\|x+y\|^2 + \|x-y\|^2, x, y \in S(X)\}$, and $f(X) = \inf\{\|x+y\|^2 + \|x-y\|^2, x, y \in S(X)\}$ be the Pythagorean constants [11];

$r(X) = \sup\{2(\|x+y\| + \|x-y\|) : x, y \in S(X)\}$ [8]; and

$A_1(X) = \frac{1}{2} inf_{x \in S(X)}\{sup_{y \in S(X)}(\|x+y\| + \|x-y\|)\}$, and $A_2(X) = \frac{1}{2} sup_{x \in S(X)}\{sup_{y \in S(X)}(\|x+y\| + \|x-y\|)\}$[1].

Actually, $r(X) = 4A_2(X)$.

A curve in a Banach space X is a continuous mapping $x : [a,b] \to X$ and in this case it is denoted by $C := \{x(t) : a \leq t \leq b\}$. A curve is called simple if it does not have multiple points. A curve is called closed if $x(a) = x(b)$. A closed curve is called symmetric about the origin if $x \in C$, then also $-x \in C$.

The concept of the length of a curve in Banach spaces resembles the same concept in Euclidean spaces. For a curve $C = \{x(t) : t \in [a,b]\}$ and a partition $P := \{t_0, t_1, t_2, \ldots, t_n\} \subset [a,b]$ where

$$a = t_0 < t_1 < t_2 < \ldots < t_i < \ldots < t_n = b,$$

let

$$l(C,P) = \sum_{i=1}^{n} \|x(t_i) - x(t_{i-1})\|.$$

The length $l(C)$ of a curve C is defined as the least upper bound of $l(C,P)$ for all partitions P of $[a,b]$, that is,

$$l(C) = \sup\{l(C,P) : P \text{ is a partition of } [a,b]\}.$$

If $l(C)$ is finite, then the curve C is called rectifiable.

Theorem 2.1. ([3], [18]) *Let X_2 be a two dimensional Banach space and K_1, K_2 be closed convex subsets of X_2 with nonvoid interiors. If $K_1 \subseteq K_2$, then $l(\partial(K_1)) \leq l(\partial(K_2))$, where $l(\partial(K_i))$ denote the lengths of the circumferences of $K_i, i = 1, 2$.*

For a normed linear space X, it is clear that $S(X_2)$ is a simple closed curve which is symmetric about the origin and unique up to orientation.

Theorem 2.2. [18] *Let X_2 be a two dimensional Banach space. The following statements are true:*

(1) $6 \leq l(S(X_2)) \leq 8$*;*
(2) $l(S(X_2)) = 8$ *if and only if* $S(X_2)$ *is a parallelogram;*
(3) $l(S(X_2)) = 6$ *if and only if* $S(X_2)$ *is an affinely regular hexagon.*

Let $O(X) = inf\{l(S(X_2), X_2 \subset X\}$ and $Q(X) = sup\{l(S(X_2), X_2 \subset X\}$ [10].

Definition 2.2. [14] Let X be a normed space. Define

$$d(X) := \inf \left\{ \begin{array}{l} \max\left\{1 - \|\frac{x_1+x_2}{2}\|, 1 - \|\frac{x_2+x_3}{2}\|, 1 - \|\frac{x_3+(-x_1)}{2}\|\right\} : \\ x_1, x_2, x_3, -x_1, -x_2, \text{ and } -x_3 \text{ are counterclockwise} \\ \text{vertices of an inscribed normal hexagon in a two} \\ \text{dimensional subspace of } X \end{array} \right\}.$$

Remark 2.1.

(1) $0 \leq \delta(1) \leq d(X) \leq C(1) \leq \frac{1}{2}$;
(2) For a Hilbert space H, $\delta(1) = d(X) = C(1) = 1 - \frac{\sqrt{3}}{2}$;
(3) Let X be l_p, or L_p space with $p, q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$, then $d(X) \leq C(1) \leq 1 - \frac{(2^p-1)^{\frac{1}{p}}}{2}$ for $p \leq 2$; and $d(X) \geq \delta(1) \geq 1 - \frac{(2^p-1)^{\frac{1}{p}}}{2}$ for $p > 2$.

Proof. The proof of (3) of Remark 2.1: From Clarkson inequality ([4], [5]): $\|x+y\|^p + \|x-y\|^p \geq (\|x\| + \|y\|)^p + (\|x\| - \|y\|)^p$ for all $x, y \in X$ and $1 < p \leq 2$, we have $\|x+y\| \geq (2^p-1)^{\frac{1}{p}}$. Therefore, 1-$\frac{\|x+y\|}{2} \leq 1 - \frac{(2^p-1)^{\frac{1}{p}}}{2}$. From definition of d(X) and $C(1)$, we have $d(X) \leq C(1) \leq 1 - \frac{(2^p-1)^{\frac{1}{p}}}{2}$.

From Clarkson inequality $\|x+y\|^p + \|x-y\|^p \leq (\|x\| + \|y\|)^p + (\|x\| - \|y\|)^p$ for all $x, y \in X$ and $p \geq 2$, we have $\|x+y\| \leq (2^p-1)^{\frac{1}{p}}$. Therefore, 1-$\frac{\|x+y\|}{2} \geq 1 - \frac{(2^p-1)^{\frac{1}{p}}}{2}$. From definition of d(X) and $\delta(1)$, we have $d(X) \geq \delta(1) \geq 1 - \frac{(2^p-1)^{\frac{1}{p}}}{2}$.

From Theorem 2.20 and Theorem 2.22 of [14], we have the following.

Theorem 2.3. *If a Banach space X fails to have weak normal structure, then for any $\varepsilon > 0$, there exists a two dimensional subspace X_2 of X such that*

(i) $l(S(X_2)) - \varepsilon \leq r(X_2) \leq l(S(X_2))$;
(ii) $4 + \frac{2}{1-d(X)} - \varepsilon \leq l(S(X_2)) \leq 4 + \frac{2}{1-d(X)} + \varepsilon$.

From Theorem 2.3, we have the following result.

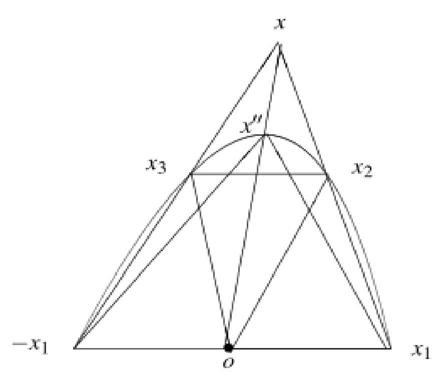
Corollary 2.1 *For a Banach space X, if $r(X) < 4 + \frac{2}{1-d(X)}$ or $A_2(X) < 1 + \frac{1}{2(1-d(X))}$, then X has weak normal structure.*

From Theorem 2.3, we also have the following result.

Corollary 2.2 *For a Banach space X, either $Q(X) < 4 + \frac{2}{1-d(X)}$, or $O(X) > 4 + \frac{2}{1-d(X)}$ then X has weak normal structure.*

Theorem 2.4. *For a Banach space X, if $J(X) < 1 + \frac{1}{2(1-d(X))}$, then X has weak normal structure.*

Proof. The idea of the proof is similar to the proof of Theorem 2. 20 of [14]. Suppose X does not have weak normal structure. For $\varepsilon > 0$, let x_1, x_2, and x_3 in $S(X)$ satisfying the three conditions in Lemma 2.1. Let X_2 be the two dimensional space spanned by x_1, x_2 and x_3. (See the Figure below).



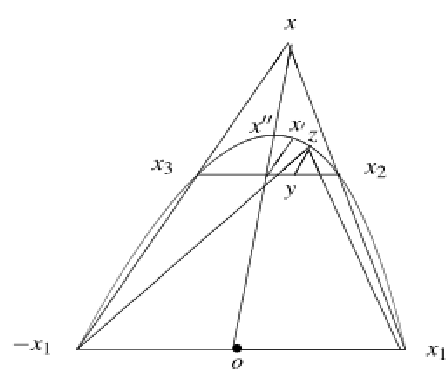
Let $x = -x_1 + 2(x_3 + x_1) = x_1 + 2(x_2 - x_1) \in X_2$, and let $x'' = [0,x] \bigcap S(X_2)$. Then we can write x'' as $x'' = \beta \frac{x_2+x_3}{2} = \beta \frac{2x_2-x_1}{2}$, where $2 \geq \beta \geq \frac{1}{1-d(X)}$, and $x'' - x_1 = \beta \frac{x_2+x_3}{2} - x_1 = \beta \frac{2x_2-x_1}{2} - x_1 = \beta x_2 - (\frac{\beta}{2} + 1)x_1 = (\frac{\beta}{2} + 1)(\frac{2\beta}{\beta+2} x_2 - x_1)$. Since $\frac{2\beta}{\beta+2} \leq 1$, from Lemma 2.1 and Lemma 2.2 we have $\|\frac{2\beta}{\beta+2} x_2 - x_1\| \geq 1 - 2\varepsilon$ and therefore $\|x'' - x_1\| \geq (\frac{\beta}{2} + 1)(1 - 2\varepsilon)$. Similarly, we have $\|x''+x_1\| \geq (\frac{\beta}{2}+1)(1-2\varepsilon)$. Hence, $J(X) \geq \|x''+x_1\| \bigwedge \|x''-x_1\| \geq (\frac{\beta}{2}+1)(1-2\varepsilon) \geq (1+\frac{1}{2(1-d(X))})(1-2\varepsilon)$. The proof is completed.

Since $g(X)J(X)=2$ [7], we have the following result.

Corollary 2.3. *For a Banach space X, if $g(X) > \frac{4(1-d(X))}{3-2d(X)}$, then X has weak normal structure.*

Theorem 2.5. *For a Banach space X, if $\delta(1+t) > \frac{t}{2} - td(X)$ for any $0 \leq t \leq 1$ then X has weak normal structure.*

Proof. Suppose X does not have weak normal structure. For $\varepsilon > 0$, let x_1, x_2 x_3, and X_2 be the same as the proof of Theorem 2.4. (See the Figure below).



Consider $y = x_2 - tx_1$ for $0 \leq t \leq 1$, $z \in S(X_2)$ such that $z - y = k_1 x_2$ for some $k_1 > 0$, and $x^{'} \in S(X_2)$ such that $x^{'} - \frac{x_2+x_3}{2} = k_2 x_2$ for some $k_2 > 0$. Then $\|x^{'} - \frac{x_2+x_3}{2}\| \geq 1 - \|\frac{x_1+x_2}{2}\| \geq d(X)$. From convexity of U(X), $\frac{\|z-y\|}{\|x^{'}-\frac{x_2+x_3}{2}\|} \geq \frac{t}{\frac{1}{2}}$, so $\|z-y\| \geq 2td(X)$. Then $z = y + \|z-y\|x_2$, $z+x_1 = y+x_1+\|z-y\|x_2 = (1+\|z-y\|)x_2+(1-t)x_1 = (2+\|z-y\|-t)\frac{(1+\|z-y\|)x_2+(1-t)x_1}{2+\|z-y\|-t}$. So $\|z+x_1\| \geq 2+\|z-y\|-t-\varepsilon \geq 2+2td(X)-t-\varepsilon$. On other hand, from Lemma 2.1, $\|z-x_1\| \geq \|y-x_1\|-\varepsilon = \|x_2-x_1+t(-x_1)\|-\varepsilon = \|x_3+t(-x_1)\|-\varepsilon = (1+t)\frac{\|x_3+t(-x_1)\|}{1+t}-\varepsilon \geq 1+t-2\varepsilon$. We have $\delta(1+t-2\varepsilon) \leq 1 - \frac{2+2td(X)-t-\varepsilon}{2} \leq \frac{t}{2} - td(X) + \varepsilon$. The proof is completed.

From Theorem 2.4 and Corollary 2.3, we have the following.

Theorem 2.6. *For a Banach space X, if $E(X) < 2(1+\frac{1}{2(1-d(X))})^2$ or $f(X) > 2(\frac{4(1-d(X))}{3-2d(X)})^2$, then X has weak normal structure.*

Definition 2.3. [15] A Banach space X is called uniformly nonsquare if there exists $\delta > 0$ such that if $x, y \in S_X$, then either $\|x+y\|/2 \leq 1-\delta$ or $\|x-y\|/2 \leq 1-\delta$.

Definition 2.4. ([5], [6]) Let X and Y be Banach spaces. We say that Y is *finitely representable in* X if for any $\varepsilon > 0$ and any finite dimensional subspace $N \subseteq Y$ there is an isomorphism $T : N \to X$ such that for any $y \in N$, $(1-\varepsilon)\|y\| \leq \|Ty\| \leq (1+\varepsilon)\|y\|$.

The Banach space X is called *superreflexive* if any space Y which is finitely representable in X is reflexive.

Remark 2.2. It is well known that if X is uniformly nonsquare then X is supper-reflexive and therefore X is reflexive.

Theorem 2.7. ([8], [11], [12], [13]) *For a Banach space X, if* $E(X) < 8, f(X) > 2, J(X) < 2, g(X) > 1$*, or* $Q(x) < 8$ *then X is uniformly nonsquare, therefore X is reflexive.*

From Remark 2.2 and Theorem 2.7, we have the following result.

Theorem 2.8. *For a Banach space X, if* $E(X) < 2(1+\frac{1}{2(1-d(X))})^2$*,* $f(X) > 2(\frac{4(1-d(X))}{3-2d(X)})^2$*,* $J(X) < 1+\frac{1}{2(1-d(X))}$*,* $g(X) > \frac{4(1-d(X))}{3-2d(X)}$*,* $Q(X) < 4+\frac{2}{1-d(X)}$ *or* $O(X) > 4+\frac{2}{1-d(X)}$*, then X has normal structure.*

Since $\delta(2) > 0$ implies X is reflexive, we have the following result.

Theorem 2.9. *For a Banach space X,if* $\delta(1+t) > \frac{t}{2} - td(X)$ *for any* $0 \leq t \leq 1$ *then X has normal structure.*

To prove the relationship among $A_1(X)$, C(X) and normal structure we need the following two results.

Lemma 2.3. *Let X be a Banach space without weak normal structure, then for any* $x_2 \in S(X)$ *and any* $\varepsilon, 0 < \varepsilon < 1$*, there exist* x_1*,* x_2*,* x_3 *in* $S(X)$ *satisfying*

(i) $x_2 - x_3 = ax_1$ *with* $|a-1| < \varepsilon$*;*

(ii) $|||x_1 - x_2|| - 1|, |||x_3 - (-x_1)|| - 1| < \varepsilon$*; and*

(iii) $||\frac{x_1+x_2}{2}||, ||\frac{x_3+(-x_1)}{2}|| > 1-\varepsilon$.

Proof. It is a direct result from the proof of Lemma 2.3 in [12] with $x_2 = z_1$.

Lemma 2.4. [10] *C(X) is a continuous function on [0, 2), and an increasing function from* $[0,2]$ *to* $[0,1]$.

Theorem 2.10. *For a Banach space X, if $A_1(X) > 1 + \frac{1}{2(1-C(1))}$, then X has weak normal structure.*

Proof. Suppose X does not have weak normal structure. For $\varepsilon > 0$ and any $x_2 \in S(X)$, let x_1, x_2, and x_3 in $S(X)$ satisfying the three conditions in Lemma 2.1. Let X_2 be the two dimensional space spanned by x_1, x_2 and x_3. Let x and x'' be the same as the proof in Theorem 2.4. Then we can write x'' as $x'' = \gamma\frac{x_2+x_3}{2} = \gamma\frac{2x_2-x_1}{2}$ where $\gamma \leq \frac{1}{1-C(1)}$, and $x'' - x_1 = \gamma\frac{x_2+x_3}{2} - x_1 = \gamma\frac{2x_2-x_1}{2} - x_1 = \gamma x_2 - (\frac{\gamma}{2}+1)x_1 = (\frac{\gamma}{2}+1)(\frac{2\gamma}{\gamma+2}x_2 - x_1)$. Since $\frac{2\gamma}{\gamma+2} \leq 1$, from Lemma 2.1 and Lemma 2.2 we have $\|\frac{2\gamma}{\gamma+2}x_2 - x_1\| \leq 1-2\varepsilon$ and therefore $\|x''-x_1\| \leq (\frac{\gamma}{2}+1)(1-2\varepsilon)$. Similarly, we have $\|x''+x_1\| \leq (\frac{\gamma}{2}+1)(1-2\varepsilon)$. Since $x_2 \in S(X)$ is arbitrary, $A_1(X) \leq \frac{1}{2}(\|x''+x_1\| + \|x''-x_1\|) \leq (\frac{\gamma}{2}+1)(1-2\varepsilon) \leq (1+\frac{1}{2(1-C(1))})(1-2\varepsilon)$. The proof is completed.

Theorem 2.10 is also a direct result of Remark 2.1, Theorem 2.3 and Corollary 2.2.

3. Uniform normal structure

Let $\mathscr{F}$ be a filter on an index set I, and let $\{x_i\}_{i\in I}$ be a subset in a Hausdorff topological space X, $\{x_i\}_{i\in I}$ is said to converge to x with respect to $\mathscr{F}$, denote by $\lim_{\mathscr{F}} x_i = x$, if for each neighborhood V of x, $\{i \in I : x_i \in V\} \in \mathscr{F}$. A filter $\mathscr{U}$ on I is called an ultrafilter if it is maximal with respect to the ordering of the set inclusion. An ultrafilter is called trivial if it is of the form $\{A : A \subseteq I, i_0 \in A\}$ for some $i_0 \in I$. We will use the fact that if $\mathscr{U}$ is an ultrafilter, then

(i) for any $A \subseteq I$, either $A \in \mathscr{U}$ or $I \setminus A \in \mathscr{U}$;

(ii) if $\{x_i\}_{i\in I}$ has a cluster point x, then $\lim_{\mathscr{U}} x_i$ exists and equals to x.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space equipped with the norm $\|(x_i)\| = \sup_{i\in I}\|x_i\| < \infty$.

Definition 3.1. [17] Let $\mathscr{U}$ be an ultrafilter on I and let

$$N_{\mathscr{U}} = \{(x_i) \in l_\infty(I, X_i) : \lim_{\mathscr{U}} \|x_i\| = 0\}.$$

The ultraproduct of $\{X_i\}_{i\in I}$ is the quotient space $l_\infty(I, X_i)/N_{\mathscr{U}}$ equipped with the quotient norm.

We will use $(x_i)_{\mathscr{U}}$ to denote the element of the ultraproduct. It follows from the assertion (ii) above, and the definition of quotient norm that

$$\|(x_i)_{\mathscr{U}}\| = \lim_{\mathscr{U}} \|x_i\|. \tag{1}$$

In the following we will restrict our index set I to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X$ for all $i \in \mathbb{N}$ for some Banach space X. For an ultrafilter $\mathscr{U}$ on $\mathbb{N}$, we use $X_{\mathscr{U}}$ to denote the ultraproduct.

We proved the following two results in ([8], [10], [11], [12], [14]).

Theorem 3.1. *For any Banach space X and for any nontrivial ultrafilter U on $\mathbb{N}$, $E(X) = E(X_{\mathscr{U}})$, $f(X) = f(X_{\mathscr{U}})$, $J(X) = J(X_{\mathscr{U}})$, $g(X) = g(X_{\mathscr{U}})$, $Q(X) = Q(X_{\mathscr{U}})$, $C(X) = C(X_{\mathscr{U}})$, and $d(X) = d(X_{\mathscr{U}})$.*

Theorem 3.2. [9] *For any Banach space X, and for any nontrivial ultrafilter U on $\mathbb{N}$, $\delta_{X_{\mathscr{U}}}(\varepsilon) = \delta_X(\varepsilon)$ for $0 \le \varepsilon \le 1$.*

By using the untra-techniques, similar to the proofs in ([8], [10], [11], [12],[14]) we have the following.

Theorem 3.3. *For any Banach space X, and for any nontrivial ultrafilter U on $\mathbb{N}$, $A_1(X) = A_1(X_{\mathscr{U}})$.*

Theorem 3.4. *For a Banach space X, if $E(X) < 2(1 + \frac{1}{2(1-d(X))})^2$, $f(X) > 2(\frac{4(1-d(X))}{3-2d(X)})^2$, $J(X) < 1 + \frac{1}{2(1-d(X))}$, $g(X) > 1 + \frac{1}{2(1-C(1))}$, or $Q(X) < 4 + \frac{2}{1-d(X)}$, then X has uniformly normal structure.*

Theorem 3.5. *For a Banach space X,if $\delta(1+t) > \frac{t}{2} - td(X)$ for any $0 \le t \le 1$ then X has uniformly normal structure.*

Finally, we remark here that Theorem 3.4 improves Theorem 5.3 for $f(X)$ and for $E(X)$ with $d(X) \ge \frac{\sqrt{10}-3}{\sqrt{10}-2}$ of [11]. Theorem 3.4 improves Theorem 5.3 of [12]. Theorem 3.4 improves Theorem 11 of [10]. Theorem 3.5 improves Theorem 8 of [9].

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